Prof. Dr. Christoph Bohle Tübingen WS 2019/20

# Geometry of Manifolds/Geometry in Physics
# Exercise Sheet 7

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**Diese Aufgaben sind schriftlich auszuarbeiten und am 5. Dezember vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.**

**Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.**

**Aufgabe 1.** Let $M \subset \mathbb{R}^N$ be a submanifold. Show that its Levi–Civita connection with respect to the induced metric is

$$\nabla_X Y = (X(Y))^T,$$

where $()^T$ stands for the pointwise orthogonal projection to the tangent space and $X(Y)$ denotes componentwise directional derivative of $Y \in \mathfrak{X}(M) \subset C^\infty(M, \mathbb{R}^N)$ in direction of $X$. (In order to check torsion freeness, use local extensions of the vector fields to open subsets of $\mathbb{R}^N$ and naturality of the Lie bracket.)

**Aufgabe 2.** Let $(M, g)$ be a Riemannian manifold and $\tilde{g} = e^{2u} g$ with smooth $u\colon M \to \mathbb{R}$ a conformally changed metric. Show that the Levi–Civita connections $\nabla$ and $\tilde{\nabla}$ of $g$ and $\tilde{g}$ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X, Y)\operatorname{grad} u,$$

where $\operatorname{grad}(u)$ denotes the vector field such that $g(\operatorname{grad}(u), .) = du$.

**Aufgabe 3.** The aim of this exercise is to prove the existence of a partition of unity subordinated to any given open covering $V_\alpha$, $\alpha \in I$ of a manifols $M$.

i) Prove the claim for compact $M$.
ii) Show that every manifold admits a family of open subsets $G_i$, $i \in \mathbb{N}$ such that for all $i$

$$\bar{G}_i \text{ compact} \qquad \text{and} \qquad \bar{G}_i \subset G_{i+1}.$$

(Start by choosing a countable basis of the topology consisting of open subsets whose closure is compact.)
iii) Choose for every $i$ a finite covering of $\bar{G}_{i+1} \backslash G_i$ by sets of the family $V_\alpha$, $\alpha \in I$. Construct a locally finite covering of $M$ by intersecting these coverings of $\bar{G}_{i+1} \backslash G_i$ with $G_{i+2} \backslash \bar{G}_{i-1}$ (where $G_0 = \emptyset$).
iv) Prove the claim for general $M$.

**Aufgabe 4.** Show that every manifold admits a Riemannian metric. (Use the existence of a partitions of unity.)