Integrable system methods for higher genus CMC surfaces in the 3-sphere

Habilitationsschrift

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Summary

1. Introduction

In this summary I collect recent achievements in applying integrable systems methods to the study of compact, oriented constant mean curvature (CMC) surfaces of higher genus in the round 3-sphere S^3 and in euclidean 3-space $\mathbb{R}^3$. Integrable systems methods have shown to be very powerful tools for studying special differential equations like the KdV and the KP equations or the harmonic map equation on Riemann surfaces. These methods are used to construct many new solutions to the corresponding equations and are also applied to classify special subclasses. In surface theory, integrable systems techniques provide a detailed understanding of the space of all CMC tori, while real analytic methods are used to prove the existence of examples with special properties (like the Lawson minimal surface [**La**] or minimizers of the Willmore energy [**MaNe, NdSchae**]). The aim of my research [**1, 2, 3, 4**] is to establish effective methods for constructing compact CMC surfaces of higher genus, and to apply these methods to obtain a better understanding of the space of all compact CMC surfaces in space forms.

CMC surfaces are the stationary points of the area functional under the constraint of fixed enclosed volume. While there are no compact minimal surfaces (characterized by zero mean curvature) in $\mathbb{R}^3$ there exist compact immersed CMC surfaces of every genus in $\mathbb{R}^3$ [**Kap90, Kap91, Kap95**], and even compact and embedded minimal surfaces in S^3 [**La, PS**]. There is a well-known unified treatment of surfaces of constant non-zero mean curvature in $\mathbb{R}^3$ and CMC surfaces (including the minimal ones) in S^3. I will mainly focus on CMC surfaces in S^3, but almost all results remain true for CMC surfaces in $\mathbb{R}^3$.

Hopf [**Hopf**] proved the first global result about CMC surfaces showing that all CMC spheres in space forms are round. To do so, he observed that the complex bilinear part of the second fundamental form of a CMC surface is a holomorphic quadratic differential and hence it vanishes on 2-spheres. The round 2-spheres are also the only compact embedded CMC surfaces in $\mathbb{R}^3$ due to Alexandrov's maximum principle [**Al**]. Hopf's conjecture that the only immersed compact CMC surfaces in $\mathbb{R}^3$ are the round spheres was actually disproven by a counterexample of Wente [**We**] in the early eighties of the last century. The birth of integrable surface theory was the explicit construction of these so-called Wente tori by Abresch [**Ab**] in terms of elliptic functions. This inspired the work of Pinkall and Sterling [**PS**] and Hitchin [**Hi90**] who proved that CMC tori and harmonic maps from tori to S^3 form integrable systems. Based on these papers Bobenko [**Bo**] gave formulas for all CMC tori in terms of theta functions.

Those papers [**PS, Hi90, Bo**] from the late eighties and early nighties initiated a decade of large progress in the (integrable systems) theory of CMC surfaces in three-dimensional space forms. Among the most important problems studied since are the Lawson conjecture, now solved by Brendle [**Br**] using different methods, the construction of new CMC surfaces, either compact or with suitable boundary conditions like Delaunay ends, and the deformation theory of CMC surfaces in the 3-sphere in order to explore the moduli space of all compact CMC surfaces. There has been some recent progress on the latter problem [**HaKiSchm, KiSchmSch**] for classes of CMC tori and CMC cylinders of finite type. There has also been some progress on the second problem, starting with the work of Dorfmeister, Pedit and Wu [**DPW**] who developed a Weierstrass type representation for all simply connected CMC surfaces in terms of a loop algebra valued holomorphic 1-form. The main ingredient of the DPW theory are loop group factorization methods. These methods have already been applied earlier to the KdV and KP equations (see [**SW**] and the references therein). The relation between the spectral curve approach and the DPW approach to CMC tori was clarified for example in [**Hel**] or [**McI**]. In marked contrast to the study of tori, no theory was developed during this period for general compact CMC surfaces.

The DPW theory has been successfully used to construct many new CMC surfaces [**KiKoRoSch05, KiKoRoSch07, KiMcISch, KiRoSch, KiSchS, RoSch, Sch**], most of which are surfaces with Delaunay ends. On the other hand, only lately have compact CMC surfaces of higher genus been constructed (numerically) by integrable system methods [**3, 4**].

After recalling some basic observations and results used throughout this summary (Section 2) I briefly review the spectral curve theory of CMC tori in Section 3. In Section 4 it is explained how to apply loop group factorization methods in order to study compact CMC surfaces of higher genus in S^3. In particular, the global version of the DPW method for compact CMC surfaces of genus 2 [**1, 2**] is discussed. Moreover, I present the recent spectral curve approach to (a class of) CMC surfaces of genus 2 [**3**] and show how to use these integrable systems techniques to explore the space of compact embedded CMC surfaces experimentally [**4**]. In the last section, I give a short outlook for further study of compact CMC surfaces of genus $g \geq 2$.

2. Preliminaries

Analytic description of CMC surfaces. All the material presented here is well-known, see for example [**Bo, Hi90**] or also [**2**].

An Immersion $f\colon M \to S^3$ induces a Riemannian metric on the surface M. As M is always assumed to be oriented, this induces a Riemann surface structure on M. An easy calculation shows that f is minimal if and only if f is harmonic (with respect to the induced Riemann surface structure):

$$d^\nabla * df = 0,$$

where ∇ is the pullback via f of the Levi-Civita connection on S^3 by f and $*$ is given by the complex structure of the Riemann surface. Moreover, f automatically satisfies the integrability equation

$$d^\nabla df = 0$$

which are a variant of the Gauss-Codazzi equations. More explicitly, these equations can be written as

$$d * \phi = 0$$

and

$$d\phi + \frac{1}{2}[\phi \wedge \phi] = 0$$

where $\phi = f^*\omega$ is the $\mathfrak{su}(2)$-valued pull-back of the Maurer-Cartan form of $S^3 = SU(2)$. Therefore, $*\phi$ integrates, at least locally, to a conformal immersion into the euclidean 3-space $\mathfrak{su}(2)$ and one easily computes that this surface has constant mean curvature. Moreover, this surface in euclidean 3-space has only translational periods and no rotational ones. This transformation can be reversed to obtain minimal surfaces (with periods) from CMC surfaces in $\mathbb{R}^3$.

The associated family of flat connections. A useful way to rewrite the Gauss-Codazzi equations for harmonic maps works as follows: First recall that $\mathfrak{su}(2) \otimes \mathbb{C} = \mathrm{End}_0(\mathbb{C}^2)$ is the space of trace-free endomorphisms. Note that the pull-back of the Levi-Civita connection $\nabla = d + \frac{1}{2}\phi$ also gives rise to a connection on the trivial $\mathbb{C}^2$-bundle known as the induced spinor connection. The pull-back $\phi = f^*\omega$ of the Maurer-Cartan form of S^3 splits into its complex linear part $2\Phi \in \Gamma(M; K\,\mathrm{End}_0(\mathbb{C}^2))$ and its complex anti-linear part $-2\Phi^* \in \Gamma(M; \bar{K}\,\mathrm{End}_0(\mathbb{C}^2))$:

$$\frac{1}{2}\phi = \frac{1}{2}f^*\omega = \Phi - \Phi^*.$$

Then, the $\mathrm{SL}(2,\mathbb{C})$ connections

$$\nabla^\lambda := \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

are flat for all $\lambda \in \mathbb{C}^*$, unitary for $\lambda \in S^1$ and trivial for $\lambda = \pm 1$ by construction. From this *associated family of flat connections* one can recover the immersion f as the gauge between $\nabla^{-1} = d$ and $\nabla^1 = d + \phi = d + f^{-1}df$ (written in the left trivialization of $\mathrm{SU}(2) = S^3$).

A slight generalization of the above discussion gives the following well-known theorem:

Theorem. *Let $f \colon M \to S^3$ be a conformal CMC immersion. Then there exists an associated family of flat* $\mathrm{SL}(2,\mathbb{C})$*-connections*

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

on a hermitian rank 2 bundle $V \to M$ which is unitary along $S^1 \subset \mathbb{C}^*$ and trivial at $\lambda_1 \neq \lambda_2 \in S^1$. Here, Φ is a nowhere vanishing complex linear endomorphism-valued 1-form which is nilpotent and Φ^* is its adjoint. The immersion f is given as the gauge between ∇^{λ_1} and ∇^{λ_2} where we identify $SU(2) = S^3$ and its mean curvature is $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. Conversely, every family of flat $\mathrm{SL}(2,\mathbb{C})$*-connections satisfying the properties above determines a conformal CMC immersion.*

In the limit $\lambda_1 \to \lambda_2$ the mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ of the immersion into S^3 blows up. In this case, such loops of flat connections can be used to obtain CMC surfaces in $\mathbb{R}^3$ without periods if ∇^{λ_1} is trivial and the derivative of the monodromy representation at λ_1 is 0. The exact formulas for CMC surfaces in $\mathbb{R}^3$ can be found in [**Bo**].

It is well known [**Hi90**], that the generic connection ∇^λ of the associated family of a compact CMC surface is not trivial unless the surface is totally umbilic and therefore (a covering of) a round sphere. Moreover, for CMC immersions from a compact Riemann surface of genus $g \geq 2$, the generic connection ∇^λ of the associated family is irreducible [**2**]. Therefore, we have a coarse classification of compact CMC surfaces into three categories: the genus 0 case with trivial monodromy, the torus case with abelian monodromy and the case of higher genus CMC surfaces with (generically) non-abelian monodromy.

2.1. Holomorphic structures. I need to recall some notions and results concerning holomorphic bundles on Riemann surfaces and their relation to flat connections. This will play a prominent role in the study of higher genus surfaces in Section 4 and is also quite useful for understanding the case of tori.

On a Riemann surface every connection ∇ splits into its complex linear part

$$\partial^\nabla = \frac{1}{2}(\nabla - i * \nabla)$$

and its complex anti-linear part

$$\bar\partial^\nabla = \frac{1}{2}(\nabla + i * \nabla).$$

The former is called a anti-holomorphic structure while the later is a holomorphic structure. For a generic holomorphic structure $\bar{\partial}$ on a complex bundle of degree 0, there exists a flat connection ∇ with the given complex anti-linear part. In this situation ∇ is also called a holomorphic connection with respect to $\bar{\partial}$. The difference Ψ between two flat connections with the same holomorphic structure is a holomorphic endomorphism-valued 1-form, a so-called Higgs field.

On a complex vector bundle equipped with a hermitian metric, there is a one-to-one correspondence between unitary connections and holomorphic structures. The unitary connection corresponding to a holomorphic structure is called its Chern connection. It is an easy application of the Weitzenböck technique to show that there are no holomorphic subbundles of positive degree on a holomorphic vector bundle which admits a flat unitary connection. Holomorphic bundles of degree 0 are called semi-stable if there are no holomorphic subbundles of positive degree, and stable if all (proper) holomorphic subbundles have strictly negative degree.

The celebrated Narasimhan-Seshadri theorem [**NaSe**] states that every stable holomorphic bundle of degree 0 admits a unique flat connection which is unitary with respect to some hermitian metric. This gives rise to the Narasimhan-Seshadri correspondence between (stable) holomorphic structures and unitary flat connections.

Hitchin abelianization. While the moduli space of holomorphic line bundles on a compact Riemann surface M is an abelian variety (the Picard group), the moduli space of holomorphic structures on bundles of higher degree is more complicated, particularly for surfaces of genus $g \geq 2$. A nice way to understand these moduli spaces is given by the Hitchin system [**Hi87**], where one equips a holomorphic bundle with a Higgs field. In general the eigenlines of a Higgs field are not well-defined on the surface M itself, but on a covering $\tilde{M} \to M$. The eigenlines give rise to points in some subspace of the Picard variety of $\tilde{M}$.

We are interested in the special case of a rank 2 bundle with trivial determinant and trace-free Higgs field Ψ: First, note that $\det \Psi$ is a holomorphic quadratic differential and I simply assume that $\det \Psi$ has simple zeros only. Consider the double covering

$$\pi\colon \tilde{M} = \{\omega_x \in K_M \mid \omega_x^2 = \det \Psi\} \to M,$$

the natural involution $\sigma\colon \tilde{M} \to \tilde{M}$ and the canonical eigenvalue section $\omega \in H^0(\tilde{M}, \pi^* K_M)$. The Riemann surface $\tilde{M}$ has genus $4g-3$, where g is the genus of M. The eigenlines

$$L_{\pm} = \ker \pi^* \Psi \mp \omega \,\mathrm{Id}$$

are well-defined and intersect at the zeros of ω, i.e., at the branch points of π. As the original bundle has trivial determinant, we obtain

$$L_+ \otimes \sigma^* L_+ = L_+ \otimes L_- = \pi^* K_M^*,$$

i.e., the eigenline bundles lie in an affine Prym variety. Note that one can take the pull-back of a spin bundle as the origin in order to identify this space with the ordinary Prym variety of $\tilde{M} \to M$.

Conversely, for a bundle L satisfying $L \otimes \sigma^* L = \pi^* K_M^*$ one obtains a holomorphic rank 2 bundle with trivial determinant together with a Higgs field Ψ such that the eigenlines of Ψ are L and $\sigma^* L$. The rank 2 bundle can be recovered as the push forward $\pi_* L^*$ of the bundle L^*. Altogether, one obtains a birational map from an open set in the Prym variety to the moduli space of stable bundles.

3. The spectral curve theory for CMC tori

In this section, I recall the basic ideas of the spectral curve approach to CMC tori in S^3. My presentation here follows [**Hi90**]. Even though the theory for CMC tori works differently than in the case of compact higher genus CMC surfaces, I think that some understanding of the genus 1 case is necessary for what follows in Section 4.

The basic observation is that flat $\mathrm{SL}(2, \mathbb{C})$ connections on a torus generically split into the direct sum of flat line bundle connections which are dual to each other. These flat line bundle connections have a basic invariant, their (abelian) monodromy representation. Fixing generators Γ_1 and Γ_2 of the fundamental group of the torus, the monodromy representation is given by two complex non-zero numbers acting on the line by multiplication. A flat connection is specified modulo gauge transformations by its monodromy representation (up to conjugation). Therefore, the gauge class of a generic flat $\mathrm{SL}(2, \mathbb{C})$ connection on a torus is fixed by two numbers representing the monodromy of oneself-dual of the parallel eigenline bundles. The two eigenline bundles can only coincide if they are self-dual or, equivalently, if their monodromy is ± 1. Therefore, the complement of the set of those flat $\mathrm{SL}(2, \mathbb{C})$ connections which split into a direct sum of flat line bundle connections is contained in the subset of the flat connections whose monodromies along Γ_1 and Γ_2 have trace ± 2.

Let $f\colon T^2 \to S^3$ be a conformal CMC immersion, and ∇^λ be its associated family of flat connections. As was shown in [**Hi90**], the generic connection ∇^λ is non-trivial. Therefore, the eigenvalues of the monodromies of the eigenlines vary non-trivially, and can be parametrized as holomorphic functions on a Riemann surface which double covers $\mathbb{C}^*$. The branch points of this covering are over those $\lambda \in \mathbb{C}^*$ at which the two eigenvalues of the individual monodromies coalesce to odd order. A fundamental observation in [**Hi90**] is that the branch points of this covering do not accumulate at $\lambda = 0$ and $\lambda = \infty$. Therefore, the Riemann surface can be compactified to a hyper-elliptic Riemann surface Σ. In the case at hand it turns out that Σ branches over 0 and ∞. Moreover, a careful analysis of the limiting behavior of the monodromies shows that the logarithmic differentials θ and $\tilde{\theta}$ of the eigenvalue functions (with respect to the two generators of the fundamental group) have second order poles over 0 and ∞. The gauge equivalence classes of the connections ∇^λ are determined by the curve Σ and by the abelian differentials of the second kind θ and $\tilde{\theta}$ (at least for those λ where the eigenvalues of the monodromies are not ± 1). Note that the hyper-elliptic curve Σ and the abelian differentials satisfy automatically a reality condition imposed by the unitarity of the connections ∇^λ along the unit circle.

In general Σ is not exactly the spectral curve as defined in [**Hi90**] and only coincides with it for generic CMC immersions. There are two reasons for working with a slightly modified definition of the spectral curve: First, when deforming CMC tori by deforming their spectral data, branch points can come together providing singular curves. And second, the spectral data should capture information about the order of intersection of the eigenline bundles. We do not want to go into details but only restrict to the general case in which Hitchin's spectral curve is the curve Σ defined above.

The knowledge of the gauge equivalence classes of ∇^λ for all $\lambda \in \mathbb{C}^*$ does not yet uniquely determine the CMC torus. In fact, one needs to consider the eigenline bundles $E_x \to \Sigma$ (for fixed points $x \in T^2$) corresponding to the eigenvalues parametrized by the points in Σ. This eigenline bundle extends holomorphically over 0 and ∞ and has (in the generic case) degree $-(g+1)$, where g is genus of the spectral curve. As a matter of fact, this eigenline bundle flows linearly in the Picard group of Σ when changing the base point $x \in T^2$. It was shown by Hitchin, that the spectral data consisting of the spectral curve, the abelian differentials, and the eigenline bundle uniquely determine the CMC torus. Moreover, for spectral data which satisfy the reality conditions one obtains a CMC immersion into S^3 (which generically has periods). The period closing problem is transcendental in terms of the spectral data $(\Sigma, \theta, \tilde{\theta})$.

Polynomial Killing fields. Before I explain the reconstruction of CMC tori out of the spectral data in more detail I want to introduce another viewpoint on the spectral data for CMC tori: For a generic flat $\mathrm{SL}(2,\mathbb{C})$-connection on an elliptic curve, the parallel eigenlines are exactly the holomorphic line subbundles of degree 0 of the induced holomorphic structure. By the same reasoning, they coincide generically with the anti-holomorphic line subbundles of degree 0 with respect to the induced anti-holomorphic structure. Moreover, the space of parallel trace-free endomorphisms is generically complex

1-dimensional and coincides with the space of holomorphic (respectively anti-holomorphic) trace-free endomorphisms.

Consider a CMC torus and its associated family of flat connections. Then, the induced family of holomorphic structures extends to $\lambda = 0$ while the induced family of anti-holomorphic structures extends through $\lambda = \infty$. As a consequence, there exists a holomorphic line bundle $\mathcal{L} \to \mathbb{CP}^1$ which coincides for generic $\lambda \in \mathbb{CP}^1$ with the space of parallel trace-free endomorphisms. This line bundle has a meromorphic section $\mathcal{E}$ without zeros or poles in $\mathbb{C} \subset \mathbb{CP}^1$. One can show that $\mathcal{L}$ has non-positive degree, which implies that $\mathcal{E}$ has poles over ∞. By the very definition, it is a (polynomial) family of endomorphisms $\mathcal{E}(\lambda)$, which are parallel with respect to ∇^λ. It is called the polynomial Killing field of the immersion. Fixing a point $x \in T^2$ and evaluating the endomorphism, the eigenlines of the holomorphic endomorphism coincide with the parallel eigenline bundle E_x over the spectral curve.

To construct a CMC immersion from spectral data $(\Sigma, \theta, \tilde{\theta}, L)$ satisfying the reality conditions we apply the push forward construction again: there exists a meromorphic function f (the eigenvalue function of $\mathcal{E}$) on Σ such that it induces for every line bundle $L \in Pic_{g+1}(\Sigma)$ an endomorphism on the trivial holomorphic rank 2 bundle over $\mathbb{CP}^1$ whose eigenline with respect to the eigenvalue f is L^*. As we already know, the eigenlines vary linearly in the Picard variety of Σ with respect to $x \in T^2$ (the conformal type of the torus T^2 is determined by the quotient of the principle parts of θ and $\tilde{\theta}$ over $\lambda = 0$; this torus sits naturally in the *real* part of the Picard variety after fixing a base point L) and we obtain an algebro-geometric construction of the parallel eigenlines for the family of flat connections. The connections on the eigenlines are already determined by the abelian differentials θ and $\tilde{\theta}$. Therefore, we have basically all informations to reconstruct the family of flat connections and hence also the CMC immersion. Concrete formulas for the CMC immersions in terms of theta functions can be found in [**Bo**].

4. Higher genus CMC surfaces

The spectral curve approach to CMC tori relies on the fact that the monodromy representation of the associated family of flat connections is abelian. This is no longer the case for compact CMC surfaces of higher genus $g \geq 2$ as shown in [**2**]. In fact, any CMC map from a compact Riemann surface into S^3 whose associated family of flat connections is reducible factors through a CMC torus or is a covering of a round 2-sphere [**Ger**]. Therefore, we cannot simply apply the methods explained in the previous section to CMC immersions of higher genus surfaces but have to develop a new theory. In the following the main achievements of [**1, 2, 3, 4**] are summarized.

