Prof. Dr. Christoph Bohle Tübingen WS 2019/20

Geometry of Manifolds/Geometry in Physics
Exercise Sheet 9

Jonas Ziefle 10. Dezember 2019

Diese Aufgaben sind schriftlich auszuarbeiten und am 19. Dezember vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Show that a covariant derivative $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ on a vector bundle $E \to M$ is a local operator and can therefore be restricted to open subsets.

Aufgabe 2. Show that every vector bundle $E \to M$ admits a covariant derivative (Hint: partition of unity). Show furthermore that the set of connections on $E \to M$ is an affine space with vector space

$$\Omega^1(M, \mathrm{End}(E)) = \Gamma(T^*M \otimes \mathrm{End}(E)),$$

where $\mathrm{End}(E) = E^* \otimes E$.

Aufgabe 3. A connection ∇ on a real vector bundle E with a bundle metric $\langle . \rangle$ is called *metric* if

$$d\langle \varphi, \psi \rangle = \langle \nabla\varphi, \psi \rangle + \langle \varphi, \nabla\psi \rangle$$

for all sections φ, $\psi \in \Gamma(E)$. Show that

a) a connection is metric iff its connection forms with respect to orthonormal frames are skew symmetric,
b) every vector bundle with bundle metric admits a metric connection, and
c) the set of metric connections on $(E, \langle . \rangle)$ is an affine space with underlying vector space equal to the space the 1–forms with values in skew symmetric endomorphisms of E.

Aufgabe 4. Denote by $E = U \times \mathbb{K}^r$ the trivial rank r vector bundle on a simply connected open subset $U \subset \mathbb{R}^n$. A connection on E is given by the connection form (with respect to the standard frame)

$$\omega = -\sum_{i=1}^{n} A_i \, dx_i$$

with $A_i \in C^\infty(U, Mat(r \times r, \mathbb{K}))$. Show that the existence of a parallel frame, i.e., a solution $\Phi \in \Gamma(U, GL_r(\mathbb{K}))$ to $\nabla\Phi = d\Phi + \omega\Phi = 0$, implies that

$$\frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i = [A_i, A_j]$$

for all i, $j \in \{1, ..., n\}$.