Prof. Dr. Christoph Bohle Tübingen WS 2019/20

Geometry of Manifolds/Geometry in Physics
Exercise Sheet 5

Jonas Ziefle 12. November 2019

Diese Aufgaben sind schriftlich auszuarbeiten und am 21. November vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Denote by g the Riemannian metric on $S^n \subset \mathbb{R}^{n+1}$ induced by the Euclidean metric. Compute the coordinate representations of g with respect to the charts defined by the stereographic projections φ_N und φ_S

Aufgabe 2. Determine the representations of the Euclidean metric on $\mathbb{R}^3$ with respect to spherical and cylindrical coordinates. Find coordinates on $\mathbb{R}^3$ with respect to which the coefficients of the metric are not diagonal.

Aufgabe 3. The Minkowski metric on $\mathbb{R}^{n+1}$ is the non–degenerate symmetric bilinear form

$$g = -dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

a) Show that the Minkowski metric induces a Riemannian metric on

$$H^n := \{x \in \mathbb{R}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_n^2 = -1,\, x_0 > 0\} \subset \mathbb{R}^{n+1}.$$

b) Show that stereographic projection with respect to $(-1, 0, \ldots, 0)$ is an isometry between H^n and the Poincaré ball model of hyperbolic space, i.e. $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$ with the metric $g_x = \frac{4}{(1-|x|^2)^2}(dx_1^2 + \ldots + dx_n^2)$.

Aufgabe 4. Let $f\colon M \to N$ be a diffeomorphism between Riemannian manifolds (M, g) and (N, h). Show that the following are equivalent:

(i) f is conformal,
(ii) f preserves angles, and
(iii) f preserves right angles.

Aufgabe 5.* A C^1–map $f\colon U \to \mathbb{R}^2 = \mathbb{C}$ on an open subset $U \subset \mathbb{R}^2 = \mathbb{C}$ is called *conformal*, if pointwise the vectors $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are perpendicular and have the same length. Show that:

a) If U is connected, then f is conformal $\Leftrightarrow$ f is holomorphic or anti–holomorphic.

b) If U is connected then f preserves lengths $\Leftrightarrow$ f is of the form $f(z) = az + b$ or $a\bar{z} + b$ with $a \in \mathbb{S}^1 \subset \mathbb{C}$ and $b \in \mathbb{C}$.