The key idea for constructing higher genus CMC surfaces by integrable system methods goes back to a paper of Dorfmeister, Pedit and Wu [**DPW**] and is part of what is nowadays known as the DPW method. They use special λ-families of holomorphic endomorphism-valued 1-forms $\xi(\lambda)$, called DPW potentials, to construct all simply connected CMC surfaces by applying loop group factorizations as follows: They consider the family of flat connections $d + \xi(\lambda)$ and take a parallel frame Φ depending holomorphically on λ. Recall that the Iwasawa decomposition of a holomorphic loop Φ of $\mathrm{SL}(2,\mathbb{C})$ matrices uniquely determines its unitary part $\mathcal{F}$ (i.e. the factor which is unitary along the unit circle) and its positive part B extending to $\lambda = 0$ by

$$\Phi = \mathcal{F}B,$$

see for example [**SW, DPW**]. Then, the unitary part $\mathcal{F}$ of the loop group Iwasawa decomposition (point-wise on the domain) is a parallel frame for the associated family of flat connections of a suitable CMC surface.

These loop group factorization methods can be generalized to give an appropriate tool for constructing and studying compact CMC surfaces of higher genus. The prototype for this method is the following theorem proven in [**3**] (see also [**DW, KiKoRoSch07**] for the corresponding DPW version).

Theorem 1. *Let $U \subset \mathbb{C}$ be an open set containing the disc of radius $1+\epsilon$. Let $\lambda \in U \setminus \{0\} \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

- *$\tilde{\nabla}^\lambda$ has the asymptotic expression at $\lambda = 0$ given by*
$$\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla} + \ldots$$
where $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ is nowhere vanishing and nilpotent;
- *the* **reality condition** *holds: for all $\lambda \in S^1 \subset U \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^\lambda$ is unitary with respect to this metric;*
- *the* **extrinsic closing condition** *holds: $\tilde{\nabla}^\lambda$ is trivial for two so-called* Sym *points $\lambda_1 \neq \lambda_2 \in S^1$.*

Then there exists a unique (up to spherical isometries) CMC surface $f\colon M \to S^3$ of mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ such that its associated family of flat connections ∇^λ and the family $\tilde{\nabla}^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g(\lambda) = \tilde{\nabla}^\lambda$ for all λ.

Note that this theorem can be generalized easily to families of flat connections with apparent singularities, i.e., singularities of the connections around which the monodromy is trivial. This is needed if one uses meromorphic connections to describe CMC surfaces of higher genus. The theorem also applies to situations, where the connections $\tilde{\nabla}^\lambda$ are not unitarizable for a finite number of points on the unit circle. One can also generalize it to the case where the connections are only defined on a small punctured disc around $\lambda = 0$ by using the r-Iwasawa decomposition (see [**KiKoRoSch07**] for more details in this direction).

The benefit of Theorem 1 is that instead of working with holomorphic curves into the infinite dimensional space of all flat $\mathrm{SL}(2, \mathbb{C})$ connections one can study holomorphic curves into finite dimensional (sub)spaces of flat connections containing sufficiently many representatives gauge equivalence classes of flat $\mathrm{SL}(2, \mathbb{C})$ connections. Moreover, one can choose "classes" of flat connections which seem to be most appropriate for the given task. I explain two particularly useful instances of this method in the following.

Compact CMC surfaces via families of meromorphic connections. In [**DPW**] families of holomorphic 1-forms are used to produce all simply connected CMC surfaces in $\mathbb{R}^3$ and S^3. For a compact surface of higher genus this does not work as can be seen by the following reasoning: The gauge equivalence classes of the connections ∇^λ depend non-trivially on λ. As the connections are unitary along the unit circle the Narasimhan-Seshadri correspondence implies that the associated family of holomorphic structures $\bar{\partial}^{\nabla^\lambda}$ varies also non-trivially with respect to λ. Hence, there is no holomorphic structure with respect to which the connections ∇^λ can be gauged to holomorphic connections simultaneously.

This problem can be avoided if one allows gauge transformations with pole like singularities and meromorphic connections - those with connection forms which are meromorphic with respect to holomorphic frames. The main question here is how many and which type of poles are needed in order to obtain a positive λ-dependent gauge which transforms the associated family of flat connections into a family of meromorphic connections (with respect to some fixed holomorphic structure). The answer to this question is needed in order to find appropriate subspaces of meromorphic connections in which one can search for holomorphic $\mathbb{C}^*$-families of connections satisfying the conditions of Theorem 1.

It turns out that it is more appropriate to take the holomorphic structure $S^* \oplus S$ of the holomorphic spin bundle of the immersion and its dual (S^* can also be obtained as the holomorphic kernel of the complex linear part Φ of the differential of the immersion) rather than the trivial holomorphic structure. The reason for this is that the holomorphic structure $S^* \oplus S$ is infinitesimal near to the gauge orbit of the holomorphic structure $\bar{\partial}^0$ at $\lambda = 0$. This allows for a better control of the positive gauge. A positive gauge is needed in order to go back from the family of meromorphic connections to the original associated family of flat connections via the Iwasawa decomposition. Note that one can easily switch to meromorphic connections on the trivial $\mathbb{C}^2$-bundle by fixing a meromorphic frame of $S^* \oplus S$ induced by a meromorphic section of the spin bundle.

In [**1, 2**] I restricted to the case of compact, oriented minimal surfaces in S^3 of genus 2. It is clear that all results immediately generalize to the case of CMC surfaces of genus 2. The main theorem of [**2**] provides a very concrete description of the general case of genus 2 CMC surfaces:

Theorem. [**2**] *Let ∇^λ be the holomorphic family of flat connections associated to a compact oriented CMC surface $f\colon M \to S^3$ of genus 2. Assume that $\bar\partial^0$ is stable and that f is homotopic to an embedding. Let S be the associated spinor bundle of f. Label the six Weierstrass points $Q_1, .., Q_6$ of M such that $KS = L(Q_1 + Q_2 + Q_3) = L(Q_4 + Q_5 + Q_6)$.*

Then there exists a positive λ-dependent gauge

$$B : \lambda \in \tilde{B}(0; \epsilon) \subset \mathbb{C} \to \hat{\Gamma}(\mathrm{End}(V))$$

with pole-like singularities at $Q_1, .., Q_6$ of order 1 such that the connections

$$\hat{\nabla}^\lambda := \nabla^\lambda \cdot B_\lambda$$

have poles of order 1 on the diagonal (with respect to the unitary decomposition $V = S^{-1} \oplus S$) at $Q_1, .., Q_6$ and poles of order 2 in the lower left entry at $Q_1, .., Q_3$ and in the upper right at $Q_4, .., Q_6$. The family $\hat{\nabla}^\lambda$ has an expansion in λ of the form

$$\hat{\nabla}^\lambda = \begin{pmatrix} \nabla_0^* & \lambda^{-1} + \omega \\ -\frac{i}{2}Q & \nabla_0 \end{pmatrix} + \textit{higher order terms},$$

where ∇_0 is a meromorphic connection on S, $\omega \in \mathcal{M}(M; \mathbb{C})$ is some meromorphic function and $Q \in H^0(K^2)$ is the Hopf field of the CMC immersion.

This gives fairly explicit knowledge of a finite dimensional space of meromorphic connections which can be used to produce CMC surfaces $f\colon M \to S^3$ of genus 2 via families of flat connections. In particular cases one can further specify the space of meromorphic connections: Consider Lawson symmetric CMC surfaces of genus 2, i.e., those which are provided with a group of (extrinsic) symmetries generated by the hyper-elliptic involution, a holomorphic $\mathbb{Z}_3$-action with 4 totally branched fixed points, and another involution with 2 fixed points. Then, the quotient $M/\mathbb{Z}_3 = \mathbb{C}\mathrm{P}^1$ is the projective line, and the associated family of flat connections ∇^λ can be gauged into $d + \eta(\lambda)$ via a positive gauge, where

$$(1)\quad \eta=\eta_{A,B}=\pi^*\begin{pmatrix} -\frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)}+\frac{A}{z} & \lambda^{-1}-\frac{(A+\frac{2}{3})(A-\frac{1}{3})}{B}z^2 \\ \frac{B}{(z^2-z_0^2)(z^2-z_1^2)}-\frac{\lambda A(A+1)z_0^2z_1^2}{z^2(z^2-z_0^2)(z^2-z_1^2)} & \frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)}-\frac{A}{z}\end{pmatrix}dz.$$

In (1) $\pm z_0$ and $\pm z_1$ are the images of the branch points of $M\to M/\mathbb{Z}_3=\mathbb{C}\mathrm{P}^1$, and A and B are holomorphic functions in λ. A proof of this formula was given in [**1**]. Note that the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on the 4-punctured sphere whose local monodromies have trace -1 is complex 2-dimensional. Then A and B provide local coordinates on this space. Thus, the formula above is the best one can hope for in the case of Lawson symmetric CMC surfaces of genus 2. An open question is how to determine the λ-dependent functions A and B (defined on an open neighborhood of the unit disc) such that the family of flat connections $d+\xi_{A(\lambda),B(\lambda)}$ is unitarizable along the unit circle. I will come back to this question in the following sections.

Spectral curves for Lawson symmetric CMC surfaces. A disadvantage of the use of meromorphic connections is that every parametrization of gauge equivalence classes of flat connections via meromorphic connections introduces coordinates on the moduli space of flat connections. These coordinates fail to cover the full moduli space of flat connections and are branched at special points. Thus, for a full theoretical understanding of CMC surfaces of higher genus one needs a better description of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on compact Riemann surfaces. The character variety, i.e., the space of representations of the fundamental group modulo conjugation, is not appropriate for our purposes as it does not enable us to impose the necessary limiting behavior of the families of flat connections when λ tends to 0 or ∞. Moreover, working with monodromies instead of flat connections representing the gauge classes does not allow for a direct computation and visualization of a CMC surface. The approach developed in [**3**] is based on the Hitchin abelianization and works as follows.

As the family of holomorphic structures $\bar\partial^\lambda$ extends to $\lambda=0$, a useful structure on the moduli space of flat connections $\mathcal{A}$ is given by the (holomorphic) projection $\mathcal{A}\to\mathcal{M}$ of the moduli space of flat connections to the moduli space of semi-stable holomorphic bundles. (Here I ignore the fact that not all flat connections give rise to stable or semi-stable holomorphic structures. The reader may look into [**3**] in order to see how to overcome this and related issues.) It turns out, that this projection is a non-trivial affine holomorphic vector bundle over the moduli space $\mathcal{M}$. The underlying holomorphic vector bundle is the cotangent bundle $T^*\mathcal{M}$. In this setup, the family of flat connections associated to a CMC immersion gives rise to a holomorphic map

$$[\bar{\partial}^\lambda]\colon \mathbb{C}^* \to \mathcal{A} \to \mathcal{M}$$

which extends to $\lambda = 0$. Its lift $[\nabla^\lambda]\colon \mathbb{C}^* \to \mathcal{A}$ has a first order pole over $\lambda = 0$. Moreover, the map $[\nabla^\lambda]$ coincides with the (real-analytic) "Narasimhan-Seshadri section" (consisting of the unitarizable flat connections)

$$\varphi_{NS}\colon \mathcal{M} \to \mathcal{A}$$

along the unit circle $S^1 \subset \mathbb{C}^*$. At the first glance this seems even less explicit and computable as the character variety approach, but one can use the Hitchin abelianization in order to understand the moduli spaces more concretely.

Consider the case of Lawson symmetric CMC surfaces $f\colon M \to S^3$ of genus 2. We need to study the moduli spaces of Lawson symmetric connections, i.e., those which are equivariant with respect to the Lawson symmetries, and the moduli space of Lawson symmetric holomorphic structures. The space of Lawson symmetric Higgs fields, with respect to a fixed (stable) Lawson symmetric holomorphic structure, is complex 1-dimensional, and the determinant of any Lawson symmetric Higgs field is a multiple (possibly zero) of the Hopf differential of the Lawson symmetric CMC surface. This holomorphic quadratic differential has 4 simple zeros. Thus, the eigenlines of a Lawson symmetric Higgs field are (Lawson) symmetric elements in the affine Prym variety of the Riemann surface $\tilde{M} \to M$ which double covers M and branches over the 4 zeros. It turns out that the eigenlines of Lawson symmetric Higgs fields are in one-to-one correspondence with the elements of the Jacobian of the elliptic curve $\tilde{M}/\mathbb{Z}_3$. As every Lawson symmetric holomorphic structure has a complex 1-dimensional space of Lawson symmetric Higgs fields and because the generic Higgs field has to different eigenlines one obtains a double covering

$$Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{M}^{Lawson-symmetric}$$

by sending an eigenline to the corresponding holomorphic structure. Similarly, one obtains a two-to-one correspondence between the moduli space of flat line bundles on the torus $\tilde{M}/\mathbb{Z}_3$ and the moduli space $\mathcal{A}^{Ls}$ of Lawson symmetric flat $\mathrm{SL}(2,\mathbb{C})$-connections, see [**3**]. Note that this correspondence provides us with an explicit description of the moduli spaces.

As in the case of tori one can parametrize $\mathbb{C}^*$-families of flat (Lawson symmetric) $\mathrm{SL}(2,\mathbb{C})$ connections via families of flat line bundles (on $\tilde{M}/\mathbb{Z}_3$). Again, this leads to the notion of a spectral curve, i.e., a double covering of the spectral plane $\mathbb{C}^*$. This spectral curve branches over $\lambda = 0$ as in the case of tori. The spectral data, i.e., the spectral curve together with the map into the moduli space of flat line bundles on the torus $\tilde{M}/\mathbb{Z}_3$, satisfy a reality condition. In contrast to the case of tori, this condition is transcendental in terms of the spectral data.

Conversely, spectral data satisfying the reality and the Sym point condition give rise to compact CMC surfaces by applying Theorem 1. Analogous to the case of tori, one cannot expect in general that spectral data uniquely determine a CMC surface. In fact, there may exist (a finite dimensional space of) dressing transformations of compact CMC surfaces which do not change the spectral data. But these transformations break the Lawson symmetries [**3**].

Experiments. In the previous section the spectral curve approach to Lawson symmetric CMC surfaces of genus 2 was described. As explained there, the reality conditions are transcendental in terms of the spectral data. Therefore, it is much more difficult to construct new examples than in the case of tori. We have performed computer experiments [**4**] in order to construct the Lawson surface of genus 2 and CMC deformations of it numerically. In this way we also obtained new insights about the moduli space of compact CMC surfaces in S^3 of higher genus.

We started by computing the Narasimhan-Seshadri section numerically, which allowed for an implementation of the reality condition in terms of the spectral data. We then successfully carried out the numerical search for the spectral data of the Lawson surface of genus 2. In order to visualize the Lawson surface (see Figure 1), we transformed the spectral data into a DPW potential of the form (1). This transformation is transcendental in nature, but is explicitly given in terms of theta functions on the elliptic curve $\tilde{M}/\mathbb{Z}_3$. The main reason for performing this transformation is due to the fact that the DPW method has already been implemented on the computer by Nicholas Schmitt. I would like to mention here, that we have not been able to perform this initial experiment on the Lawson surface entirely within the DPW approach, mainly because there are no values $A, B \in \mathbb{C}$ such that the connection $d + \eta_{A,B}$ has trivial monodromy. In contrast, it is quite easy to impose the extrinsic closing conditions within the spectral curve approach. Moreover, it was unclear whether the DPW potential of the Lawson surface exists on the whole punctured λ-disc. Hence a meaningful computer experiment testing the unitarizability of the connections along the unit circle in the DPW setup was not possible.

As explained above, there exists a transformation between the spectral curve approach and the DPW approach. By translating the numerical spectral data of the Lawson surface into a DPW potential for the Lawson surface, we overcame the problems of the DPW approach. In particular, we were able to impose a generalized Sym point condition in the DPW approach [**4**].

Building on the expertise gained from the experiments on the Lawson surface we were able to study CMC deformations of the Lawson surface in the 3-sphere. We were able to construct two disjoint 1-parameter families of Lawson symmetric CMC surfaces of genus 2 numerically (see Figure 1 and Figure

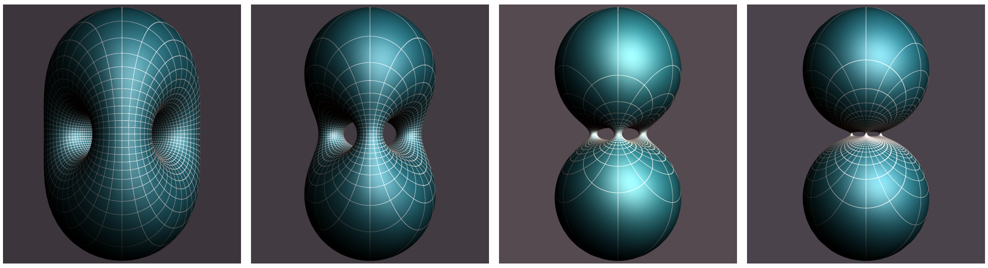

FIGURE 1. The constant mean curvature deformation of Lawson's genus 2 minimal surface $\xi_{2,1}$. Images by Nicholas Schmitt.

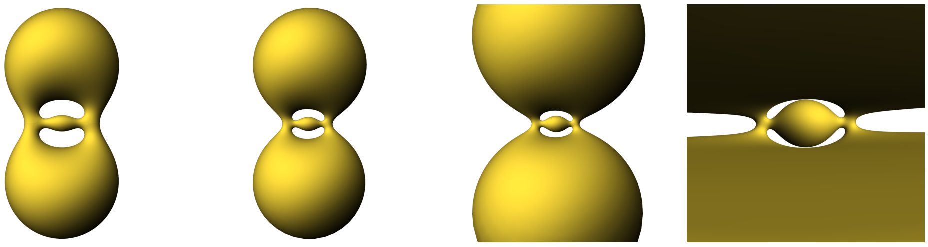

FIGURE 2. Unlike the CMC Lawson family in Figure 1, this family of genus two CMC surfaces in the 3-sphere is not connected to Lawson's minimal surface $\xi_{2,1}$. Images by Nicholas Schmitt.

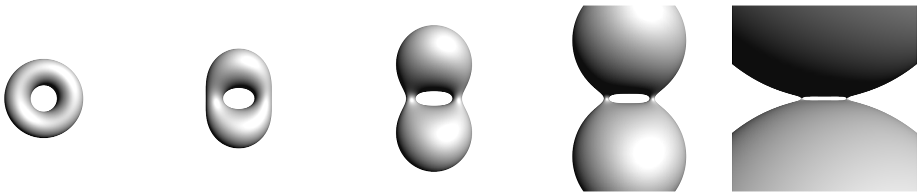

FIGURE 3. The family of 2-lobe Delaunay tori in the 3-sphere, starting at a homogeneous torus and limiting to a doubly covered minimal 2-sphere. Images by Nicholas Schmitt.

2). The first family is reminiscent of the CMC deformation of the Clifford torus through homogeneous CMC tori while the second family behaves similarly to the CMC family of 2-lobe Delaunay tori (see Figure 3). These experiments have provided for the first time a detailed picture of the space of embedded (Lawson symmetric) CMC surfaces of genus 2 in the 3-sphere. The experiments generalize to the Lawson surfaces $\xi_{g,1}$ and produce 1-parameter families of Lawson symmetric CMC surfaces in S^3. In the case of tori, our experiments are in accord with the well-developed theory of CMC tori.

5. Outlook

There are several open questions regarding the spectral curve approach (and more generally the integrable systems approach) to compact CMC surfaces of higher genus. The first one is how to extend the spectral curve approach to all CMC surfaces of higher genus. It is clear that such a general spectral curve theory cannot provide effective formulas for all CMC surfaces of arbitrary genus. Instead it should be more like a framework in which particularly interesting subclasses (like symmetric CMC surfaces or CMC surfaces of low genus) can be dealt with.

The most important question is how to handle the reality condition of the spectral data in a way that is explicit enough to allow an existence proof of the experimentally found CMC surfaces of higher genus. A promising approach is based on the hyper-Kähler geometry of the moduli space $\mathcal{A}$ of flat $\mathrm{SL}_2(\mathbb{C})$ connections. The relation between the reality condition of the spectral data and the hyper-Kähler geometry of the moduli space is made most transparent by considering the fact [**BisRag, Tyu**] that the natural symplectic form [**Gold**] on the moduli space of flat connections identifies with the $(0,1)$ part of the derivative of the "Narasimhan-Seshadri section" $\mathcal{S}\colon \mathcal{M} \to \mathcal{A}$ on the space $\mathcal{M}$ of stable bundles. In [**3**] I determined the asymptotically dominant parts of $\mathcal{S}$ and of the Goldman symplectic form. These dominant parts behave similar to their counterparts in the case of tori, and these observations will hopefully help us better understand the conditions under which spectral data satisfying the reality condition exist.

