Geometry of Manifolds/Geometry in Physics
Exercise Sheet 12

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Diese Aufgaben sind schriftlich auszuarbeiten und am 23. Januar vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1. Let γ be a regular curve in a Riemannian manifold (M, g). Then γ is a *pregeodesic*, i.e., a curve that admits a reparametrization making it a geodesic, if and only if $\nabla_{\gamma'}\gamma' \| \gamma'$, i.e., $\nabla_{\gamma'}\gamma'$ is parallel to γ'.

Aufgabe 2. Let f and h be isometries of a connected Riemannian manifold (M, g). Show that $f \equiv h$ if for one point p we have $f(p) = h(p)$ and $df_p = dh_p$. (Hint: start by proving a local result using the exponential map.)

Aufgabe 3. Show that:

a) $SO(n)$ is a submanifold of the $(n \times n)$–matrices. What is its tangent space in Id?
b) If X is a skew symmetric matrix, the curve $\gamma(t) = e^{tX}$ is a geodesic of $SO(n)$ with respect to the induced metric. (Here e^A stands for the usual matrix exponential function.)

(Hint: the Euclidean metric on the space of $n \times n$–matrices is $\langle A, B\rangle = tr(A^T B)$. Both the left and right action of the group $SO(n)$ on $n \times n$–matrices is by isometries. Determine the normal space to $SO(n)$ in the point Id. Determine the tangent and normal spaces to $SO(n)$ in the point $G \in SO(n)$. Conclude that $\gamma''(t)$ is contained in the normal space to $SO(n)$ in the point $\gamma(t)$.)

Aufgabe 4. The aim of this exercise is to determine the geodesics of the hyperbolic plane using Poincaré's half plane model

$$H^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \qquad g = \frac{1}{y^2}(dx^2 + dy^2).$$

a) Prove that the shortest path between two points on the y–axis is along the line segment between the points.
b) Parametrize the positive part of the y–axis with respect to hyperbolic arc length.
c) Prove that the geodesics of the hyperbolic plane are precisely the half circles and half lines in the upper half plane that intersect the x–axis orthogonally and are parametrized with respect to hyperbolic arc length. (Hint: use that $SL(2, \mathbb{R})$ acts as a group of isometries on the hyperbolic plane via

$$\begin{bmatrix} z = x + iy \\ 1 \end{bmatrix} \mapsto A \begin{bmatrix} z = x + iy \\ 1 \end{bmatrix}$$

for $A \in SL(2, \mathbb{R})$. Under this action every pair of points in the upper half plane can be mapped to a pair of points on the y–axis.