Prof. Dr. Christoph Bohle Tübingen SS 2019

Geometry of Manifolds II : Exercise Sheet 3

Jonas Ziefle 30. April 2019

Diese Aufgaben sind schriftlich auszuarbeiten und am 9. Mai vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Compute $H^1_{dR}(\mathbb{R}^2\backslash\{0\})$.

Aufgabe 2. Show that

a) the exterior product induces a product on $H^*_{dR}(M)$ and
b) if M is a compact, orientable manifold of dimension n without boundary,

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R} \qquad ([\omega],[\eta]) \to \int_M \omega \wedge \eta$$

is well defined.

Aufgabe 3. Prove that:

a) Given a cohomology class $[\omega] \in H^k_{dR}(M)$ and a map $f\colon N \to M$ defined on a k–dimensional compact, oriented manifold N without boundary,

$$\int_{f(N)} [\omega] := \int_N f^*\omega$$

is well defined.

b) If there is an oriented, compact manifold $\tilde{N}$ with boundary $\partial\tilde{N} = N_1 \sqcup -N_2$ and $F\colon \tilde{N} \to M$ such that $F_{|N_1} = f_1$ and $F_{|N_2} = f_2$, then

$$\int_{f_1(N_1)} [\omega] = \int_{f_2(N_2)} [\omega].$$

Aufgabe 4. Let M be a compact, connected, orientable manifold of dimension n without boundary. Prove that every form $\omega \in \Omega^n(M)$ with $\int_M \omega \neq 0$ generates $H^n_{dR}(M)$, in particular

$$H^n_{dR}(M) \cong \mathbb{R}.$$

(Hint: a) Show that an n–form $\omega \in \Omega^n(M)$ with $\int_M \omega = 0$ that has compact support in the domain of a chart diffeomorphic to an open cube in $\mathbb{R}^n$ is exact. b) Deduce that $[\omega_1] = [\omega_2]$ for a pair of forms $\omega_i \in \Omega^n(M)$ with $\int_M \omega_i = 1$, $i = 1, 2$ and compact support in open sets U_i, $i = 1, 2$ with $U_1 \cap U_2 \neq \emptyset$ that are diffeomorphic to open cubes in $\mathbb{R}^n$. c) Using that M is arcwise connected, show that the preceding also holds if $U_1 \cap U_2 = \emptyset$. d) Prove the claim using a suitable partition of unity.)