A related question is whether there exist natural flows (like the Whitham flow for CMC tori) on the space of spectral data which generate the moduli space of CMC surfaces of higher genus. Our experiments indicate that there should indeed exist such a flow, at least on the space of Lawson symmetric CMC surface of genus 2. Theoretically, such a flow should be determined as follows: First, one deforms the isomonodromic flow, i.e., the monodromy preserving deformation, of the family of flat connections (along a curve into the space of Riemann surfaces) in such a way that the asymptotic behavior (the complex linear part must have a first order pole while the complex anti-linear part extends to $\lambda = 0$) of the family of flat connections is preserved. The proof of existence of such a deformation should again be related to the hyper-Kähler geometry of the moduli space of flat connections. And in a second step one singles out those deformation directions (in the space of Riemann surfaces) such that the corresponding flow retain the extrinsic closing conditions (at varying Sym points). An effective deformation theory would not only give rise to many new examples of compact CMC surfaces but also help to understand the moduli space of all compact CMC surfaces.

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APPENDIX A

Lawson's genus two surface and meromorphic connections

by Sebastian Heller
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Lawson's genus two surface and meromorphic connections

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Abstract We investigate the Lawson genus 2 surface by methods from integrable system theory. To this surface we derive a globally defined meromorphic DPW potential, which is determined up to two unknown functions depending only on the spectral parameter.

Keywords Minimal surface of higher genus · DPW method

Mathematics Subject Classification (2000) Primary 53A10 · 53C42 · 53C43; Secondary 14H60

1 Introduction

Concrete examples of the right type have always been fruitful in mathematics. The construction of constant mean curvature (CMC) tori by Wente [21] has stimulated the work on CMC tori in 3-dimensional space forms by many authors. After Abresch's [1] analytical description of the Wente tori, a complete classification of CMC tori in terms of holomorphic data was given by Pinkall and Sterling [19] and Hitchin [10] independently. This lead to the construction of all CMC tori in terms of theta functions by Bobenko [3].

There are also examples of compact minimal surfaces and CMC surfaces in R^3 of higher genus, see [12,15] or [13]. The genus 2 minimal surface $M \subset S^3$ of Lawson [15], which we are going to study here, might be the most simple one. But none of these surface is known explicitly and the construction of them gives no hint how to describe all compact CMC surfaces in space forms. The aim of this paper is to study Lawson's genus 2 minimal surface

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M in a more explicit way. The hope is, that this provides some insight into a theory of higher genus surfaces.

There is a general method due to Dorfmeister et al. [7], which produces, in principal, all CMC surfaces (and, more generally, harmonic maps into symmetric spaces). A CMC surface in a 3-dimensional space form can be described by their associated family of flat connections ∇^ζ on a complex rank 2 bundle V. The idea of the DPW method is to gauge ∇^ζ into a family of meromorphic connections of a special form, the so-called DPW potential, in a way which can be reversed. The advantage is that one can write down meromorphic connections easily. On simply connected domains, each minimal surface can be obtained from such a family of meromorphic connections. To obtain a surface one takes a ζ-depending parallel hence holomorphic frame and splits it into the unitary and positive parts via Iwasawa decomposition in the loop group. Then the unitary part is a parallel frame of a family of unitary connections describing a minimal surface. The surface obtained in this fashion depends on the ζ-depending initial condition of the parallel frame. Dressing, i.e. changing this initial condition, will give different surfaces. If one wants to make surfaces with topology via DPW, one has to ensure that one can patch simply connected domains together. This has been worked out only in very special cases, for example for trinoids, the genus zero CMC surfaces with three Delauney ends, or CMC tori. Up to now there are no examples of closed higher genus surfaces. We show how the DPW method can be applied to the case of the Lawson surface M, and prove that a globally defined DPW potential for the Lawson surface does exists on M. We determine this potential almost explicitly.

In the first part we recall the gauge theoretic description of minimal surfaces in S^3. We give an explicit link to the local description of minimal surfaces via the extended frame. In the third section we shortly explain Lawson's construction of compact minimal surfaces in S^3. We collect all the symmetries and all holomorphic data of Lawson's genus 2 surface. Especially, we will determine the spin bundle S, and we show that the associated rank 2 bundle V with the holomorphic structure $(\nabla^0)''$ is stable.

The main part deals with the construction of a DPW potential for the Lawson surface. We prove that one can find a globally defined gauge with pole like singularities at the Weierstrass points of M, such that the family of connections obtained by gauging is a meromorphic family of connections with respect to the fixed direct sum spin holomorphic structure $S^* \oplus S$ on V. The gauge is positive in the loop group, i.e. it extends to $\zeta = 0$ in a special form, so that one can get back the Lawson surface by the DPW method. Using the symmetries of the surface, we can show that the DPW potential has corresponding symmetries. In fact, there exists a corresponding family of meromorphic connections on $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ with regular singularities at the branch points, and apparent singularities at the images of the Weierstrass points. Moreover, the symmetries are enough to determine the DPW potential on M (and on $\mathbb{CP}^1$) up to two unknown functions, the accessory parameters, depending on ζ only (see Theorem 4.2). These two functions are almost determined by the properties that the holonomy is unitary and that the resulting surface has all symmetries.

The author thanks Aaron Gerding, Franz Pedit and Nick Schmitt for helpful discussions.

2 Minimal Surfaces in S^3

We shortly describe a way of treating minimal surfaces in S^3 due to Hitchin [10]. For more details, one can also consult [11].

We consider the round 3-sphere S^3 with its tangent bundle trivialized by left translation $TS^3 = S^3 \times \Im\mathbb{H}$ and Levi Civita connection given, with respect to the above trivialization, by

$\nabla = d + \frac{1}{2}\omega$. Here $\omega \in \Omega^1(S^3, \Im\mathbb{H})$ is the Maurer–Cartan form of S^3 which acts via adjoint representation. It is well-known that S^3 has a unique spin structure. We consider the associated complex spin bundle $V = S^3 \times \mathbb{H}$ with complex structure given by right multiplication with $i \in \mathbb{H}$. There is a complex hermitian metric $(.,.)$ on it given by the trivialization and by the identification $\mathbb{H} = \mathbb{C}^2$. The Clifford multiplication is given by $TS^3 \times V \to V$; $(\lambda, v) \mapsto \lambda v$ where $\lambda \in \Im\mathbb{H}$ and $v \in \mathbb{H}$. This is clearly complex linear. The induced complex unitary connection is given by

$$\nabla = \nabla^{spin} = d + \frac{1}{2}\omega, \tag{2.1}$$

where the $\Im\mathbb{H}$-valued Maurer–Cartan form acts by left multiplication in the quaternions. Via this construction the tangent bundle TS^3 identifies as the skew symmetric trace-free complex linear endomorphisms of V.

Let M be a Riemann surface and $f \colon M \to S^3$ be a conformal immersion. Then the pullback $\phi = f^*\omega$ of the Maurer–Cartan form satisfies the structural equations

$$d\phi + \frac{1}{2}[\phi \wedge \phi] = 0.$$

Another way to write this equation is

$$d^{\nabla}\phi = 0, \tag{2.2}$$

where $\nabla = f^*\nabla = d + \frac{1}{2}\phi$, with $\phi \in \Omega^1(M; \Im\mathbb{H})$ acting via adjoint representation. From now on we only consider the case of f being minimal. Under the assumption of f being conformal f is minimal if and only if it is harmonic. This is exactly the case when

$$d^{\nabla} * \phi = 0. \tag{2.3}$$

The complex rank 2 bundle $V := f^*V \to M$ can be used to rewrite the equations: Consider $\phi \in \Omega^1(M; f^*TS^3) \subset \Omega^1(M; \mathrm{End}_0(V))$ via the interpretation of TS^3 as the bundle of trace-free skew hermitian endomorphisms of V. Then

$$\frac{1}{2}\phi = \Phi - \Phi^*$$

decomposes into K and $\bar{K}$ parts, i.e. $\Phi = \frac{1}{2}(\phi - i * \phi) \in \Gamma(K\mathrm{End}_0(V))$ and $\Phi^* = \frac{1}{2}(\phi + i * \phi) \in \Gamma(\bar{K}\mathrm{End}_0(V))$. Moreover, f is conformal if and only if $\operatorname{tr}\Phi^2 = 0$. In view of a rank 2 bundle V and $\operatorname{tr}\Phi = 0$ this is equivalent to

$$\det \Phi = 0. \tag{2.4}$$

Note that f is an immersion if and only if Φ is nowhere vanishing. The Eqs. (2.2) and (2.3) are equivalent to

$$\nabla''\Phi = 0, \tag{2.5}$$

where $\nabla'' = \frac{1}{2}(d^{\nabla} + i * d^{\nabla})$ is the underlying holomorphic structure of the pull-back of the spin connection on V. Of course Eq. (2.5) does not contain the property that $\nabla - \frac{1}{2}\phi = d$ is trivial. Locally, or on simply connected sets, this is equivalent to

$$F^{\nabla} = [\Phi \wedge \Phi^*] \tag{2.6}$$

as one easily computes.

Conversely, given an unitary rank 2 bundle $V \to M$ over a simply connected Riemann surface with a special unitary connection ∇ and a trace free field $\Phi \in \Gamma(K\mathrm{End}_0(V))$ without zeros, which satisfy (2.4), (2.5) and (2.6), we get a conformally immersed minimal surface as follows: By Eqs. (2.5) and (2.6), the unitary connections $\nabla^L = \nabla - \Phi + \Phi^*$ and $\nabla^R = \nabla + \Phi - \Phi^*$ are flat. Because M is simply connected they are gauge equivalent. Due to the fact that tr $\Phi = 0$, the determinant bundle $\Lambda^2 V$ is trivial with respect to all these connections. Hence, the gauge is $SU(2) = S^3$-valued with differential $\phi = 2\Phi - 2\Phi^*$. Thus it is a conformal immersion. The harmonicity follows from (2.5).

From Eqs. (2.5) and (2.6) one sees that the associated family of connections

$$\nabla^\zeta := \nabla + \zeta^{-1}\Phi - \zeta\Phi^* \tag{2.7}$$

is flat for all $\zeta \in \mathbb{C}^*$. This family contains all the informations about the surface. It is often much easier to describe the family of connections than the minimal surface explicitly, for example in the case of tori, see [4] or [10], or in the case of a 3-punctured sphere, see [14]. The aim of this paper is to study the associated family of flat connections for the Lawson genus 2 surface, which will be done in Sect. 4.

The geometric significance of the spin structure of an immersion $f : M \to S^3$ is described in Pinkall [18]. We consider the bundle V with its holomorphic structure $\bar\partial := \nabla''$. As we have seen the complex part Φ of the differential of a conformal minimal surface satisfies tr $\Phi = 0$ and $\det \Phi = 0$, but is nowhere vanishing. We obtain a well-defined holomorphic line subbundle

$$L := \ker \Phi \subset V.$$

Because Φ is nilpotent the image of Φ satisfies $\mathrm{im}\Phi \subset K \otimes L$. Consider the holomorphic section

$$\Phi \in H^0(M; \mathrm{Hom}(V/L, KL))$$

without zeros. The holomorphic structure $\bar\partial - \Phi^*$ turns $V \to M$ into the holomorphically trivial bundle $\underline{\mathbb{C}}^2 \to M$. As tr $\Phi^* = 0$, the determinant line bundle $\Lambda^2 V$ of $(V, \bar\partial)$ is holomorphically trivial. This implies $V/L = L^{-1}$ and we obtain

$$\mathrm{Hom}(V/L, KL) = L^2 K$$

as holomorphic line bundles. Because $L^2 K$ has a holomorphic section Φ without zeros, we get

$$L^2 = K^{-1}.$$

Hence, its dual bundle $S = L^{-1}$ is a spinor bundle of the Riemann surface M. Clearly, S^{-1} is the only Φ-invariant line subbundle of V. Moreover, one can show that S^{-1} is the $-i$-eigenbundle of the complex quaternionic structure $\mathcal{J}$ given by quaternionic right multiplication with the right normal vector $R : M \to S^2 \subset \Im\mathbb{H}$, see [5] and [11]. This shows that S gives the spin structure of the immersion.

Let $V = S^{-1} \oplus S$ be the unitary decomposition. With respect to this decomposition the pull-back of the spin connection on S^3 can be written as

$$\nabla = \begin{pmatrix} \nabla^{spin^*} & -\frac{i}{2}Q^* \\ -\frac{i}{2}Q & \nabla^{spin} \end{pmatrix}, \tag{2.8}$$

where ∇^{spin} is the spin connection corresponding to the Levi-Civita connection on M, $Q \in H^0(M, K^2)$ is the Hopf field of the immersion f, and $Q^* \in \Gamma(M, \bar K K^{-1})$ is its adjoint, see [11] for details.

The Higgsfield $\Phi \in H^0(M, K\mathrm{End}_0(V))$ can be identified with

$$\Phi = 1 \in H^0(M; K\mathrm{Hom}(S, S^{-1})),$$

and its adjoint Φ^* is given by the volume form vol of the induced Riemannian metric.

2.1 Local description

Let $U \subset M$ be a simply connected open subset and $z\colon U \to \mathbb{C}$ be a holomorphic chart. Write $g = e^{2u}|dz|^2$ for an appropriate function $u : U \to R$. Choose a local holomorphic section $s \in H^0(U; S)$ with $s^2 = dz$, and let $t \in H^0(U, S^{-1})$ be its dual holomorphic section. Then

$$(e^{-u/2}t, e^{u/2}s)$$

is a special unitary frame of $V = S^{-1} \oplus S$ over U. Write the Hopf field as $Q = q(dz)^2$ for some local holomorphic function $q : U \to \mathbb{C}$.

The connection form of the spin connection ∇^{spin} on the spin bundle $S \to M$ with respect to the local frame s is given by $-\partial u$, and with respect to $e^{u/2}s$, it is given by $\frac{1}{2}i * du$. From formula 2.8 the connection form of ∇ with respect to $(e^{-u/2}t,\ e^{u/2}s)$ is

$$\begin{pmatrix} -\frac{1}{2}i * du & -\frac{i}{2}e^{-u}\bar{q}d\bar{z} \\ -\frac{i}{2}e^{-u}qdz & \frac{1}{2}i * du \end{pmatrix}.$$

The Higgsfield Φ and its adjoint Φ^* are given by

$$\Phi = \begin{pmatrix} 0 & e^u dz \\ 0 & 0 \end{pmatrix}, \quad \Phi^* = \begin{pmatrix} 0 & 0 \\ e^u d\bar{z} & 0 \end{pmatrix}$$

with respect to the frame $(e^{-u/2}t, e^{u/2}s)$. These formulas are well-known, see [6], or, in slightly different notation, [3].

3 Lawson's genus 2 surface

We recall Lawson's construction [15] of the genus 2 minimal surface $f : M \to S^3$. We describe the symmetries of the this surface. We use these symmetries to determine the underlying Riemann surface and the holomorphic structures on its associated bundle. Most of this is well-known, but our arguments in the next sections rely on this.

3.1 Construction of the Lawson surface

For two points $A, B \in S^3$ with distance $dist(A, B) < \pi$ we denote by AB the minimal oriented geodesic from A to B. If A and B are antipodal, i.e. $dist(A, B) = \pi$, and $C \in S^3 \setminus \{A, B\}$, we denote by ACB the unique oriented minimal geodesic from A to B through the point C. For a geodesic γ and a totally geodesic sphere S we denote the reflections across γ and S by r_γ and r_S, respectively.

Let M be an oriented surface with boundary γ, and complex structure J. Let γ be oriented and $X \in T_p\gamma$ with $X > 0$. We say γ represents the oriented boundary if $JX \in T_pM$ represents the exterior normal of the surface for all $p \in \gamma \subset M$.

Consider the round 3-sphere

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C} \oplus \mathbb{C}$$

and the geodesic circles $C_1 = S^3 \cap (\mathbb{C} \oplus \{0\})$ and $C_2 = S^3 \cap (\{0\} \oplus \mathbb{C})$ on it. Take the six points

$$Q_k = (e^{i\frac{\pi}{3}(k-1)}, 0) \in C_1$$

in equidistance on C_1, and the four points

$$P_k = (0, e^{i\frac{\pi}{2}(k-1)}) \in C_2$$

in equidistance on C_2. Consider the closed geodesic convex polygon $\Gamma = P_1Q_2P_2Q_1$ in S^3 with vertices P_1, Q_1, P_2, Q_2 and oriented edges P_1Q_2, Q_2P_2, P_2Q_1, and Q_1P_1. Then there exists an unique solution for the Plateau problem with boundary Γ, i.e. a smooth surface which is area minimizing under all surfaces with boundary Γ. This surface is the fundamental piece of the Lawson surface. One can reflect this solution at the geodesic P_1Q_1 to obtain a smooth surface with piecewise smooth boundary given by the polygon $P_1Q_6P_4Q_1P_2Q_2P_1$. The surface obtained in this way can be rotated around P_1P_2 by $\frac{2}{3}\pi$ two times, to obtain a new minimal surface, call it R, with possible singularity at P_1, and with oriented boundary given by the oriented edges $P_2Q_1P_4$, $P_4Q_6P_2$, $P_2Q_5P_4$, $P_4Q_4P_2$, $P_2Q_3P_4$, and $P_4Q_2P_2$. As Lawson has proven, the point P_1 is a smooth point on this surface. Now, one can continue, and reflect the resulting surface across the geodesic C_1. Again, the surface $R \cup r_{C_1}(R)$ obtained in this way is smooth at each of its points. Moreover it is embedded and orientable. The surface is closed as one can see as follows: The Q_k are fixpoints of r_{C_1}, and r_{C_1} interchanges P_1 and P_3, P_2 and P_4. Moreover this reflection acts orientation preserving on the surface. Therefore the oriented boundary edges $P_2Q_1P_4, P_4Q_6P_2, \ldots, P_4Q_2P_2$ of R are mapped to the oriented boundary edges $P_4Q_1P_2, P_2Q_6P_4, \ldots, P_2Q_2P_4$ of $r_{C_2}(R)$. But by the meaning of the boundary orientation described above one sees that $R \cap r_{C_1}(R)$ is closed.

It is proven by Lawson that the zeros of the Hopf differential Q are exactly at the points $P_1, \ldots, P_4$ of order 1. Then $\deg K^2 = 4g - 4$ gives $g = 2$ for the genus of the Lawson surface.

3.2 Symmetries of the Lawson surface

There are two types of symmetries of the Lawson surface: the first type consists of the symmetries (i.e. reflections at geodesics) which were used to construct the Lawson surface from the fundamental piece. It is clear that they give rise to isometries of the surface. The other symmetries are isometries of S^3 which map the polygon Γ to itself. Then, by the uniqueness of the Plateau solution, they give rise to isometries of the Lawson surface, too.

A generating system of the symmetry group of the Lawson surface is given by

- the $\mathbb{Z}^2$-action generated by Φ_2 with $(a, b) \mapsto (a, -b)$; it is orientation preserving on the surface and its fix points are $Q_1, \ldots, Q_6$;
- the $\mathbb{Z}_3$-action generated by the rotation Φ_3 around P_1P_2 by $\frac{2}{3}\pi$, i.e. $(a, b) \mapsto (e^{i\frac{2}{3}\pi}a, b)$, which is holomorphic on M with fix points $P_1, \ldots, P_4$;
- the reflection at P_1Q_1, which is antiholomorphic; it is given by $\gamma_{P_1Q_1}(a, b) = (\bar{a}, \bar{b})$;
- the reflection at the sphere S_1 corresponding to the real hyperplane spanned by $(0, 1), (0, i), (e^{\frac{1}{6}\pi i}, 0)$, with $\gamma_{S_1}(a, b) = (e^{\frac{\pi}{3}i}\bar{a}, b)$; it is antiholomorphic on the surface,
- the reflection at the sphere S_2 corresponding to the real hyperplane spanned by $(1, 0), (i, 0), (0, e^{\frac{1}{4}\pi i})$, which is antiholomorphic on the surface and satisfies $\gamma_{S_2}(a, b) = (a, i\bar{b})$.

Note that all these actions commute with the $\mathbb{Z}_2$-action. The last two fix the polygon Γ. They and the first two map the oriented normal to itself. The third one maps the oriented normal to its negative.

3.3 The Riemann surface

Using the symmetries, one can determine the Riemann surface structure of the Lawson surface $f : M \to S^3$. One way to describe the Riemann surface structure is to factor out the $\mathbb{Z}_2$-action which is exactly the hyperelliptic involution of the genus 2 surface. Instead of doing this we factor out the $\mathbb{Z}_3$-symmetry which will be much more useful later on.

