Prof. Dr. Christoph Bohle | Tübingen WS 2019/20

Geometry of Manifolds/Geometry in Physics
Exercise Sheet 1

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Diese Aufgaben sind schriftlich auszuarbeiten und am 24. Oktober vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Show that a metric space admits a countable basis of the topology if and only if it admits a countable dense subset. Conclude that $\mathbb{R}^n$ admits a countable basis.

Aufgabe 2. Show that $\mathbb{R}^n/\mathbb{Z}^n$ equipped with the quotient topology (cf. Exercise 6) is a topological manifold. Show that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle $\mathbb{S}^1$.

Aufgabe 3. Show that $\mathbb{R}^2/\mathbb{Z}^2$ with the quotient topology (as in Exercise 2) is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ with the product topology (cf. Exercise 7).

Aufgabe 4. Show that the *projective spaces*

$$\mathbb{KP}^n = (\mathbb{K}^{n+1} \backslash \{0\})_{/\sim}, \qquad v \sim w \Leftrightarrow v = w\lambda \text{ for } \lambda \in \mathbb{K}$$

over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with the quotient topology (cf. Exercise 6) are topological manifolds.

Additional exercises (for the first exercise session)

Aufgabe 5. Show that a bijective continuous map

$$f\colon X \to Y$$

from a compact space X to a Hausdorff space Y is a homeomorphism.

Aufgabe 6. Given a topological space X and a surjective map

$$p\colon X \to \tilde{X},$$

the *quotient topology* on $\tilde{X}$ is defined by

$$O \subset \tilde{X} \text{ is open} \quad \Longleftrightarrow \quad p^{-1}(O) \subset X \text{ is open.}$$

Show that:

i) The quotient topology is the maximal topology on $\tilde{X}$ such that p is continuous.

ii) The quotient topology is the only topology on $\tilde{X}$ such that
   - p is continuous and
   - for any map $f\colon \tilde{X} \to Z$ with values in a topological space Z:

$$f \text{ is continuous} \quad \Longleftrightarrow \quad f \circ p \text{ is continuous.}$$

iii) It is possible that X is Hausdorff, but the quotient topology on $\tilde{X}$ is not.

Aufgabe 7. Given topological spaces X_1 and X_2, the *product topology* on

$$X_1 \times X_2$$

is the topology with basis $\{O_1 \times O_2 \mid O_i \subset X_i \text{ open}\}$. Show that:

i) The product topology is the smallest topology such that both projections $\pi_i\colon X_1 \times X_2 \to X_i$, $i = 1, 2$ are continuous.

ii) The product topology is the only topology on $X_1 \times X_2$ such that
   - the projections $\pi_i\colon X_1 \times X_2 \to X_i$, $i = 1, 2$ are continuous and
   - for any map $f\colon Z \to X_1 \times X_2$ from a topological space Z to $X_1 \times X_2$

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \pi_i \circ f, i = 1, 2 \text{ are continuous.}$$

iii) If X_1, X_2 are Hausdorff, then $X_1 \times X_2$ is Hausdorff.