Prof. Dr. Christoph Bohle Tübingen WS 2019/20

Geometry of Manifolds/Geometry in Physics
Exercise Sheet 13

Jonas Ziefle 23. Januar 2020

Diese Aufgaben sind schriftlich auszuarbeiten und am 30. Januar vor der Vorlesung abzugeben. Für jede Aufgabe gibt es 4 Punkte.

Zweierabgaben sind erlaubt. Bitte bei der ersten Abgabe Matrikelnummer(n) angeben.

Aufgabe 1.* Let $f(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Compute $f^*(\omega_1)$ and $f^*(\omega_2)$ for

$$\omega_1 = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx \qquad \text{and} \qquad \omega_2 = x\,dx + y\,dy.$$

Aufgabe 2.* Compute the exterior differential of the following differential forms:

a) $xy\,dx \wedge dy + 2x\,dy \wedge dz + 2y\,dx \wedge dz$,
b) $z^2\,dx \wedge dy + (z^2+2y)\,dx \wedge dz$,
c) $13x\,dx + y^2\,dy + xyz\,dz$,
d) $e^x\cos(y)\,dx - e^x\sin(y)\,dy$,
e) $x\,dy \wedge dz + y\,dx \wedge dz + z\,dx \wedge dy$.

Aufgabe 3.* Show that

$$\begin{aligned} d\omega(X_0,\dots,X_k) = &\sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k) \end{aligned}$$

for $\omega \in \Omega^k(M)$ and $X_0,\dots,X_k \in \Gamma(TM)$.

Aufgabe 4.* Prove that

$$d^M(f^*\omega) = f^*(d^N\omega)$$

for $f\colon M \to N$ and $\omega \in \Omega^k(N)$.

Aufgabe 5.* An *orientation* of a manifold M of dimension n is given by an equivalence class of positive atlasses, where an atlas is called positive if all its changes of charts have positive Jacobi determinants. Prove that there is a 1–1–correspondence between orientations on M and equivalence classes of nowhere vanishing forms $\omega \in \Omega^n(M)$, where $\omega \sim \tilde{\omega}$ if and only if there is a positive function $\lambda \in C^\infty(M)$ such that

$$\tilde{\omega} = \lambda\omega.$$

(Hint: Use a partition of unity.)

Aufgabe 6.* Prove that an oriented Riemannian manifold (M, g) of dimension n admits a unique form $\omega \in \Omega^n(M)$, its *volume form*, with the property that $\omega_p(X_1, ..., X_n) = 1$ for all $p \in M$ and every positive orthonormal basis X_1,...,X_n of T_pM. Show that with respect to a positive chart $(U, \varphi = (x_1, ..., x_n))$

$$\omega_{|U} = \sqrt{\det(g_{ij})}dx_1 \wedge ... \wedge dx_n,$$

where g_{ij} denotes the coefficient matrix of g with respect to the chart.

Aufgabe 7.* Prove that an orientation preserving diffeomorphism between two oriented Riemannian manifolds is an isometry if and only if it is conformal and volume preserving.

Aufgabe 8.* How does the volume form

$$\omega^2 = dx \wedge dy$$

of $\mathbb{R}^2$ look with respect to polar coordinates? How does the volume form

$$\omega^3 = dx \wedge dy \wedge dz$$

of $\mathbb{R}^3$ look with respect to spherical and cylindrical coordinates?