The quotient $M/\mathbb{Z}_3$ has an unique structure of a Riemann surface such that $\pi : M \to M/\mathbb{Z}_3$ is holomorphic. The degree of this map is 3 and its fixpoints are $P_1, \ldots, P_4$ with branch order 2. Thus $M/\mathbb{Z}_3 = \mathbb{CP}^1$ by Riemann–Hurwitz. We fix this map by the properties $\pi(Q_1) = 0, \pi(P_1) = 1$ and $\pi(Q_2) = \infty \in \mathbb{CP}^1$. Then we have $\pi(Q_3) = \pi(Q_5) = 0$, and $\pi(Q_4) = \pi(Q_6) = \infty$ automatically. A symmetry τ on M gives rise to an action on $\mathbb{CP}^1 = M/\mathbb{Z}_3$ if and only if $\tau(p)$ and p lie in the same $\mathbb{Z}_3$-orbit for all $p \in M$. This happens for all symmetries described above.

The symmetry Φ_2 defines a holomorphic map $\Phi_2 : \mathbb{CP}^1 \to \mathbb{CP}^1$ which fixes 0 and ∞ and satisfies $\Phi_2^2 = \mathrm{Id}$, thus $\Phi_2(z) = -z$. In particular we have $\pi(P_3) = -1$. Similarly, the induced action of γ_{S_2} is antiholomorphic on $\mathbb{CP}^1$, fixes 0 and ∞ and satisfies $\gamma_{S_2}^2 = \Phi_2$. Therefore $\gamma_{S_2}(z) = \pm i\bar{z}$. In fact $\gamma_{S_2}(z) = i\bar{z}$, and we obtain $\pi(P_2) = i$ and $\pi(P_4) = -i$.

We collect the symmetries induces on $\mathbb{CP}^1$:

- the $\mathbb{Z}_2$-action induces $z \mapsto -z$;
- the reflection at $P_1 Q_1$ induces the antiholomorphic map $z \mapsto \bar{z}$;
- the reflection at the sphere S_1 gives $z \mapsto \frac{1}{\bar{z}}$;
- the reflection at the sphere S_2 gives rise to the antiholomorphic map $z \mapsto i\bar{z}$.

These observations easily imply the first part of

Proposition 3.1 *The Riemann surface M associated to the Lawson genus 2 surface is the three-fold covering $\pi : M \to \mathbb{CP}^1$ of the Riemann sphere with branch points of order 2 over $\pm 1, \pm i \in \mathbb{CP}^1$. The Hopf differential of the Lawson genus 2 surface is given by*

$$Q = \pi^* \frac{ir}{z^4 - 1} (dz)^2$$

for a nonzero real constant $r \in R$.

Proof The Hopf differential is $\mathbb{Z}_3$-invariant and has simple zeros at $P_1, \ldots, P_4$. Therefore Q is a non-zero complex multiple of $\pi^* \frac{1}{z^4-1}(dz)^2$. The Hopf differential is the K^2-part of the second fundamental form, i.e. $II = Q + Q^*$ for minimal surfaces. The straight line from 0 to 1 in $\mathbb{CP}^1$ corresponds to the geodesics $Q_1 P_1$, $Q_3 P_1$ and $Q_5 P_1$ in S^3 lying on M. So the geodesic curvature of $Q_1 P_1 \subset S^3$ vanishes which implies the assertion. □

3.4 The holomorphic structures

We use the symmetries to compute the spinor bundle $S \to M$ associated to the Lawson genus 2 minimal surface and the holomorphic structure ∇'' on V.

Proposition 3.2 *Let $f : M \to S^3$ be a conformal minimal immersion. Let $\Psi : S^3 \to S^3$ and $\psi : M \to M$ be orientation preserving isometries such that $f \circ \psi = \Psi \circ f$. Let $S \to M$ be the spinor bundle associated to f. Then*

$$\psi^* S = S$$

as holomorphic bundles.

Proof Because S^3 is simply-connected there is only one $Spin$-structure on S^3. Hence,

$$(\Psi^* V, \Psi^* \nabla^{Spin}) \cong (V, \nabla^{Spin}).$$

From the assertion one sees

$$\psi^* V = \psi^* f^* V = f^* \Psi^* V \cong f^* V = V$$

as unitary bundles with unitary connections on M. Now S^{-1} is the $-i$ eigenbundle of the complex quaternionic structure $\mathcal{J}$ induced by left multiplication with $-R$. But $-R$ is invariant under ψ. The holomorphic structure of $S^{-1} \subset V$ is given by ∇'', which is also invariant under ψ. Therefore, the holomorphic structure of S^{-1} is invariant under ψ . □

The correspondence between equivalence classes of divisors and holomorphic line bundles is classical, see [9]. As above, we denote by $Q_1, \dots, Q_6$ the Weierstrass points of M. Because $g(M) = 2$ there are exactly $\#H^1(M; \mathbb{Z}_2) = 16$ different spin structures on M,i.e. holomorphic line bundles L satisfying $L^2 = K$. We list all of them below: on the left side are the different spin bundles, and on the right side are their pullbacks under symmetry Φ_3 (for the computations we use that $2Q_i - 2Q_j, \quad i, j = 1, \dots, 6$, and $Q_1 + Q_3 + Q_5 - Q_2 - Q_4 - Q_6$ are principal divisors):

L	$\Phi_3^* L$
$L(Q_1)$	$L(Q_5)$
$L(Q_2)$	$L(Q_6)$
$L(Q_3)$	$L(Q_1)$
$L(Q_4)$	$L(Q_2)$
$L(Q_5)$	$L(Q_3)$
$L(Q_6)$	$L(Q_4)$
$L(Q_2 + Q_3 - Q_1)$	$L(Q_6 + Q_1 - Q_5) = L(Q_5 + Q_6 - Q_1)$
$L(Q_2 + Q_4 - Q_1)$	$L(Q_6 + Q_2 - Q_5) = L(Q_3 + Q_4 - Q_1)$
$L(Q_2 + Q_5 - Q_1)$	$L(Q_6 + Q_3 - Q_5) = L(Q_2 + Q_4 - Q_1)$
$L(Q_2 + Q_6 - Q_1)$	$L(Q_6 + Q_4 - Q_5) = L(Q_2 + Q_3 - Q_1)$
$L(Q_3 + Q_4 - Q_1)$	$L(Q_1 + Q_2 - Q_5) = L(Q_2 + Q_5 - Q_1)$
$L(Q_3 + Q_5 - Q_1)$	$L(Q_1 + Q_3 - Q_5) = L(Q_3 + Q_5 - Q_1)$
$L(Q_3 + Q_6 - Q_1)$	$L(Q_1 + Q_4 - Q_5) = L(Q_4 + Q_5 - Q_1)$
$L(Q_4 + Q_5 - Q_1)$	$L(Q_2 + Q_3 - Q_5) = L(Q_4 + Q_6 - Q_1)$
$L(Q_4 + Q_6 - Q_1)$	$L(Q_2 + Q_4 - Q_5) = L(Q_3 + Q_6 - Q_1)$
$L(Q_5 + Q_6 - Q_1)$	$L(Q_3 + Q_4 - Q_5) = L(Q_2 + Q_6 - Q_1)$

From this table one gets that the only Φ_3-invariant spinor bundle is $L(Q_1 + Q_3 - Q_5)$. Because Φ_3 satisfies the conditions of Proposition 3.2 we obtain

Theorem 3.3 *The spinor bundle $S \to M$ of the Lawson genus 2 surface is given by*

$$S = L(Q_1 + Q_3 - Q_5).$$

The holomorphic structure ∇'' is given by $\bar\partial = \begin{pmatrix} \bar\partial^* & -\frac{i}{2}Q^* \\ 0 & \bar\partial \end{pmatrix}$ on $V = S^{-1} \oplus S$, where $\bar\partial$ and $\bar\partial^*$ are the holomorphic structures on S and S^{-1} given by Theorem 3.3, and $Q^* \in \Gamma(M; \bar K K^{-1})$ is the adjoint of the Hopf differential. A holomorphic bundle over a Riemann surface of rank 2 and degree 0 is called stable if it does not contain proper holomorphic subbundles of degree greater or equal 0. We refer to [17] for details about extensions and stable bundles. Because $\int_M (Q^*, Q) \neq 0$ Serre duality implies that $Q^* \in \Gamma(M; \bar K K^{-1})$ is not in the image of the corresponding $\bar\partial$-operator. From this observation one sees that there are no holomorphic subbundles of positive degree of $(V, \bar\partial)$. By [17], V is non-stable if and only if there exits a point $x \in M$ such that $Q^* \otimes s_x \in \Gamma(M; \bar K K^{-1} L(x))$ is in the image of the corresponding $\bar\partial$-operator. Here $s_x \in H^0(L(x))$ is the canonical section of $L(x)$ which has exactly a simple zero at x. By Serre duality, this condition is satisfied exactly in the case, that

$$\int_M (Q^* \otimes s_x, \alpha) = 0$$

for all $\alpha \in H^0(K^2 L(-x))$. Otherwise said, Q^* is perpendicular to the 2-dimensional subspace of holomorphic quadratic differentials which have a zero at some arbitrary but fixed point $x \in M$ if and only if V is non-stable. Let $P_1, \ldots, P_4$ be the umbilics of the Lawson surface, and $\omega_1, \omega_2 \in H^0(M; K)$ be the hyper-elliptic differentials with $(\omega_1) = P_1 + P_3, (\omega_2) = P_2 + P_4$. Using the hyperelliptic picture of M, which can be obtained from the symmetry Φ_2, and the symmetries of the Lawson surface one can easily compute

$$\int_M (Q^*, \omega_1^2) = \int_M (Q^*, \omega_2^2) = 0,$$

and

$$\int_M (Q^*, \omega_1 \omega_2) \neq 0.$$

Therefore, the space of holomorphic quadratic differentials which are perpendicular to Q^* has no common zero. We have proven

Theorem 3.4 *The holomorphic rank 2 bundle (V, ∇'') associated to the Lawson genus 2 surface is stable.*

Remark 3.5 The holomorphic structure of a bundle V in a short exact sequence $0 \to S^{-1} \to V \to S$ is determined by the line of its extension class $[-\frac{i}{2}Q^*] \in PH^1(K^{-1})$. This line is already determined by $\int_M (Q^*, \omega_1^2) = \int_M (Q^*, \omega_2^2) = 0$ and $\int_M (Q^*, \omega_1\omega_2) \neq 0$.

Remark 3.6 This theorem shows that the method of [11] to get a global DPW potential works for the Lawson surface. In order to get more informations about this potential, we will go another way in Sect. 4.

4 A DPW potential for Lawson's genus 2 surface

The idea of the DPW method is to gauge the family ∇^ζ (see Eq. 2.7) into a family of meromorphic connections in a way which can be reversed. In principle, one can construct all minimal surfaces in S^3 by this method. But in concrete situations, it is very difficult to produce surfaces with prescribed properties. For example, there is no compact minimal surface of genus $g \geq 2$ constructed via the DPW method up to now. Nevertheless, there is some work of the author [11] which shows, that the DPW method should work fine for compact surfaces of genus 2: there it is proven that there exists a globally defined DPW potential which gives back the minimal surface. Here, we consider the special case of the Lawson genus 2 surface, and we can show (Theorem 4.3) the existence of a globally defined DPW potential, whose behavior on the surface is completely described. The freedom of the potential is given by two unknown functions in ζ.

Definition 4.1 A meromorphic connection ∇ on a holomorphic vector bundle $(V, \bar\partial)$ is a connection with singularities which can be written with respect to a local holomorphic frame as $d + w$ where w is an meromorphic endomorphism-valued 1-form.

Of course, on Riemann surfaces meromorphic connections are flat. For line bundles there exists the degree formula

$$res(\nabla) = -deg(L)$$

on Riemann surfaces, where the $res(\nabla)$ is the sum of all local residui.

Theorem 4.2 *Let ∇^ζ be the holomorphic family of flat connections on V associated to Lawson's genus 2 surface $f : M \to S^3$. Let $Q_1, \ldots, Q_6$ be the Weierstrass points of M. Then there exists a holomorphic map*

$$\tilde B : \zeta \in B(0;\epsilon) \subset \mathbb{C} \to \Gamma(M \setminus \{Q_1, \ldots, Q_6\}, \mathrm{End}(V))$$

which satisfies $B_0 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ *and* $\det B_\zeta = 1$ *for all ζ, such that the gauged connection*

$$\nabla^\zeta \cdot B_\zeta$$

is a holomorphic family of meromorphic connections $\hat\nabla^\zeta$ for $\zeta \in B(0;\epsilon) \setminus \{0\} \subset \mathbb{C}$ on the (fixed) holomorphic vector bundle $(V = S^{-1} \oplus S, \bar\partial^{spin})$.

More precisely the family has an expansion

$$\hat\nabla^\zeta = \begin{pmatrix} \nabla_0^* & \zeta^{-1} \\ -\frac{i}{2}Q & \nabla_0 \end{pmatrix} + \text{higher order terms in } \zeta,$$

for some meromorphic connection ∇_0 on S and the Hopf field $Q \in H^0(K^2)$ of the surface. The connections have poles of order 1 on the diagonal at $Q_1, \ldots, Q_6$ and of order 2 in the upper right and lower left corner at Q_2, Q_4, Q_6 respectively Q_1, Q_3, Q_5.

Proof The condition that $\nabla^\zeta \cdot B_\zeta$ is a holomorphic family of meromorphic connections on the holomorphic bundle $S^{-1} \oplus S$ translates easily to

$$\bar\partial^{spin} B = \begin{pmatrix} \frac{i}{2}Q^* & +\zeta\Phi^* \end{pmatrix} B, \tag{4.1}$$

with $Q^* \in \Gamma(\bar K K^{-1})$ and $\Phi^* \in \Gamma(\bar K K)$. Writing

$$B = \sum_{k\geq 0} B_k \zeta^k$$

and

$$B_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

one obtains the equations

$$\begin{aligned} \bar{\partial} a_k &= \frac{i}{2} Q^* c_k \\ \bar{\partial} c_{k+1} &= \Phi^* a_k \\ \bar{\partial} b_k &= \frac{i}{2} Q^* d_k \\ \bar{\partial} d_{k+1} &= \Phi^* b_k \end{aligned} \tag{4.2}$$

for $a_k, d_k \in \Gamma(M; \underline{\mathbb{C}}), b_k \in \Gamma(M; K^{-1})$ and $c_k \in \Gamma(M; K)$. If we would take $a_0 = d_0 = 1, c_0 = 0$, then, by Serre duality, there does not exists a smooth b_0 satisfying the equation above. To overcome this problem, we search for solutions with singularities. Set

$$a_0 = 1, d_0 = 1, c_0 = 0.$$

Take the divisors $D = Q_1 + Q_3 + Q_5$ and $\tilde{D} = Q_2 + Q_4 + Q_6$ which are invariant under the $\mathbb{Z}_2$ and under the $\mathbb{Z}_3$ action. Note that $L(D) = L(\tilde{D}) = KS$. Now consider

$$\begin{aligned} \tilde{a}_k &= a_k \otimes s_D \in \Gamma(KS) \\ \tilde{b}_k &= b_k \otimes s_{\tilde{D}} \in \Gamma(S) \\ \tilde{c}_k &= c_k \otimes s_D \in \Gamma(K^2 S) \\ \tilde{d}_k &= d_k \otimes s_{\tilde{D}} \in \Gamma(KS). \end{aligned} \tag{4.3}$$

We get new equations

$$\begin{aligned} \bar{\partial} \tilde{a}_k &= \frac{i}{2} Q^* \tilde{c}_k \in \Gamma(\bar{K} KS) \\ \bar{\partial} \tilde{c}_{k+1} &= \Phi^* \tilde{a}_k \in \Gamma(\bar{K} K^2 S) \\ \bar{\partial} \tilde{b}_k &= \frac{i}{2} Q^* \tilde{d}_k \in \Gamma(\bar{K} S) \\ \bar{\partial} \tilde{d}_{k+1} &= \Phi^* \tilde{b}_k \in \Gamma(\bar{K} KS). \end{aligned} \tag{4.4}$$

Again, Serre duality tells us that there does always exist a solution for each of these equations. But we need more: we want the ζ-series $\sum \tilde{B}_k \zeta^k$ to be convergent and $\det B_\zeta = 1$. We explain how this can be achieved. Note that all occurring bundles inherit canonical unitary metrics from the surface metric. These give us fixed Sobolev norms and spaces. By Poincare inequality there exists a constant $c > 0$ such that the solution s for any of the above equations, which is unique by the property of being orthogonal to the kernel of the corresponding $\bar{\partial}$-operator, satisfies

$$\parallel s \leq c \parallel |\bar{\partial} s \parallel .$$

Note that, if the right hand side of any of these equations is symmetric with respect to the $\mathbb{Z}_2$ or $\mathbb{Z}_3$ symmetry, the unique solution has also this symmetry. We take always this unique solution, and solve for $\tilde{a}_{k+1}, \ldots, \tilde{d}_{k+1}$ inductively. Thus we obtain $\| \tilde{B}_k \| < C^k$ for some constant C, which implies smooth convergence for small ζ. Set

$$B := \sum_{k \geq 0} \begin{pmatrix} \tilde{a}_k \otimes s_{-D} & \tilde{b}_k \otimes s_{-\tilde{D}} \\ \tilde{c}_k \otimes s_{-D} & \tilde{d}_k \otimes s_{-\tilde{D}} \end{pmatrix} \zeta^k.$$

Then B satisfies Eq. (4.1). Because of this equation the determinant $\det(B_\zeta)$ is a meromorphic function on M for all ζ. Moreover it is invariant under the $\mathbb{Z}_2$ action. Clearly, $det(B_\zeta) \geq -D - \tilde{D}$ for all ζ. But the only $\mathbb{Z}_2$-invariant functions with this divisor inequality are the constants. Thus there exists $h : B(0; \epsilon) \to \mathbb{C}$ with $\det(B_\zeta) = h(\zeta)$. Therefore

$$B_\zeta \begin{pmatrix} \frac{1}{h(\zeta)} & 0 \\ 0 & 1 \end{pmatrix}$$

is the gauge we were looking for. □

As we have seen in the proof of the previous theorem, the meromorphic connections $\hat{\nabla}^\zeta$ have some of the symmetries of the Lawson surface by construction. We use these symmetries to write down the corresponding DPW potential almost explicitly. To do so, we trivialize $S^* \oplus S \to M \setminus \{Q_2, Q_4, Q_6, P_1, \ldots, P_4\}$ using the meromorphic sections

$$s = s_{Q_2+Q_4+Q_6-P_1-P_2-P_3-P_4} \in \mathcal{M}(S^*)$$

and

$$t = s_{-Q_2-Q_4-Q_6+P_1+P_2+P_3+P_4} \in \mathcal{M}(S).$$

Theorem 4.3 *Let $\pi : M \to \mathbb{CP}^1$ be the three-fold covering of the Riemann sphere. Then with respect to the meromorphic frame (s, t) of $S^* \oplus S$ and up to a diagonal gauge only depending on ζ the family of connections given by Theorem 4.2 can be written as $d + \xi$ with*

$$\xi = \pi^* \begin{pmatrix} -\frac{4}{3}\frac{z^3}{z^4-1} + \frac{A}{z} & \zeta^{-1} + Bz^2 \\ \frac{G}{(z^4-1)} + \frac{\zeta H}{z^2(z^4-1)} & \frac{4}{3}\frac{z^3}{z^4-1} - \frac{A}{z} \end{pmatrix} dz$$

for some ζ depending even functions A, B, G, H which satisfy $H = A + A^2$ and $B = -\frac{1}{G}(-\frac{1}{3} + A + (\frac{1}{3} - A)^2)$.

Proof Note that the upper right corner with respect to the holomorphic decomposition $S^* \oplus S$ of ∇^ζ has the invariant meaning of a meromorphic function on M with at most double poles at Q_2, Q_4, Q_6. There is a well-defined holomorphic function $h(\zeta)$ on $B(0; \epsilon) \setminus \{0\}$ which is the constant part (but depending on ζ) of the upper right corner. From the starting condition $a_0 = 1 = d_0$ and $c_0 = 0$ we get the Laurent expansion $h(\zeta) = \zeta^{-1} + h_0 + \cdots$. For ϵ small enough we can take a square root g of $\zeta h(\zeta)$. Instead of working with $\hat{\nabla}^\zeta$ we gauge it by $\begin{pmatrix} g(\zeta) & 0 \\ 0 & 1/g(\zeta) \end{pmatrix}$, so that the part of the upper right corner, which is constant along M, is given by ζ^{-1}.

