Geometry of Manifolds II : Exercise Sheet 4

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Diese Aufgaben sind schriftlich auszuarbeiten und am 16. Mai vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Compute the cohomology of $\mathbb{RP}^2$. (Hint: $\mathbb{RP}^2$ can be decomposed into a disc and a Möbius strip.)

Aufgabe 2. Let

$$0 \to A^* \to B^* \to C^* \to 0$$

be a short exact sequence of cochain complexes. Prove: if two of the three complexes have finite dimensional cohomology, the third has finite dimensional cohomology as well. In particular, then

$$\chi(B^*) = \chi(A^*) + \chi(C^*),$$

where for a cochain complex A^* with finite dimensional cohomology

$$\chi(A^*) = \sum_k (-1)^k \dim(H^k(A^*))$$

denotes its *Euler–Charakteristic*.

Aufgabe 3. Show that

$$\chi(M) = \chi(M_1) + \chi(M_2) - 2$$

for the connected sum $M = M_1 \# M_2$ of two compact, oriented surfaces M_1 and M_2.

Aufgabe 4. The *degree* $\deg(f)$ of a differentiable map $f\colon M \to N$ between compact, connected, and oriented manifolds M and N of dimension n without boundary is

$$\deg(f) := \int_M f^*\omega,$$

where $\omega \in \Omega^n(N)$ is an n–form with $\int_N \omega = 1$. Show that $\deg(f)$ is well defined, does not change under (differentiable) homotopies, and satisfies $\deg(f) \in \mathbb{Z}$. (Hint: use that, by Exercise 4 on Sheet 3,

$$\int_M : H^n_{dR}(M) \to \mathbb{R} \qquad [\omega] \mapsto \int_M \omega$$

is an isomorphism and same for N. Moreover, use that, by Sard's Theorem, for generic $q \in N$ the differential df_p is invertible for all $p \in f^{-1}(\{q\})$.)