Prof. Dr. Christoph Bohle Tübingen SS 2019

Geometry of Manifolds II : Exercise Sheet 10

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Diese Aufgaben sind schriftlich auszuarbeiten und am 11. Juli vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Show that

a) the Ricci tensor of a semi–Riemannian manifold is symmetric and
b) in dimension $n = 3$ the Ricci tensor determines the curvature tensor.

Aufgabe 2. Prove that the length of

$$X \wedge Y$$

for linearly independent vectors X, Y equals the area of the parallelogram generated by X and Y.

Aufgabe 3. Given a semi Riemannian manifold M of dimension $n \geq 3$, prove that if the sectional curvature $K(\sigma_p)$ at every point $p \in M$ is independent of the 2–plane $\sigma_p \subset T_pM$, it moreover doesn't depend on the point p. (Hint: Apply the second Bianchi identity $d^\nabla \mathcal{R} = 0$ to the Riemann curvature tensor, which here is of the form

$$\mathcal{R}(X, Y)Z = K(g(Y, Z)X - g(X, Z)Y)$$

for some function K. The computation can be simplified by using that, for every $p \in M$, there locally exists a *synchronous basis*, i.e., a local orthonormal frame $(X_1, ..., X_n)$ of TM for which $(\nabla X_i)(p) = 0$, $i = 1, ..., n$. Its existence can be seen by parallel transporting an orthonormal frame at p along radial geodesics.)

Aufgabe 4. Using the fundamental theorem of submanifold theory, prove that a Riemannian manifolds of constant sectional curvature $K \neq 0$ is locally isometric to a sphere (if $K > 0$) or hyperbolic space (if $K < 0$).