With respect to the given trivialization the connection 1-form ξ, the so-called DPW potential, is a $\mathfrak{sl}(2; \mathbb{C})$-valued family of meromorphic 1-forms. We want to deduce the symmetries of the potential ξ from the symmetries of the family of connections ∇^ζ and of the gauge B_ζ. We start with the $\mathbb{Z}_3$ symmetries. Note that $\Phi_3^*(s, t) = (s, t)$. By construction the family of connections $\hat{\nabla}^\zeta$ is also invariant under $\mathbb{Z}_3$. Thus ξ is invariant under the $\mathbb{Z}_3$-action, i.e. $\Phi_3^*\xi = \xi$. Similarly, the family of connections $\hat{\nabla}^\zeta$ is also invariant under $\mathbb{Z}_2$. But as $\Phi_2^*(s,t) = (-is, it)$ the generator Φ_2 of the $\mathbb{Z}_2$-action satisfies

$$\Phi_2^*\xi = \xi \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Next, we look at $\tau = \gamma_{S_2} \circ \gamma_{P_1Q_1}$ which is holomorphic on the surface but changes orientation in space. Note that $\tau^*(s,t) = (e^{-\frac{\pi i}{4}}s, e^{\frac{\pi i}{4}}t)$, and $\tau^*\nabla^\zeta = \nabla^{-\zeta}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. From $\tau^*Q^* = -Q^*$, $\tau^*\Phi^* = \Phi^*$ and from the recursive construction of B one sees that

$$\tau^*B(\zeta) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} B(-\zeta) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Altogether we obtain the following symmetry

$$\tau^*\xi(\zeta) = \xi(-\zeta) \cdot \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}$$

for the DPW potential ξ.

The symmetry $\tilde{\tau} = \gamma_{S_1} \circ \gamma_{P_1Q_1}$ is more difficult to handle. The reason for this is that it interchanges the points Q_1, Q_3, Q_5 and Q_2, Q_4, Q_6. As it is holomorphic on the surface and orientation reversing in space, we have again $\tilde{\tau}^*\nabla^\zeta = \nabla^{-\zeta}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Moreover $\tilde{\tau}^*(s,t) = (-izs, i\frac{1}{z}t)$. We claim that the symmetry of the potential takes the following form:

$$\tilde{\tau}^*\xi(\zeta) = \xi(-\zeta) \cdot g \cdot \begin{pmatrix} z \circ \pi & 0 \\ 0 & \frac{1}{z \circ \pi} \end{pmatrix},$$

for some (globally defined) automorphism g. In fact, g is given by

$$g = \begin{pmatrix} a_0 + a_1 z^2 & \frac{1}{z} b_1(z^4 - 1) \\ \frac{c_{-1}}{z} & \frac{d_{-1}}{z^2} + d_0 \end{pmatrix} \circ \pi$$

for some ζ depending functions $a_0, \ldots, d_0$. This can be deduced from the fact that $\tilde{\tau}^*\xi(\zeta)$ can be obtained by the same method as ξ, see the proof of Theorem 4.2, with the difference that the gauge has singularities in the first column at Q_2, Q_4, Q_6 and in the second column at Q_1, Q_3, Q_5. We will not give the details as it turns out below that the symmetry $\tilde{\tau}$ does not give any new information for the potential ξ.

Next, we list the poles of the potential ξ. It has

- poles on the diagonal of order 1 at $Q_1, \ldots, Q_6$;
- poles on the diagonal of order 1 at $P_1, \ldots, P_4$ with residuum ∓ 1;
- poles in the lower left corner up to order 2 at Q_1, Q_3, Q_5;
- poles in the lower left corner up to order 2 at $P_1, \ldots, P_4$;
- poles in the upper right corner up to order 4 at Q_2, Q_4, Q_6.

Note that the poles at $P_1, \ldots, P_4$ and the poles of order 4 instead of 2 in the upper right corner come from the chosen trivialization (s,t).

We have enough informations to determine the potential: because ξ and the trivialization have both the $\mathbb{Z}^3$-symmetry, ξ is the pullback of an ζ-depending $\mathfrak{sl}(2;\mathbb{C})$-valued meromorphic 1-form $\tilde{\xi}$ on $\mathbb{CP}^1$. This 1-form on $\mathbb{CP}^1$ can be considered as the connection 1-form with respect to the frame $(s_\infty, s_{-\infty})$ of a holomorphic family of connections on $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$.

Clearly $\tilde{\xi}$ has corresponding symmetries and pole behavior, for example, the poles on the diagonal have residuum $\pm\frac{1}{3}$ at the 4th order roots of 1.

The most general form of a potential on $\mathbb{CP}^1$ with the symmetries for Φ_2, Φ_3 and τ and singularities as above is up to a diagonal gauge only depending on ζ

$$\begin{pmatrix} -\frac{4}{3}\frac{z^3}{z^4-1}+\frac{A}{z} & \zeta^{-1}+Bz^2 \\ \frac{G}{(z^4-1)}+\frac{\zeta H}{z^2(z^4-1)} & \frac{4}{3}\frac{z^3}{z^4-1}-\frac{A}{z} \end{pmatrix} dz \tag{4.5}$$

for some ζ depending even functions A, B, G, H.

The singularities at $z = 0$ and $z = \infty$ are apparent. In fact, the construction of the potential shows that there must be a ζ-depending meromorphic gauge g which is diagonal at $\zeta = 0$, and lower triangular respectively upper triangular for general ζ, such that g gauges the singularities at $z = 0$ respectively $z = \infty$ away. Then, a simple computation shows

$$\begin{aligned} H &= A + A^2 \\ B &= -\frac{1}{G}\left(-\frac{1}{3} + A + (\frac{1}{3} - A)^2\right) \end{aligned} \tag{4.6}$$

for all ζ.

Again, a short computation shows, that each potential of the form (4.5) with functions A, B, G, H satisfying the Eq. (4.6) posses the symmetry for $\tilde{\tau}$. □

The next task would be to determine the functions A and G. As we have seen, they cannot be computed out of the symmetries. These functions satisfy more complicated equations. Namely, the DPW potential must be unitarizable. This means, that the holonomy representation of the family of connections for some ζ-depending starting condition must extend to $\mathbb{C} \setminus \{0\}$, and must be unitary for $\zeta \in S^1 \subset \mathbb{C}$. This starting condition is called dressing. If one gauges the singularity at $z = 0$ away with an lower triangular gauge and then starts integrating at $z = 0$, i.e. finding a parallel frame, then the dressing must be diagonal as one can see from the symmetries. The holonomy depends transcendentally on A and G, so it is a very hard problem to determine the exact form of A and G. The space of representations of $\pi_1(\mathbb{CP}^1 \setminus p_1, \ldots, p_4$ in $\mathrm{SL}(2, \mathbb{C})$ modulo conjugation with fixed conjugacy classes around $p_1, \ldots, p_4$ is a cubic surface in $\mathbb{C}^3$, see [2]. One can show, that for fixed $\zeta \in \mathbb{C}^*$ the holonomy representation depends on A and G independently. Thus, one obtains an open nonempty subset of all possible representations from the potential ξ. But, one cannot obtain all possible representations as the proof of the following theorem shows.

Theorem 4.4 *The family of meromorphic connections $\hat{\nabla}$ and the DPW potential ξ do not extend to the whole unit circle.*

Proof As the gauge B is well-defined on the surface M and not multi-valued, the monodromy representations of ∇^ζ and of $\hat{\nabla}^\zeta$ are equivalent for all $\zeta \in \mathbb{C}^*$. If $\hat{\nabla}^{\pm 1}$ would exist, the monodromy would be trivial. This means there would exist two linear independent meromorphic sections $v, w \in \mathcal{M}(M, S^* \oplus S)$ parallel with respect to $\hat{\nabla}^1$. As $\hat{\nabla}^1$ has its only poles at $Q_1, \ldots, Q_6$ the same holds for v and w. Write $v = x \oplus y$ with respect to $S^* \oplus S$. From the special form of ξ one sees that x has simple poles at Q_2, Q_4, Q_6 and y has simple poles at Q_1, Q_3, Q_5. Therefore, y is a constant multiple of the meromorphic section $s_{-Q_1-Q_3-Q_5+P_1+\cdots+P_4} = \frac{1}{z \circ \pi} t \in \mathcal{M}(M, S)$. The same argument holds for a decomposition of w, which shows that a parallel frame v, w would not be linear independent at the points $P_1, \ldots, P_4$, which is a contradiction. □

In order to determine which connection in Theorem 4.3 has unitarizable holonomy it may be useful to gauge these connections to better known differential equations, namely to Fuchsian systems.

Theorem 4.5 *The connection of the form $d+\xi$, where ξ is given as in Theorem 4.3 is gauge equivalent to a Fuchsian system of the form*

$$\begin{aligned} d + &\begin{pmatrix} p & -p^2+\frac{1}{9} \\ 1 & -p \end{pmatrix}\frac{dz}{z-1} + \begin{pmatrix} -p & \frac{-p^2+\frac{1}{9}}{r} \\ r & p \end{pmatrix}\frac{dz}{z-i} \\ &+ \begin{pmatrix} p & p^2-\frac{1}{9} \\ -1 & -p \end{pmatrix}\frac{dz}{z+1} + \begin{pmatrix} -p & \frac{p^2-\frac{1}{9}}{r} \\ -r & p \end{pmatrix}\frac{dz}{z+i}, \end{aligned} \tag{4.7}$$

where p and r are given in terms of the spectral parameter ζ, A and $F = \frac{G}{\zeta}$ as follows:

$$p = \frac{1}{12}\frac{-A^2+2A^3+3A^4-3F^2}{FA}$$
$$r = -i\frac{F+A^2+A}{-F+A^2+A}.$$

Remark 4.6 The gauge necessarily has poles at $0, \infty \in \mathbb{CP}^1$. In fact these are the only poles. Note that the gauge depends meromorphically on ζ and has a pole at $\zeta = 0$. This implies that we cannot apply the Iwasawa decomposition to the family of Fuchsian equations in order to obtain a minimal surface.

Proof The connections of Theorem 4.3 have apparent singularities at $0, \infty \in \mathbb{CP}^1$. The singularity at $z = 0$ vanishes by applying the gauge transformation

$$g_1 = \begin{pmatrix} 1 & 0 \\ -\frac{A\zeta}{z} & 1 \end{pmatrix}.$$

Then, the remaining singularity at $z = \infty$ of the connection obtained in this way can be gauged away with

$$g_2 = \begin{pmatrix} -\frac{1}{3G}(3A-1)A\zeta z^2 + 1 & \frac{1}{A\zeta}z) \\ -\frac{1}{3G}(3A-1)A^2\zeta^2 z & 1 \end{pmatrix}.$$

Up to a diagonal gauge which only depends on A, G and ζ, this is the Fuchsian system . □

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APPENDIX B

Higher genus minimal surfaces in S^3 and stable bundles

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Higher genus minimal surfaces in S^3 and stable bundles

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Abstract. We consider compact minimal surfaces $f: M \to S^3$ of genus 2 which are homotopic to an embedding. We prove that such surfaces can be constructed from a globally defined family of meromorphic connections by the DPW method. The poles of the meromorphic connections are at the Weierstrass points of the Riemann surface and are at most quadratic. For the existence proof of the DPW potential, we give a characterization of stable extensions $0 \to S^{-1} \to V \to S \to 0$ of spin bundles S by its dual S^{-1} in terms of an associated element of $PH^0(M; K^2)$. We also show that the family of holomorphic structures associated to a minimal surface of genus $g \geq 2$ in S^3 is generically stable.

1. Introduction

The systematic investigation of harmonic maps from Riemann surfaces to S^3 (or more generally to symmetric spaces) has started with the introduction of the associated $\mathbb{C}^\times$-family of flat connections ∇^ζ, see for example [17] or [9]. Using this family there have been many deep results concerning harmonic 2-spheres and harmonic 2-tori. For example, the space of constant mean curvature (CMC) tori in $\mathbb{R}^3$ and minimal tori in S^3 is well understood, see [2, 9, 11, 16].

On the other hand, there is no satisfactory treatment of compact CMC or minimal surfaces of higher genus using the methods of integrable systems. Nevertheless, there is a general method due to Dorfmeister, Pedit and Wu [6], which, at least in principal, produces all such surfaces. The idea of the DPW method is to gauge ∇^ζ into a family of meromorphic connections in a way which can be reversed: It is shown that one can gauge the holomorphic family of connections by a "positive" gauge into a family of meromorphic connections of a special form on simply connected domains. A "positive" gauge is a ζ-depending family of endomorphisms of determinant 1 which holomorphically extends into $\zeta = 0$. From such a family of meromorphic connections one obtains minimal surfaces as follows: Take a ζ-depending parallel frame and split it into the unitary and the positive part by loop Iwasawa decomposition. Then the unitary part is a parallel frame of a family of unitary connections describing a minimal surface. The surface obtained in this fashion depends on the ζ-depending starting condition of the parallel frame. Dressing, that is changing this starting condition, will give new surfaces.

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Making compact surfaces (or even surfaces with topology) via this method is more complicated. One has to ensure, by choosing the family of meromorphic connections and the right dressing, that the frame of the unitary part is well-defined up to (unitary) holonomy, and that the surface closes. This has been worked out only in very special cases, for example for trinoids, the genus 0 CMC surfaces with three Delaunay ends, and tori. But so far there are no examples for (closed) higher genus surfaces produced by this method.

The aim of this paper is to show that for compact oriented minimal surfaces in S^3 of genus 2 it is possible to find a DPW potential with a "nice" behavior on the Riemann surface. Our method applies in the following situation: We assume that the minimal surface is homotopic to an embedding. This is equivalent to saying that the associated spin bundle S has no holomorphic sections, see [15]. The second assumption is that the holomorphic structure $(\nabla^{\zeta})''$ at $\zeta = 0$ is stable. This condition is needed in technical details. For the only known example, the Lawson genus 2 surface, it is satisfied. Moreover, the holomorphic structure $(\nabla^{\zeta})''$ is stable for generic $\zeta \in \mathbb{C}$ as we show in Section 5. Under these assumptions the family of connections ∇^{ζ} can be gauged globally to a family of meromorphic connections on M with constant holomorphic structure given by the trivial extension of S by S^{-1}. The poles of these meromorphic connections are exactly at the Weierstrass points of the Riemann surface and of order at most 2, see Theorem 1.

In the first part of this paper we recall the gauge theoretic description of minimal surfaces $f\colon M \to S^3$ in S^3 due to Hitchin [9]. We give explicit formulas of the occurring connections and sections in terms of geometric quantities like the Hopf field and the spinor connection. We also give a link to the local description of surfaces preferred by other authors.

In the second section we introduce the associated family of flat connections ∇^{ζ}. We gauge this family by a ζ-depending B with special singularities such that the meromorphic connections have a constant holomorphic structure. This can be achieved by solving $\bar{\partial}$-equations on M: The gauge B must be a section in a bundle $E \to U \subset \mathbb{C} \ni \zeta$ whose fibers are finite dimensional spaces of holomorphic sections in a bundle over M (varying in ζ), see Lemma 1. The difficulty is in proving that the gauge has constant determinant $\det(B) = 1$ as to avoid additional and possibly ζ-depending singularities. The determinant is given by a map into a finite dimensional space, $\det\colon E \to H^0(M; K^3)$, compare with Lemma 2. We use the implicit function theorem to find a gauge with $\det(B) = 1$. The surjectivity of the differential of det, restricted to the $\zeta = 0$ slice, reduces to an algebro-geometric condition on the holomorphic structure $(\nabla^0)''$, see Lemma 2 and Theorem 3. For a more detailed understanding of this condition, we study non-trivial extensions $0 \to S^{-1} \to V \to S \to 0$ over compact surfaces M of genus 2 with the property that S has no holomorphic sections. There is a natural $1:1$ correspondence between non-trivial extension of this form and elements of $PH^0(M; K^2)$. In this setup we will identify the non-stable extensions with the bundles which do not satisfy the above mentioned condition (Theorem 3).

In the last chapter we consider compact oriented immersed minimal surfaces of genus $g \geq 2$. We prove that the holonomy representation of the family ∇^{ζ} is generically non-abelian and that the holomorphic bundle $(V, (\nabla^{\zeta})'')$, associated to the connection ∇^{ζ}, is stable for generic $\zeta \in \mathbb{C}$. This shows that the methods which worked so well in the characterization of minimal tori cannot work for compact minimal surfaces of genus $g \geq 2$, see also [7].

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2. Minimal surfaces in S^3

First we shortly describe a gauge-theoretic way of treating minimal surfaces in S^3 due to Hitchin [9]. We refer to [13] for details about Clifford algebras and spinors, and to [9] for the main source of the material described below.

Consider the round 3-sphere S^3 with its tangent bundle trivialized by left translation

$$TS^3 = S^3 \times \operatorname{Im}\mathbb{H}$$

and Levi-Civita connection given, with respect to the above trivialization, by

$$\nabla = d + \frac{1}{2}\omega.$$

Here $\omega \in \Omega^1(S^3, \operatorname{Im}\mathbb{H})$ is the Maurer–Cartan form of S^3 which acts via adjoint representation on the Lie algebra $\operatorname{Im}\mathbb{H} = \mathrm{su}(2)$. This formula is equivalent to the well-known characterization of the Levi-Civita connection by the property that for left-invariant vector fields X, Y it satisfies $\nabla_X Y = \frac{1}{2}[X, Y]$.

There are two equivalent complex representations of the spin group S^3 induced from the Clifford representation

$$\mathrm{Cl}(\mathbb{R}^3) = \mathbb{H} \otimes \mathbb{C} \oplus \mathbb{H} \otimes \mathbb{C}$$

on the complex vector space $\mathbb{H}$ with complex structure given by right multiplication with i. It is well known that S^3 has an unique spin structure. We consider the associated complex spin bundle

$$V = S^3 \times \mathbb{H}$$

with complex structure given by right multiplication with $i \in \mathbb{H}$. We have a complex hermitian metric $(\cdot,\cdot)$ on it given by the trivialization and by the identification $\mathbb{H} = \mathbb{C}^2$. The Clifford multiplication is given by

$$TS^3 \times V \to V; \quad (\lambda, v) \mapsto \lambda v,$$

where $\lambda \in \operatorname{Im}\mathbb{H}$ and $v \in \mathbb{H}$. This is clearly complex linear. The unitary spin connection is given by

$$\nabla = \nabla^{\mathrm{spin}} = d + \frac{1}{2}\omega,$$

where the $\operatorname{Im}\mathbb{H}$-valued Maurer–Cartan form acts by left multiplication on the quaternions. Via this construction the tangent bundle TS^3 identifies as the skew hermitian trace-free complex linear endomorphisms of V.

Let M be a Riemann surface and $f\colon M \to S^3$ be a conformal immersion. Then the pullback $\phi = f^*\omega$ of the Maurer–Cartan form satisfies the structural equations

$$d\phi + \frac{1}{2}[\phi \wedge \phi] = 0.$$

Another way to write this equation is

$$d^{\nabla}\phi = 0, \tag{2.1}$$

where $\nabla = f^*\nabla = d + \frac{1}{2}\phi$, with $\phi \in \Omega^1(M; \operatorname{Im}\mathbb{H})$ acting via adjoint representation. Conversely, every solution $\eta \in \Omega^1(M; \operatorname{Im}\mathbb{H})$ to $d^{\nabla^\eta}\eta = 0$, where $\nabla^\eta = d + \frac{1}{2}\eta$, gives rise to a map $f\colon \hat{M} \to S^3$ from the universal covering $\hat{M}$ of M unique up to translations in S^3.

From now on we only consider the case of f being minimal. Under the assumption of f being conformal, f is minimal if and only if it is harmonic. This is exactly the case when

$$d^{\nabla} * \phi = 0. \tag{2.2}$$

Consider $\phi \in \Omega^1(M; f^*TS^3) \subset \Omega^1(M; \mathrm{End}_0(V))$ via the interpretation of TS^3 as the bundle of trace-free skew hermitian endomorphisms of V. Decompose $\frac{1}{2}\phi = \Phi + \bar{\Psi}$ into K and $\bar{K}$ parts, i.e. $\Phi = \frac{1}{2}(\phi - i * \phi) \in \Gamma(K\,\mathrm{End}_0(V))$ and $\bar{\Psi} = \frac{1}{2}(\phi + i * \phi) \in \Gamma(\bar{K}\,\mathrm{End}_0(V))$. The property of ϕ being skew symmetric translates to $\bar{\Psi} = -\Phi^*$, i.e.

$$\frac{1}{2}\phi = \Phi - \Phi^*.$$

Then f is conformal if and only if $\operatorname{tr}\Phi^2 = 0$, see [9]. In view of a rank 2 bundle V and $\operatorname{tr}\Phi = 0$ this is equivalent to

$$\det \Phi = 0. \tag{2.3}$$

Note that f is an immersion if and only if Φ is nowhere vanishing. In other words, the branch points of f are exactly the zeros of Φ. Moreover the equations (2.1) and (2.2) are equivalent to

$$\bar{\partial}^{\nabla}\Phi = 0, \tag{2.4}$$

where $\bar{\partial}^{\nabla} = \frac{1}{2}(d^{\nabla} + i * d^{\nabla})$ is the induced holomorphic structure on $KV \to M$.

Of course equation (2.4) does not contain the property $\nabla = d + \frac{1}{2}\phi$, i.e. that $\nabla - \frac{1}{2}\phi$ is trivial on V. Locally, or on simply connected sets, this is equivalent to the flatness of $\nabla - \frac{1}{2}\phi$, which is, as one easily computes, the same as the following formula:

$$F^{\nabla} = [\Phi \wedge \Phi^*]. \tag{2.5}$$

Conversely, given an unitary rank 2 bundle $V \to M$ over a simply connected Riemann surface with special unitary connection ∇ and trace free field $\Phi \in \Gamma(K\,\mathrm{End}_0(V))$ without zeros, which satisfy the equations (2.3), (2.4) and (2.5), we get a conformally immersed minimal surface as follows: By equation (2.4) and (2.5), the unitary connections $\nabla^L = \nabla - \Phi + \Phi^*$ and $\nabla^R = \nabla + \Phi - \Phi^*$ are flat. Because M is simply connected, they are gauge equivalent. Due to the fact that $\operatorname{tr}\Phi = 0$, the determinant bundle $\Lambda^2 V$ is trivial with respect to all these connections. Hence, the gauge is $\mathrm{SU}(2) = S^3$-valued with differential $\phi = 2\Phi - 2\Phi^*$. Thus it is a conformal immersion. The harmonicity follows from equation (2.4).

2.1. The spinor bundle of a minimal surface. We describe how the spin structure of the immersion $f: M \to S^3$ can be seen in this setup. The geometric significance of the spin structure is described in Pinkall [15], see also the literature therein. We give formulas which relate the data on V obtained in the previous part to the data usually used to describe a surface, for example the Gauss map and the Hopf field. Again, this part is based on [9].

In this section we consider the bundle V with its holomorphic structure $\bar{\partial} := \nabla''$. As we have seen the complex part Φ of the differential of a conformal minimal surface satisfies $\operatorname{tr}\Phi = 0$ and $\det\Phi = 0$, but is nowhere vanishing. We obtain a well-defined holomorphic line subbundle

$$L := \ker\Phi \subset V.$$

Because Φ is nilpotent, the image of Φ satisfies $\operatorname{Im}\Phi \subset K \otimes L$. Consider the holomorphic section

$$\Phi \in H^0(M; \operatorname{Hom}(V/L, KL))$$

without zeros. The holomorphic structure $\bar\partial - \Phi^*$ turns $V \to M$ into the holomorphically trivial bundle $\underline{\mathbb{C}}^2 \to M$. As $\operatorname{tr}\Phi^* = 0$, the determinant line bundle $\Lambda^2 V$ of $(V, \bar\partial)$ is holomorphically trivial. This implies $V/L = L^{-1}$ and we obtain

$$\operatorname{Hom}(V/L, KL) = L^2 K$$

as holomorphic line bundles. Because $L^2 K$ has a holomorphic section Φ without zeros, we get

$$L^2 = K^{-1}.$$

Hence, its dual bundle $S = L^{-1}$ is a spinor bundle of the Riemann surface M. Clearly, S^{-1} is the only Φ-invariant line subbundle of V.

There is another way to obtain the bundles S and S^{-1} which provides a link to the quaternionic holomorphic geometry, see [4]. Let $R: M \to \operatorname{Im}\mathbb{H}$ be the normal of f with respect to the trivialization of the tangent bundle $TS^3 = S^3 \times \operatorname{Im}\mathbb{H}$. Here, R stands for the right normal vector when considering the surface as lying in $S^3 \subset \mathbb{H}$. As we have seen we can consider V as the trivial quaternionic line bundle $\underline{\mathbb{H}} \to M$. Note that scalar multiplications with quaternions is from the right in order to commute with the Clifford multiplication. We define a complex quaternionic linear structure $\mathcal{J}$ by

$$v \mapsto -Rv.$$

Note that $\mathcal{J}$ can be seen as the operator given by Clifford multiplication with the negative of the determinant, i.e. for a positive oriented orthonormal basis $X, Y \in TM$ we have $\mathcal{J}(v) = Y \cdot X \cdot v$, where $\cdot$ is Clifford multiplication. Then V splits into $\pm i$ eigenspace of $\mathcal{J}$:

$$V = E \oplus \bar E := \{v \in V \mid \mathcal{J} = vi\} \oplus \{v \in V \mid \mathcal{J} = -vi\}.$$

This decomposition is orthogonal with respect to the (complex) unitary metric on $V = \underline{\mathbb{H}}$.

Proposition 1. *The kernel S^{-1} of Φ is given by the $-i$ eigenspace of $\mathcal{J}$.*

Proof. It is sufficient to prove that $\bar E \subset \ker\Phi$. Note that for all $v \in \underline{\mathbb{H}}$ the vector $v + \mathcal{J}v = v - Rvi$ is an element of $\bar E$. With $4\Phi = \phi - i * \phi$, the proof is simply a matter of computation. □

We have seen that there exists a holomorphic subbundle S^{-1} of $(V, \bar\partial)$, and that the determinant line bundle $\Lambda^2 V$ is trivial. Therefore, $(V, \bar\partial)$ is a non-trivial extension of S by S^{-1}:

$$0 \to S^{-1} \to V \to S \to 0.$$

This means that with respect to the decomposition $V = S^{-1} \oplus S$ the holomorphic structure $\bar\partial$ can be written as

$$\bar\partial = \begin{pmatrix} \bar\partial^{\mathrm{spin}^*} & \bar b \\ 0 & \bar\partial^{\mathrm{spin}} \end{pmatrix},$$

where $\bar{b} \in \Gamma(\bar{K}\,\mathrm{Hom}(S, S^{-1})) = \Gamma(\bar{K}K^{-1})$, and $\bar{\partial}^{\mathrm{spin}}$ and $\bar{\partial}^{\mathrm{spin}^*}$ are the holomorphic structures on S and S^{-1}, respectively. It is well known ([9]) that there exists a relation between $\bar{b}$ and the Hopf differential Q of the minimal surface. We want to determine the exact form of this relation. More generally, we want to find out geometric formulas for the connection ∇ on the pullback $V \to M$ of the spinor bundle of S^3.

To do so recall that the pullback of the Levi-Civita connection ∇ splits, with respect to the decomposition $f^*TS^3 = TM \oplus \mathbb{R}$ into tangential and normal part, into

$$\nabla = \begin{pmatrix} \nabla^M & -II^* \\ II & d \end{pmatrix}$$

with ∇^M being the Levi-Civita connection on M. Here II is the second fundamental form of the surface which is a symmetric bilinear form $II \in \Gamma(T^*M \otimes T^*M)$. The Weingarten operator is given by $A = -II^* \in \mathrm{End}(TM)$, i.e. $\langle A(X), Y\rangle = -II(X,Y)$ for tangent vectors $X, Y \in TM$. The K^2-part of the second fundamental form is called the Hopf field Q. For minimal surfaces in S^3 it is holomorphic, i.e. $Q \in H^0(M; K^2)$. Its zeros are exactly the umbilics of the surface.

The connection

$$\tilde{\nabla} = \begin{pmatrix} \nabla^M & 0 \\ 0 & d \end{pmatrix}$$

is an SO(3)-connection, too, and it induces a unitary connection $\tilde{\nabla}$ on the spinor bundle V. But it reduces to an SO(2)-connection, the Levi-Civita connection on M, so the corresponding connection on $V = S^{-1} \oplus S$ is the spin connection of M. Its $\bar{K}$-part is given by

$$\tilde{\nabla}'' = \begin{pmatrix} \bar{\partial}^{\mathrm{spin}^*} & 0 \\ 0 & \bar{\partial}^{\mathrm{spin}} \end{pmatrix}.$$

The difference of these two connections on V is the trace-free skew adjoint operator

$$\nabla - \tilde{\nabla} = \frac{1}{2}(\mathcal{J} \circ A)\cdot .$$

This means for all $X \in TM$, $\psi \in \Gamma(V)$ we have $\nabla_X\psi - \tilde{\nabla}_X\psi = \frac{1}{2}(\mathcal{J}A(X))\cdot\psi$ with $\cdot$ being the Clifford multiplication and A the Weingarten operator. Note that this difference is an off-diagonal endomorphism. One can compute that its K-part vanishes on S, and as an operator in $K\,\mathrm{Hom}(S^{-1}, S) = K^2$ it is exactly $-\frac{i}{2}Q$. The adjoint Q^* of $Q \in H^0(K^2)$ with respect to the hermitian product $(\cdot,\cdot)$ is determined by

$$(Q(X)v, w) = (v, Q^*(X)w)$$

for all $X \in TM$, $v \in S^{-1}$, and $w \in S$. This gives a well-defined section $Q^* \in \Gamma(\bar{K}K^{-1})$. As $\frac{1}{2}\mathcal{J} \circ A$ is skew adjoint, the extension class $[\bar{b}] \in H^1(M; K^{-1})$ of V is given by the representative

$$\bar{b} = -\frac{i}{2}Q^* \in \Gamma(\bar{K}K^{-1}).$$

Because Q is holomorphic, one can deduce that the extension class $[\bar{b}] \in H^1(M, K^{-1})$ is non-zero (or $Q = 0$, which corresponds to a totally geodesic 2-sphere), see [9]. Altogether we obtain

Proposition 2. *Let $f: M \to S^3$ be a conformal minimal immersion with associated complex unitary rank 2 bundle (V, ∇). Let $V = S^{-1} \oplus S$ be the unitary decomposition, where $S^{-1} = \ker \Phi \subset V$ and Φ is the K-part of the differential of f. With respect to this decomposition the connection can be written as*

$$\nabla = \begin{pmatrix} \nabla^{\text{spin}^*} & -\frac{i}{2}Q^* \\ -\frac{i}{2}Q & \nabla^{\text{spin}} \end{pmatrix},$$

where ∇^{spin} is the spin connection corresponding to the Levi-Civita connection on M and Q is the Hopf field of f.

The Higgsfield $\Phi \in H^0(M, K\,\text{End}_0(V))$ can be identified with

$$\Phi = 1 \in H^0(M; K\,\text{Hom}(S, S^{-1})),$$

and its adjoint Φ^* is given by the volume form vol *of the induced Riemannian metric.*

2.2. Local description. Next we give a link of the gauge theoretic description of minimal surfaces in S^3 with the local treatment of CMC surfaces in $\mathbb{R}^3$ or S^3. The later is usually used by people working with the DPW method. Moreover, the construction of minimal surfaces out of a meromorphic potential uses the local description: The Iwasawa decomposition of a (local) parallel frame of the meromorphic connection, which is after a trivialization given by the meromorphic potential, splits out a so-called extended frame $\mathcal{F}$ depending on ζ. This extended frame is nothing but the frame of the family of connections ∇^{ζ} with respect to a corresponding trivialization.

Let $U \subset M$ be a simply connected open subset and $z: U \to \mathbb{C}$ be a holomorphic chart. Write $g = e^{2u}|dz|^2$ for a function $u: U \to \mathbb{R}$. Choose a local holomorphic section $s \in H^0(U; S)$ with $s^2 = dz$, and let $t \in H^0(U, S^{-1})$ be its dual holomorphic section. Then

$$(e^{-u/2}t,\ e^{u/2}s)$$

is a special unitary frame of $V = S^{-1} \oplus S$ over U. Write the Hopf field $Q = q(dz)^2$ for some local holomorphic function $q: U \to \mathbb{C}$.

The Levi-Civita connections of conformally equivalent metrics $g = e^{2\lambda}g_0$ and g_0 differ on the canonical bundle K by the form $-2\,\partial\lambda = -(d\lambda - i * d\lambda) \in \Gamma(K)$. Therefore, the connection form of ∇^{spin} with respect to the local frame s is given by $-\partial u$, and with respect to $e^{u/2}s$, it is given by $\frac{1}{2}i * du$. From Proposition 2 the connection form of ∇ with respect to $(e^{-u/2}t,\ e^{u/2}s)$ is

$$\begin{pmatrix} -\frac{1}{2}i * du & -\frac{i}{2}e^{-u}\bar{q}d\bar{z} \\ -\frac{i}{2}e^{-u}q dz & \frac{1}{2}i * du \end{pmatrix}.$$

The Higgsfield Φ and its adjoint Φ^* are given by

$$\Phi = \begin{pmatrix} 0 & e^u dz \\ 0 & 0 \end{pmatrix}, \qquad \Phi^* = \begin{pmatrix} 0 & 0 \\ e^u d\bar{z} & 0 \end{pmatrix}$$

with respect to the frame $(e^{-u/2}t,\ e^{u/2}s)$. These formulas are well known, see [5], or, in slightly other notation, [2]. Therefore, the associated family of flat connections, see equation (3.1), takes locally the same form which is used to compute minimal surfaces in S^3 out of a meromorphic potential.

3. DPW: From minimal surfaces to meromorphic connections

We restrict our considerations to compact oriented minimal surfaces in S^3 of genus 2. There is some hope that surfaces of genus $g \geq 3$ can be treated similar to surfaces of genus two, but there will be much more technical difficulties in general. We assume that the surface is homotopic to an embedding, and that the holomorphic bundle (V, ∇'') is stable. We use the notations of the previous section.

From equations (2.5) and (2.4) we see that the curvature of

$$\nabla^\zeta := \nabla + \zeta^{-1}\Phi - \zeta\Phi^* \tag{3.1}$$

vanishes for all $\zeta \in \mathbb{C} \setminus \{0\}$. The connections are special unitary for $\zeta \in S^1 \subset \mathbb{C}$, and $\mathrm{SL}(2,\mathbb{C})$-connections for $\zeta \in \mathbb{C}$. This family of connections plays a very important role in the theory of harmonic maps, as one might see from [2, 6, 9, 11, 17] or others.

The DPW method, see [6], shows that, on simply connected domains, every family of connections ∇^ζ of the form (3.1) can be obtained from meromorphic data, namely the DPW potential. We will not describe the details of this construction, one might consult [5] or [6]. The idea is, that a local parallel frame Ψ (with respect to a trivialization) of the meromorphic connection can be split by Iwasawa decomposition

$$\Psi = \mathcal{F} B$$

into a unitary part $\mathcal{F}$ and a positive part B. The positive part will be the singular gauge described below (in the trivialization). The unitary part $\mathcal{F}$ is the parallel frame (in a corresponding trivialization) of a family of connections ∇^ζ of the form (3.1) obtained for minimal surfaces.

We will prove here that for compact oriented minimal surface in S^3 of genus 2, under the conditions described above, one can gauge the holomorphic family of connections ∇^ζ globally to a family of meromorphic connections on M. The $\bar\partial$-part of this meromorphic family is constant and given by the trivial extension of S by S^{-1}. We prove that the poles of the connections are exactly at the Weierstrass points of the Riemann surface of order at most 2.

Remark. The condition that the surface is homotopic to an embedding translates to the property that the spin bundle $S \to M$ has no holomorphic sections, see [15].

Definition. A meromorphic connection on a holomorphic vector bundle $(V, \bar\partial)$ over a Riemann surface is a connection with singularities which can be written with respect to a local holomorphic frame as $d + \xi$, where ξ is an meromorphic endomorphism-valued 1-form.

Of course, meromorphic connections are flat on surfaces. For line bundles $L \to M$ there is a class of meromorphic connections which are in $1:1$ correspondence with meromorphic sections of L by declaring the section to be parallel. Moreover there exists the degree formula

$$\mathrm{res}(\nabla) = -\deg(L)$$

on Riemann surfaces, where $\mathrm{res}(\nabla)$ is the sum of all local (well-defined) residua (of the locally defined connection forms).

The condition that $\nabla^\zeta \cdot B$, the gauge of ∇^ζ by B, is a holomorphic family of meromorphic connections on the holomorphic bundle $S^{-1} \oplus S$ translates easily to

$$\bar\partial^{\mathrm{spin}} B = \Big(\frac{i}{2} Q^* + \zeta\Phi^*\Big) B, \tag{3.2}$$

where $Q^* \in \Gamma(\bar{K}K^{-1})$ and $\Phi^* \in \Gamma(\bar{K}K)$ are given as in the previous section, and $\bar\partial^{\text{spin}}$ is the holomorphic structure of the endomorphism bundle of the direct sum bundle $S^{-1} \oplus S$.

With respect to the unitary splitting $V = S^{-1} \oplus S$ we write

$$B = b_\zeta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, d \in \Gamma(M \times U; \underline{\mathbb{C}})$, $b \in \Gamma(M \times U; K^{-1})$ and $c \in \Gamma(M \times U; K)$, where $U \subset \mathbb{C}$ is a small neighborhood of 0 in the ζ-plane. Then equations (3.2) are equivalent to

$$\bar\partial a = \frac{i}{2} Q^* c, \qquad \bar\partial c = \zeta \Phi^* a, \qquad \bar\partial b = \frac{i}{2} Q^* d, \qquad \bar\partial d = \zeta \Phi^* b, \tag{3.3}$$

where the $\bar\partial$-operators are the obvious ones on the trivial holomorphic bundle $\underline{\mathbb{C}}$, on the canonical bundle K and on its dual K^{-1}. One cannot solve these equations globally (on compact surfaces) without singularities. For example, if one makes a ζ-expansion of a, b, c, d and starts with $d_0 = 1$, then, by Serre duality, there does not exist a solution to $\bar\partial b_0 = \frac{i}{2} Q^* 1$, because $[Q^*] \in H^1(M, K^{-1})$ is non-zero. What one has to do is to allow singularities of the following kind:

Definition. Let $W \to M$ be a holomorphic vector bundle over a Riemann surface. A local section $s \in \Gamma(U \setminus \{p\}, W)$ has a pole-like singularity of order k at $p \in U$ if it can locally be written as $\frac{t}{z^k}$ for a locally non-vanishing section t and a holomorphic chart z centered at p. The space of all sections with pole-like singularities will be denoted by $\hat\Gamma(M, W)$.

To solve the $\bar\partial$ problem at hand, we allow pole-like singularities at the Weierstrass points of the Riemann surface. More concrete, we take two divisors $D \neq \tilde{D}$ with

$$L(D) = L(\tilde{D}) = KS.$$

In fact one can take $D = Q_1 + Q_2 + Q_3$ and $\tilde{D} = Q_4 + Q_5 + Q_6$, where $Q_1, \dots, Q_6$ are the Weierstrass points of the Riemann surface (in the right order corresponding to the spin structure S). To see this, take a Weierstrass point Q_1. Then there exist two uniquely determined points $P_1, P_2 \in M$ such that $L(P_1 + P_2 - Q_1) = S$ by Riemann–Roch. Since S has no holomorphic sections, we see $L(P_1 + P_2) \neq K$, and one easily obtains that $P_1 = Q_2$ and $P_2 = Q_3$ are Weierstrass points. It is clear that Q_1, Q_2 and Q_3 must be pairwise disjoint. Now take another Weierstrass point Q_4 and the corresponding Weierstrass points Q_5, Q_6 such that $L(Q_5 + Q_6 - Q_4) = S$. Again, it is clear that $Q_1, \dots, Q_6$ are pairwise disjoint.

Now, we multiply with s_D and $s_{\tilde{D}}$ to guarantee the existence of solutions: We consider sections

$$\tilde{a} = a \otimes s_D \in \Gamma(M \times U; KS), \quad \tilde{c} = c \otimes s_D \in \Gamma(M \times U; K^2 S),$$

and

$$\tilde{b} = b \otimes s_{\tilde{D}} \in \Gamma(M \times U; S), \quad \tilde{d} = d \otimes s_{\tilde{D}} \in \Gamma(M \times U; KS).$$

In order to solve the equations (3.3) $\tilde{a} \oplus \tilde{c}$ and $\tilde{b} \oplus \tilde{d}$ have to be holomorphic sections of the holomorphic bundles described in the following lemma.

Lemma 1. *There exist $\epsilon > 0$ and a holomorphic bundle $V_1 \to B(0;\epsilon) \subset \mathbb{C}$ of rank 6 with the property $V_1(\zeta) := \ker(\bar\partial_1^\zeta)$, where*

$$\bar\partial_1^\zeta = \begin{pmatrix} \bar\partial & -\frac{i}{2}Q^* \\ -\zeta\Phi^* & \bar\partial \end{pmatrix} : \Gamma(M; KS \oplus K^2S) \to \Gamma(M; \bar K KS \oplus \bar K K^2 S).$$

Similarly, there exists a holomorphic bundle $V_2 \to B(0;\epsilon) \subset \mathbb{C}$ of rank 2 with the property $V_2(\zeta) := \ker(\bar\partial_2^\zeta)$, where

$$\bar\partial_2^\zeta = \begin{pmatrix} \bar\partial & -\frac{i}{2}Q^* \\ -\zeta\Phi^* & \bar\partial \end{pmatrix} : \Gamma(M; S \oplus KS) \to \Gamma(M; \bar K S \oplus \bar K KS).$$

Proof. Note that for a (holomorphic) family of elliptic operators the minimal kernel dimension is attained on an open set. Over this open set the kernel bundle is holomorphic. For details see [1] or [3].

It remains to prove that $\ker(\bar\partial_1^0)$ and $\ker(\bar\partial_2^0)$ have minimal dimension. By Riemann–Roch

$$\operatorname{index}(\bar\partial_1^\zeta) = 6.$$

With Serre duality one obtains that

$$\ker(\bar\partial_1^0) \cong H^0(M; KS) \oplus H^0(M; K^2S)$$

has dimension 6. Similarly,

$$\operatorname{index}(\bar\partial_2^\zeta) = 2,$$

and

$$\ker(\bar\partial_2^0) \cong H^0(M; KS)$$

has dimension 2. □

From Lemma 1 we see that we can find a gauge B as follows: Take a holomorphic section

$$\tilde B = (B_1, B_2) \in H^0(B(0;\epsilon), V_1 \oplus V_2).$$

Then

$$B := (B_1 \otimes s_{-D}, B_2 \otimes s_{-\tilde D})$$

is a (ζ-depending) section of $\operatorname{End}(V)$ with pole-like singularities at $Q_1, \dots, Q_6$. If we can choose $\tilde B$ such that B has constant determinant $\det(B) = 1$, then B gauges ∇^ζ into a holomorphic family of meromorphic connections with constant $\bar\partial$-part given by the trivial extension of S by S^{-1}.

In order to ensure $\det(B) = 1$, we have to study the following determinant:

Lemma 2. *The determinant map*

$$\det\colon V_1(0) \oplus V_2(0) \to H^0(M, K^3); \quad \det\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) := ad - bc$$

has surjective differential at the point

$$\begin{pmatrix} s_D & s'_{\tilde{D}} \\ 0 & s_{\tilde{D}} \end{pmatrix},$$

where $s'_{\tilde{D}} \in \Gamma(M; S)$ *is the unique solution of* $\bar\partial s'_{\tilde{D}} = \frac{i}{2} Q^* s_{\tilde{D}}$, *exactly in the case that* (V, ∇'') *is stable.*

Proof. First of all det maps to $H^0(M; K^3)$ as a consequence of the equations (3.3). So it is well-defined and holomorphic. Its derivative d_p det at

$$p := \begin{pmatrix} s_D & s'_{\tilde{D}} \\ 0 & s_{\tilde{D}} \end{pmatrix}$$

is given by the map

$$\begin{pmatrix} \alpha + q' & \beta' \\ q & \beta \end{pmatrix} \mapsto s_D \beta + s_{\tilde{D}} \alpha + s_{\tilde{D}} q' - q s'_{\tilde{D}}$$

for $\alpha, \beta \in H^0(M; KS)$, $q \in H^0(M; K^2 S)$ and solutions $q' \in \Gamma(M; KS)$ of $\bar\partial q' = \frac{i}{2} Q^* q$ and $\beta' \in \Gamma(M; S)$ of $\bar\partial \beta' = \frac{i}{2} Q^* \beta$. Note that $\dim H^0(M; KS) = 2$, $\dim H^0(M; K^2 S) = 4$, and $\dim H^0(M; K^3) = 5$.

Looking at the zeros, one sees that $s_D \beta + s_{\tilde{D}} \alpha = 0$ exactly in the case that $\beta = \lambda s_{\tilde{D}}$ and $\alpha = -\lambda s_D$ for some $\lambda \in \mathbb{C}$. Therefore, the differential d_p det maps the subspace given by $\{q = 0,\ q' = 0\}$ to a 3-dimensional subspace of $H^0(M; K^3)$.

Consider a basis $(q_1, \dots, q_4)$ of $H^0(M; K^2 S)$ with divisors given by

$$(q_1) = D + 2Q_1, \qquad (q_2) = D + 2Q_4, \qquad (q_3) = \tilde{D} + 2Q_1, \qquad (q_4) = \tilde{D} + 2Q_4.$$

It can be easily seen that the differential d_p det maps q_3 and q_4 into the 3-dimensional subspace described above. Any element in the subspace spanned by q_1, q_2 can be written as ωs_D for some $\omega \in H^0(M; K)$. Its image lies in the 3-dimensional subspace of $H^0(M; K^3)$ exactly in the case that there exists $\alpha, \beta \in H^0(M; KS)$ such that

$$\omega(s_D s'_{\tilde{D}} - s'_D s_{\tilde{D}}) = s_D \beta + s_{\tilde{D}} \alpha. \tag{3.4}$$

The decomposition (3.4) is possible for non-zero ω exactly in the case of a non-stable bundle (V, ∇''), see Theorem 3 and Remark 4. □

Theorem 1. *Let* ∇^ζ *be the holomorphic family of flat connections* (3.1) *on* V *associated to a compact oriented immersed minimal surface* $f\colon M \to S^3$ *of genus* 2. *Assume that* $(V, \nabla'' = (\nabla^0)'')$ *is stable and that* f *is homotopic to an embedding. Let* S *be the associated spinor bundle of* f. *Then, there exists an order* $Q_1, \dots, Q_6$ *of the six Weierstrass points of* M *such that* $KS = L(Q_1 + Q_2 + Q_3) = L(Q_4 + Q_5 + Q_6)$.

There exists a holomorphically ζ*-dependent gauge*

$$B : \zeta \in \tilde{B}(0; \epsilon) \subset \mathbb{C} \to \hat\Gamma(\mathrm{End}(V))$$

with pole-like singularities at Q_1, Q_2, Q_3 up to order 1 in the first column (with respect to the unitary decomposition $V = S^{-1} \oplus S$) and pole like singularities at Q_4, Q_5, Q_6 up to order 1 in the second column such that $\det B_\zeta = 1$ for all ζ and such that the gauged connection

$$\hat{\nabla}^\zeta := \nabla^\zeta \cdot B_\zeta$$

is a holomorphic family of meromorphic connections $\hat{\nabla}^\zeta$ for $\zeta \in B(0;\epsilon) \setminus \{0\} \subset \mathbb{C}$ on the (fixed) direct sum holomorphic vector bundle $S^{-1} \oplus S$. Moreover,

$$B(0) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

which implies that B is a positive gauge.

The connections $\hat{\nabla}^\zeta$ have poles up to order 1 on the diagonal (with respect to the unitary decomposition $V = S^{-1} \oplus S$) at $Q_1, \ldots, Q_6$ and poles up to order 2 in the lower left entry at Q_1, Q_2, Q_3 and in the upper right at Q_4, Q_5, Q_6. The family $\hat{\nabla}^\zeta$ has an expansion in ζ of the form

$$\hat{\nabla}^\zeta = \begin{pmatrix} \nabla_0^* & \zeta^{-1} + \omega \\ -\frac{i}{2} Q & \nabla_0 \end{pmatrix} + \text{higher order terms},$$

where ∇_0 is a meromorphic connection on S, $\omega \in \mathcal{M}(M;\mathbb{C})$, and $Q \in H^0(K^2)$ is the Hopf field of the minimal surface.

Proof. By Lemma 2 and the implicit function theorem we can find locally around $\zeta = 0$ a holomorphic section (B_1, B_2) of $V_1 \oplus V_2 \to B(0;\epsilon)$ with $B_1(0) = s_{Q_1+Q_2+Q_3}$ and $\det(B_1, B_2)(\zeta) = s_{Q_1+Q_2+Q_3} s_{Q_2+Q_4+Q_6}$ for small ζ. Here det is defined on $V_1 \oplus V_2$ analog as in Lemma 2. Then the gauge given by

$$B := (B_1 \otimes s_{-Q_1-Q_2-Q_3}, B_2 \otimes s_{-Q_4-Q_5-Q_6})$$

is of the desired form. The expansion of the family of connections $\hat{\nabla}^\zeta$ and its pole behavior can be easily computed. □

By this theorem one knows what kind of DPW potential one should use to construct genus 2 minimal surfaces $f : M \to S^3$. With respect to a meromorphic trivialization of $S^{-1} \oplus S$ the DPW potential ξ takes values (for each $\zeta \in \mathbb{C}^*$) in an explicitly known finite dimensional vector space (depending only on M). Of course, this theorem does not give new information about the behavior of the potential ξ into the ζ-direction.

Remark 1. The condition on the stability, being essential in the proof presented here, is natural in some sense. First of all, the only known example in genus 2, Lawson's genus 2 surface ([12]), has a stable holomorphic bundle (V, ∇''), see [8] or Example 2 below. Moreover, stability is an open condition, compare with Theorem 3 and Section 5.

Remark 2. In principle it should be possible to prove a theorem of this kind for all compact surfaces of higher genus. One of the main problems will be to find out such detailed information about the poles of the meromorphic connections.

Remark 3. Theorem 1 also applies for compact oriented CMC surfaces in $\mathbb{R}^3$ or S^3 of genus 2, as one sees from the discussion in Section 2.2.

Example 1. In concrete situations one should be able to find out more information about the meromorphic connections, for example in the case that the surface has many symmetries. This has already been done by the author ([8]) in the case of Lawson's genus 2 surface: The Riemann surface M is given by a threefold covering

$$\pi: M \to \mathbb{CP}^1$$

whose branch points lie over $\{1, i, -1, -i\} \subset \mathbb{CP}^1$. The Weierstrass points are the points Q_1, Q_2, Q_3 lying over 0 and Q_4, Q_5, Q_6 lying over ∞, and the spinor bundle is $S = L(Q_1 + Q_2 - Q_3)$. The umbilics $P_1, \dots, P_4$ of the minimal surface are the branch points. In a meromorphic trivialization of $S^{-1} \oplus S$ given by the meromorphic sections

$$s = s_{Q_4+Q_5+Q_6-P_1-P_2-P_3-P_4} \in \mathcal{M}(M, S^{-1}),$$
$$t = s_{-Q_6-Q_5-Q_6+P_1+P_2+P_3+P_4} \in \mathcal{M}(M, S),$$

the connection form of $\hat{\nabla}^\zeta$ is given by

$$\pi^* \begin{pmatrix} -\frac{4}{3}\frac{z^3}{z^4-1} + \frac{A}{z} & \zeta^{-1} + Bz^2 \\ \frac{G}{(z^4-1)} + \frac{\zeta H}{z^2(z^4-1)} & \frac{4}{3}\frac{z^3}{z^4-1} - \frac{A}{z} \end{pmatrix} dz.$$

Here A, B, G, H are ζ-depending holomorphic functions well-defined at $\zeta = 0$ which satisfy $H = A + A^2$ and $B = -\frac{1}{G}(-\frac{1}{3} + A + (\frac{1}{3} - A)^2)$.

4. Stable extensions $0 \to S^{-1} \to V \to S \to 0$

For general information and details about extensions and stable bundles we refer to [14]. We restrict to the case that S is a spinor bundle over a compact Riemann surface of genus 2 which has no holomorphic sections.

It is well known that non-trivial extensions

$$0 \to S^{-1} \to V \to S \to 0$$

correspond to elements of $PH^1(M; K^{-1})$ as follows: any two sections $b_1, b_2 \in \Gamma(M; \bar{K}K^{-1})$ give rise to holomorphic isomorphic extensions via

$$\bar{\partial} = \begin{pmatrix} \bar{\partial} & b_k \\ 0 & \bar{\partial} \end{pmatrix}$$

on $S^{-1} \oplus S$ if and only they are in the same class in $PH^1(M; K^{-1})$.

Theorem 2. *There exists a projective isomorphism*

$$\Phi: PH^1(M; K^{-1}) \to PH^0(M; K^2),$$

which does only depend on the spin bundle S. Hence, the space $PH^0(M; K^2)$ classifies non-trivial extensions $0 \to S^{-1} \to V \to S \to 0$.

Proof. Consider a section $b \in \Gamma(M; \bar{K}K^{-1})$ which defines a non-trivial extension $0 \to S^{-1} \to V \to S \to 0$. Let $\alpha, \beta \in H^0(M; KS)$ be a basis of $H^0(M; KS)$ and $\alpha', \beta' \in \Gamma(M; S)$ be the unique solutions of $\bar{\partial}\alpha' = b\alpha$ and $\bar{\partial}\beta' = b\beta$. Then

$$\mathcal{Q} := \alpha'\beta - \alpha\beta' \in H^0(M; K^2).$$

Clearly, the line $\mathbb{C}\mathcal{Q} \in PH^0(M; K^2)$ does not depend on the chosen basis α, β. Moreover, if we consider $\tilde{b} = b + \bar{\partial}X$ for $X \in \Gamma(M; K^{-1})$ we see that $\tilde{\alpha}' = \alpha' + X\alpha$ and $\tilde{\beta}' = \beta' + X\beta$ are the corresponding solutions. Hence $\mathbb{C}\mathcal{Q}$ does only depend on the class $[b] \in H^1(M; K^{-1})$.

Because the solutions clearly depend linearly on $b \in \Gamma(M; \bar{K}K^{-1})$, it remains to show that $\mathcal{Q} \neq 0$ for $0 \neq [b] \in H^1(M; K^{-1})$. To see this note that α and β have no common zeros. Therefore, if $\mathcal{Q}$ would vanish, the solution β' would have zeros at the zeros of β. Here, and later on in this section, we say that a smooth section s has a zero at p of order k if and only if $s \otimes (s_{-p})^k$ is smooth. Hence, there would be a solution of

$$\bar{\partial}t = b$$

for

$$t (= \beta' \otimes \beta^{-1}) \in \Gamma(M; K^{-1}) = \Gamma(M; S \otimes (KS)^{-1}).$$

Here, $\beta^{-1} \in \mathcal{M}(M; (KS)^{-1})$ is the dual meromorphic section of β. By Serre duality and the non-vanishing of $[b] \in H^1(M; K^{-1})$ there cannot be a solution t. □

Remark. One should not mistake the line $\mathbb{C}\mathcal{Q}$ in the space of holomorphic quadratic differentials associated to a non-trivial extension for the Hopf field Q of a minimal surface. But in the case of the Lawson genus 2 surface, the Hopf differential generates the line associated to the holomorphic structure ∇'', see Example 2.

The advantage of the description given by Theorem 2 is the following.

Theorem 3. *A non-trivial extension $0 \to S^{-1} \to V \to S \to 0$ over a compact Riemann surface of genus 2 with corresponding $\mathbb{C}\mathcal{Q} \in PH^0(M; K^2)$, such that S has no holomorphic sections, is stable if and only if there exist $0 \neq \omega \in H^0(M; K)$ and holomorphic sections $\alpha, \beta \in H^0(M; KS)$ such that*

$$\omega\mathcal{Q} = \alpha\beta.$$

Remark 4. Note that each element ψ of the 3-dimensional subspace $W \subset H^0(M; K^3)$ spanned by products of two holomorphic sections in $H^0(M; KS)$ can be written as a product $\psi = \alpha\beta$ for holomorphic sections $\alpha, \beta \in H^0(M; KS)$.

Proof. A non-trivial extension $0 \to S^{-1} \to V \to S \to 0$ given by $b \in \Gamma(M; \bar{K}K^{-1})$ is non-stable if and only if there exists a point $P \in M$ such that

$$b^{\perp} = \{q \in H^0(M, K^2) \mid q(P) = 0\},$$

where

$$b^{\perp} = \Big\{q \in H^0(M, K^2) \mid \int_M (b, q) = 0\Big\},$$

see [14, Lemma 5.2].

We need to characterize the zeros of $\mathcal{Q} := \Phi([b])$. Note that each holomorphic quadratic differential is the product of two holomorphic differentials. Let $(\mathcal{Q}) = P_1 + \cdots + P_4$ for $P_k \in M$, such that $L(P_1 + P_2) = L(P_3 + P_4) = K$. For each point $P \in M$ there exists an unique pair of points $\tilde{P}, \hat{P} \in M$ such that $KS = L(P + \tilde{P} + \hat{P})$. We claim that P is a zero of $\mathcal{Q}$ if and only if

$$\int_M (b, q) = 0$$

for all $q \in H^0(M; K^2)$ with $q(\tilde{P}) = q(\hat{P}) = 0$, counted with multiplicities (only important for the case of $\tilde{P} = \hat{P}$). Because $L(\hat{P} + \tilde{P}) \neq K$, the space of $q \in H^0(M; K^2)$ with $q(\tilde{P}) = q(\hat{P}) = 0$ is 1-dimensional, and it is determined by P. For $k = 1, \dots, 4$ we consider a basis

$$s = s_{P_k + \tilde{P}_k + \hat{P}_k},\ t$$

of $H^0(M; KS)$. Because s and t have no common zeros, the solution s' of $\bar{\partial} s' = bs$ has a zero at P_k, too. Therefore, there exists a solution

$$\varphi (= s' \otimes s_{-P_k}) \in \Gamma(M; SL(-P_k))$$

of

$$\bar{\partial}\varphi = bs_{\tilde{P}_k + \hat{P}_k} = bs \otimes s_{-P_k} \in \Gamma(M; \bar{K}SL(-P_k)).$$

But by Serre duality, there exists such a solution if and only if

$$\int_M (bs_{\tilde{P}_k + \hat{P}_k}, \omega) = 0$$

for all $\omega \in H^0(M; SL(P_k))$. This space is 1-dimensional because S has no holomorphic sections. Then $s_{\tilde{P}_k + \hat{P}_k}\omega \in H^0(M; KSL(-P_k)SL(P_k))$ is a holomorphic quadratic differential which spans the 1-dimensional space

$$\{q \in H^0(M; K^2) \mid q(\tilde{P}_k) = q(\hat{P}_k) = 0\}.$$

This proves the assertion for the zeros of $\mathcal{Q}$.

There exists a holomorphic differential ω and $\alpha, \beta \in H^0(M; KS)$ with $\omega\mathcal{Q} = \alpha\beta$ if and only if $\{\tilde{P}_1, \hat{P}_1\} \cap \{P_3, P_4\}$ is non-empty. This can be easily deduced from the facts that $L(\tilde{P}_1 + \hat{P}_1) \neq K$ and that S has no holomorphic sections.

If $\{\tilde{P}_1, \hat{P}_1\} \cap \{P_3, P_4\}$ is non-empty, we can assume that $\tilde{P}_1 = P_3$. Then we obtain $\{\tilde{P}_3, \hat{P}_3\} = \{P_1, \hat{P}_1\}$. Let us first assume that $P_1 \neq P_3$. By the characterization of the zeros of $\mathcal{Q}$ this implies that

$$\int_M (b, q) = 0$$

for all $q \in H^0(M; K^2)$ with $q(\tilde{P}_1) = q(\hat{P}_1) = 0$ or $q(P_1) = q(\hat{P}_1) = 0$. But these holomorphic quadratic differentials span the 2-dimensional space

$$\{q \in H^0(M, K^2) \mid q(\hat{P}_1) = 0\},$$

and we see that the extension is non-stable. If $P_1 = P_3$, then $P_2 = P_4$, too, and we have $L(2P_1 + \hat{P}_1) = KS$. With the same methods as above one can show that the property that $\mathcal{Q}$

has a zero of order 2 at P_1 implies that $\int_M (b, q) = 0$ for all holomorphic quadratic differentials with $q(\hat{P}_1) = 0$. Again, this implies that the extension is non-stable.

Conversely, assume that the extension is non-stable. Therefore, there exists a point $P \in M$ such that $\int_M (b, q) = 0$ for all holomorphic quadratic differentials with $q(P) = 0$. By the characterization of the zeros of $\mathcal{Q}$ one easily sees that $\tilde{P}$ and $\hat{P}$ are zeros of $\mathcal{Q}$. First assume that $\tilde{P} \neq \hat{P}$. Let ω be a non-zero holomorphic differential with $\omega(P) = 0$. Then

$$D := (\omega) + (\mathcal{Q}) - P - \tilde{P} - \hat{P}$$

is a positive divisor with $L(D) = KS$, and $\omega\mathcal{Q} = s_{P+\tilde{P}+\hat{P}} s_D$ is a decomposition as required. Now assume $\tilde{P} = \hat{P}$. Because $\int_M (b, q) = 0$ for all holomorphic differentials with $q(P) = 0$, one can show that $\tilde{P}$ is a zero of order 2 of $\mathcal{Q}$. Let ω be a non-zero holomorphic differential with $\omega(P) = 0$. Again

$$D := (\omega) + (\mathcal{Q}) - P - 2\tilde{P}$$

is a positive divisor with $L(D) = KS$, and $\omega\mathcal{Q} = s_{P+2\tilde{P}} s_D$ is a decomposition as required. □

Example 2. We claim that the line $\mathbb{C}\mathcal{Q} \in PH^0(M, K^2)$ associated to the holomorphic structure of Lawson's genus 2 surface (see [12]) is given by its Hopf differential Q. Let $P_1, \dots, P_4$ be the umbilics of the surface, i.e. the zeros of Q. They correspond to the points lying over 0 and ∞ in the hyper-elliptic picture

$$y^2 = z^6 - 1$$

of the Riemann surface. Then

$$\omega_1 = \frac{1}{\sqrt{z^6 - 1}} dz, \qquad \omega_2 = \frac{z}{\sqrt{z^6 - 1}} dz,$$

is a basis of the space of holomorphic differentials. As in [8], Q is given by a multiple of $\omega_1\omega_2$, and Q^* is perpendicular to $(\omega_1)^2$ and $(\omega_2)^2$ as a consequence of the symmetries of the Lawson surface. By the proof of Theorem 3 the zeros of $\mathbb{C}\mathcal{Q}$ are the zeros of the Hopf differential. Moreover, one sees from [14] or from the characterization of Theorem 3, that the holomorphic structure ∇'' is stable.

5. The family of holomorphic structures

In the last section we consider immersed compact oriented minimal surfaces in S^3 of genus $g \geq 2$. We will prove that the holomorphic structure

$$\bar{\partial}^\zeta := (\nabla^\zeta)'' = \begin{pmatrix} \bar{\partial}^{\text{spin}*} & -\frac{i}{2}Q^* \\ \zeta\Phi^* & \bar{\partial}^{\text{spin}} \end{pmatrix}$$

on V is stable for generic $\zeta \in \mathbb{C}$. We need the following result.

Proposition 3. *Any holomorphic subbundle L of degree 0 of $(V, \bar{\partial}^\zeta)$ for $\zeta \in S^1 \subset \mathbb{C}^*$ is parallel with respect to ∇^ζ.*

Proof. The holomorphic bundle L^* has a unique unitary flat connection ∇^{L^*}. Then, the subbundle $L \subset V$ gives rise to a holomorphic section $i \in H^0(M; L^* \otimes V)$. We denote the induced flat unitary connection on $L^* \otimes V$ by $\nabla = \partial + \bar\partial$. As in [9] we obtain from flatness

$$\bar\partial \partial i = 0$$

and

$$\int_M (\partial i, \partial i) = \int_M (\bar\partial \partial i, i) = 0.$$

Thus i is parallel, and L is a parallel subbundle of (V, ∇^ζ). □

If $(V, \bar\partial^\zeta)$ would not be stable for generic $\zeta \in \mathbb{C}$, the holonomy of ∇^ζ would be abelian for all $\zeta \in \mathbb{C}$: For $\zeta \in S^1 \subset \mathbb{C}$ this follows easily from the fact that with $L \subset V$ parallel, also $Lj \subset V$ is parallel. Because the holonomy depends holomorphically on ζ, this implies the assertion. But the following theorem shows that this is not possible for $g \geq 2$.

Theorem 4. *The holonomy representation of* ∇^ζ *is non-abelian for generic* $\zeta \in \mathbb{C}^*$. *As a consequence,* $(V, \bar\partial^\zeta)$ *is stable for generic* $\zeta \in \mathbb{C}^*$.

Proof. The proof is based on the observation that many arguments of Hitchin ([9]) remain true for higher genus surfaces under the assumption of abelian holonomy: First of all one would obtain a splitting

$$V = L_\zeta \oplus L_\zeta^*$$

into parallel subbundles of ∇^ζ for ζ in a punctured neighborhood $\hat U$ of $\zeta = 0$. Of course, this decomposition is holomorphic in ζ locally, but it might be that the subbundles interchange as ζ goes around 0. We can also assume that L_ζ^2 is not holomorphically trivial for $\zeta \in \hat U$. As in [9] we get a holomorphic decomposition of the trace free endomorphisms

$$\mathrm{End}_0(V, \bar\partial^\zeta) = L_\zeta^2 \oplus \underline{\mathbb{C}} \oplus L_\zeta^{-2},$$

where the $\underline{\mathbb{C}}$-part corresponds to (trace free) diagonal endomorphisms corresponding to the decomposition $V = L_\zeta \oplus L_\zeta^*$. From this one sees that $\dim H^0(M, \mathrm{End}_0(V, \bar\partial^\zeta)) = 1$ for $\zeta \in \hat U$. Moreover, a generator of this 1-dimensional space is parallel with respect to ∇^ζ. The bundle

$$H^0(M, \mathrm{End}_0(V, \bar\partial^\zeta)) \to \hat U$$

extends to $\zeta = 0$. Consider a local trivializing section Ψ, i.e. a holomorphic family of holomorphic sections

$$\zeta \in \mathbb{C} \mapsto \Psi_\zeta \in H^0(M, \mathrm{End}_0(V, \bar\partial^\zeta))$$

which is non-vanishing for small ζ. This section is covariant constant with respect to ∇^ζ for small $\zeta \neq 0$. Expanding $\Psi_\zeta = \Psi^0 + \zeta \Psi^1 + \cdots$ around $\zeta = 0$ implies

$$[\Psi^0, \Phi] = 0.$$

This yields that Ψ^0 is a (non-zero) holomorphic section in $\mathrm{Hom}(S, S^{-1}) \subset \mathrm{End}_0(V, \bar\partial^\nabla)$. But $\deg \mathrm{Hom}(S, S^{-1}) = 2 - 2g < 0$ for $g \geq 2$, and we obtain a contradiction. □

Because of this theorem it is not possible to define an eigenline spectral curve which does not depend on the chosen generator $\gamma \in \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\} \subset \pi^1(M)$. The eigenlines for different γ do not coincide, and the whole machinery which was so successful for tori cannot be applied for higher genus $g \geq 2$.

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APPENDIX C

A spectral curve approach to Lawson symmetric CMC surfaces of genus 2

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A SPECTRAL CURVE APPROACH TO LAWSON SYMMETRIC CMC SURFACES OF GENUS 2

SEBASTIAN HELLER

Abstract. Minimal and CMC surfaces in S^3 can be treated via their associated family of flat $\mathrm{SL}(2,\mathbb{C})$-connections. In this the paper we parametrize the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on the Lawson minimal surface of genus 2 which are equivariant with respect to certain symmetries of Lawson's geometric construction. The parametrization uses Hitchin's abelianization procedure to write such connections explicitly in terms of flat line bundles on a complex 1-dimensional torus. This description is used to develop a spectral curve theory for the Lawson surface. This theory applies as well to other CMC and minimal surfaces with the same holomorphic symmetries as the Lawson surface but different Riemann surface structure. Additionally, we study the space of isospectral deformations of compact minimal surface of genus $g \geq 2$ and prove that it is generated by simple factor dressing.

Contents

1. Introduction

The study of minimal surfaces in three dimensional space forms is among the oldest subjects in differential geometry. While minimal surfaces in euclidean 3-space are never compact, there exist compact minimal surfaces in S^3. In fact, it has been shown by Lawson [L] that for every genus g there exists at least one embedded closed minimal surface in the 3-sphere. A slightly more general surface class is given by constant mean curvature (CMC) surfaces. Due to the Lawson correspondence the partial differential equations

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describing minimal and CMC surfaces in S^3 can be treated in a uniform way. Compact minimal and CMC surfaces of genus 0 and 1 are well-understood by now: The only CMC 2-spheres in S^3 are the totally umbilic spheres as the Hopf differential vanishes. Furthermore, Brendle [Br] has recently shown that the only embedded minimal torus in S^3 is the Clifford torus up to isometries. This was extended by Andrews and Li [AL] who proved that the only embedded CMC tori in S^3 are the unduloidal rotational Delaunay tori. Nevertheless, there exist compact immersed minimal and CMC tori in S^3 which are not congruent to the Clifford torus respectively to the Delaunay tori. First examples have been constructed by Hitchin [H] via integrable systems methods. Moreover, all CMC tori in S^3 are constructed from algebro-geometric data defined on their associated spectral curve, see [H, PS, B].

The study of minimal surfaces via integrable system methods is based on the associated $\mathbb{C}^*$-family of flat $\mathrm{SL}(2,\mathbb{C})$-connections ∇^λ, $\lambda \in \mathbb{C}^*$. Flatness of ∇^λ for all λ in the spectral plane $\mathbb{C}^*$ is the gauge theoretic reformulation of the harmonic map equation. Knowing the family of flat connections is tantamount to knowing the minimal surface, as the minimal surface is given by the gauge between the trivial connections ∇^1 and ∇^{-1}. Slightly more general, there also exists a family of flat connections associated to CMC surfaces in S^3. They are given as the gauge between ∇^{λ_1} and ∇^{λ_2} for $\lambda_1 \neq \lambda_2 \in S^1 \subset \mathbb{C}^*$ and have mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. In the abelian case of CMC 2-tori ∇^λ splits for generic λ into a direct sum of flat connections on a line bundle and its dual. Therefore, the $\mathbb{C}^*$-family of flat $\mathrm{SL}(2,\mathbb{C})$-connections associated to a CMC torus is characterized by a spectral curve parametrizing the corresponding family of flat complex line bundles. On surfaces of genus $g \geq 2$ flat $\mathrm{SL}(2,\mathbb{C})$-connections are generically irreducible and therefore they have non-abelian monodromy. In fact, every (compact) branched CMC surface of genus $g \geq 2$ whose associated family of flat connections has abelian holonomy factors through a CMC torus or is a branched conformal covering of a round sphere [Ge]. Thus the abelian spectral curve theory for minimal and CMC tori are no longer applicable in the case of compact immersed minimal and CMC surfaces of genus $g \geq 2$.

The aim of this paper is to develop what might be called an integrable systems theory for compact higher genus minimal and CMC surfaces in S^3 based on its associated family of flat connections. The main benefit of this approach is that one can divide the construction and the study of minimal or CMC surfaces into three steps:

1. Write down (enough) flat $\mathrm{SL}(2,\mathbb{C})$-connections on a given Riemann surface.
2. Construct a family $\tilde\nabla^\lambda$ of flat $\mathrm{SL}(2,\mathbb{C})$-connections gauge equivalent (where the gauge is allowed to depend on the spectral parameter λ) to a family of flat connections associated to a CMC surface in S^3. To ensure this, $\tilde\nabla^\lambda$ needs to be unitarizable for $\lambda \in S^1$, and trivial for $\lambda_1 \neq \lambda_2 \in S^1 \subset \mathbb{C}^*$ and must have a special asymptotic behavior as $\lambda \to 0$.
3. Reconstruct an associated family of flat connections of a CMC surface from the gauge equivalent family.

In a certain sense these steps occur in the integrable system approach to CMC tori [H]. Here the gauge class of a generic flat $\mathrm{SL}(2,\mathbb{C})$-connection is determined by the holonomy of one of the eigenlines. The spectral curve parametrizes these holonomies and the gauge to the associated family can be determined with the help of the eigenline bundle on the spectral curve.
Similarly, the loop group approach to CMC surfaces put forward in [DPW], sometimes called the DPW method, starts with a family of holomorphic (or meromorphic) $\mathrm{SL}(2,\mathbb{C})$-connections on a Riemann surface. Typically, these connections are given by a λ-dependent

holomorphic (or meromorphic) $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form called the DPW potential. The DPW potential has a special asymptotic behavior for $\lambda \to 0$ which guarantees the construction of a minimal surface as follows: a (λ-dependent) parallel frame for the family of holomorphic (or meromorphic) flat connections can be split into its unitary and positive parts by the loop group Iwasawa decomposition. The unitary part is characterized by the property that it is unitary on the unit circle $S^1 \subset \mathbb{C}^*$ and the positive part extends to $\lambda = 0$ in a special way. Then, the positive part is the gauge one is looking for, or equivalently, the unitary part is a (λ-dependent) parallel frame for a family of flat connections associated to a minimal surface.
It is well-known that every flat (smooth) $\mathrm{SL}(2,\mathbb{C})$-connection on a compact Riemann surface is gauge equivalent (via a gauge which might have singularities) to a flat meromorphic connection, i.e., to a connection whose connection 1-form with respect to an arbitrary holomorphic frame is meromorphic. Nevertheless, it is impossible to parametrize meromorphic connections in a way such that one obtains a unique representative for every gauge class of flat $\mathrm{SL}(2,\mathbb{C})$-connections. Therefore, the DPW potential does not need to exist for all $\lambda \in \mathbb{C}^*$. Moreover, the meromorphic connections (given by the DPW potential) need to be unitarizable for $\lambda \in S^1$ (i.e., unitary with respect to an appropriate λ-dependent unitary metric). This reality condition leads to the problem of computing the monodromies of meromorphic connections, which cannot be done by now. The aim of this paper is to overcome these problems, at least partially.

The moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a compact Riemann surface of genus 2 has, at its smooth points, dimension $6g-6$. There exist singular points, corresponding to reducible flat connections, which have to be dealt with carefully, see [G]. As we are studying holomorphic families of connections (in the sense that the connection $1-$forms with respect to a fixed connection depend holomorphically on λ), the moduli space needs to be equipped with a compatible complex structure. Moreover, we need to determine the asymptotic behavior of the family of (gauge equivalence classes of) flat connections for $\lambda \to 0$. This seems to be difficult in the setup of character varieties, i.e., if we identify a gauge equivalence class of flat connections with the conjugacy class of the induced holonomy representation of the fundamental group of the compact Riemann surface. A more adequate picture of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections is given as an affine bundle over the "moduli space" of holomorphic structures of rank 2 with trivial determinant. The projection of this bundle is given by taking (the isomorphism class of) the complex anti-linear part of the connection. This complex anti-linear part is a holomorphic structure, and for a generic flat connections it is even stable. Elements in a fiber of this affine bundle, which can be represented by two flat connections with the same induced holomorphic structure, differ by a holomorphic 1-form with values in the trace free endomorphism bundle. These 1-forms are called Higgs fields and, as a consequence of Serre duality, they are in a natural way the cotangent vectors of the moduli space of holomorphic structures, at least at its smooth points. The bundle is an affine holomorphic bundle and not isomorphic to a holomorphic vector bundle because it does not admit a holomorphic section. Nevertheless, by the Theorem of Narasimhan and Seshadri [NS], it has a smooth section (over the semi-stable part) which is given by the one to one correspondence between stable holomorphic structures and unitary flat connections.

In addition to the study of the moduli spaces, we want to construct families of flat connections explicitly. This can be achieved by using Hitchin's abelianization, see [H1, H2]. The eigenlines of Higgs fields (with respect to some holomorphic structure $\bar\partial$) whose determinant is given by the Hopf differential of the CMC surface are well-defined on a double

covering of the Riemann surface. They determine points in an affine Prym variety and as line subbundles they intersect each other over the umbilics of the minimal surface. Moreover, a flat connection with holomorphic structure $\bar{\partial}$ determines a meromorphic connection on the direct sum of the two eigenlines of the Higgs field. The residue of this meromorphic connection can be computed explicitly, and the flat meromorphic connection is determined by algebraic geometric data on the double covering surface. Moreover, $\mathbb{C}^*$-families of flat connections can be written down in terms of a spectral curve which double covers the spectral plane $\mathbb{C}^*$. A double covering is needed as a holomorphic structure with a Higgs field corresponds to two different eigenlines and these eigenlines come together at discrete spectral values.
The spectral curve parametrizes the eigenlines of Higgs fields $\Psi_\lambda \in H^0(M, K\,\mathrm{End}_0(V, \bar{\partial}^\lambda))$ with $\det \Psi_\lambda = Q$, where the holomorphic structure $\bar{\partial}^\lambda$ is the complex anti-linear part of the connection ∇^λ. In order to fix the (gauge equivalence classes of the) flat connections ∇^λ additional spectral data are needed. They are given by anti-holomorphic structures on the eigenlines, or, after fixing a special choice of a flat meromorphic connection on a line bundle in the affine Prym variety, by a lift into the affine bundle of gauge equivalence classes of flat line bundle connections. Then, analogous to the case of tori, the asymptotic behavior for $\lambda \to 0$ of the family of flat connections can be understood explicitly: the spectral curve branches over 0 and the family of flat line bundle connections has a first order pole over $\lambda = 0$, see Theorem 5. The spectral data must satisfy a certain reality condition imposed by the property that the connections ∇^λ are unitary for $\lambda \in S^1$. In contrast to the case of CMC tori this reality condition is hard to determine explicitly. Nevertheless, the reality condition is closely related to the geometry of the moduli spaces, see Theorem 2. Once one has constructed such families of (gauge equivalence classes of) flat connections, one can construct minimal and CMC surfaces in S^3 by loop group factorization methods analogous to the DPW method. It would be very interesting to see whether these loop group factorizations can be made as explicit as in the case of tori via the eigenline construction of Hitchin [H].

In this paper we only carry out the details of the details of this program for the Lawson surface of genus 2. These methods easily generalize to the case of Lawson symmetric minimal and CMC surfaces of genus 2, i.e., those surfaces with the same holomorphic and space orientation preserving symmetries as the Lawson surface but with possibly different conformal type (determined by the cross ratio of the branch images of the threefold covering $M \to M/\mathbb{Z}_3 \cong \mathbb{P}^1$). We shortly discuss this generalization in chapter 7. As explained there one could in principle always exchange *minimal* by *CMC* and *Lawson surface* by *Lawson symmetric surface* within the paper. Moreover, the definition of the spectral curve makes sense even in the case of a compact minimal or CMC surface of genus $g \geq 2$ as long as the Hopf differential has simple zeros. In that case the asymptotic of the spectral data is analogous to Theorem 5. The main difference to the general case is that we can describe the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures explicitly, see Theorem 1.

In Section 2 we study the moduli space of those holomorphic structures of rank 2 with trivial determinant that admit a flat connection whose gauge equivalence classes are invariant under the symmetries of the Lawson surface of genus 2. We show that this space is a projective line with a double point. In Section 3 we parametrize representatives of each isomorphism class in the above moduli space by using the eigenlines of special Higgs fields. This method is called Hitchin's abelianization. In our situation the space of all eigenlines is given by the 1-dimensional square torus. By Hitchin's abelianization this

torus double covers the moduli space of holomorphic structures away from the double point. This covering map will be crucial for the construction of a spectral curve later on. We use this description in Section 4 in order to parametrize the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections whose gauge equivalence classes are invariant under the symmetries of the Lawson surface. In Theorem 1 we prove an explicit 2:1 correspondence (away from a co-dimension 1 subset corresponding to the double point of the moduli space of holomorphic structures) between flat $\mathbb{C}^*$-connections on the above mentioned square torus and the moduli space of flat connections whose gauge equivalence classes are invariant under the symmetries of the Lawson surface of genus 2. This study will be completed in Section 5 where we consider flat connections whose underlying holomorphic structures do not admit Higgs fields whose determinant is equal to the Hopf differential of the Lawson surface.
In Section 6 we define the spectral curve associated to a minimal surface in S^3 which has the conformal type and the holomorphic symmetries of the Lawson surface of genus 2 (Proposition 6.1). The spectral curve is equipped with a meromorphic lift into the affine bundle of isomorphy classes of flat line bundle connections on the square torus. This lift determines the gauge equivalence classes of the flat connections. The spectral data satisfy two important properties, see Theorem 5. Firstly, they have a certain asymptotic at $\lambda = 0$. Moreover, the spectral data must satisfy a reality condition which is related to the geometry of the moduli space of stable bundles, see Theorem 2. We prove a general theorem (6) about the reconstruction of minimal surfaces out of those families of flat connections $\tilde{\nabla}^\lambda$ as described in step 2 above.
Similar to the case of tori, compact minimal and CMC surfaces of higher genus are in general not uniquely determined by the knowledge of the gauge equivalence classes of ∇^λ for all $\lambda \in \mathbb{C}^*$. We show in Theorem 7 that all different minimal immersions with the same map $\lambda \mapsto [\nabla^\lambda]$ into the moduli space of flat connections and with the same induced Riemann surface structure are generated by simple factor dressing (as defined for example in [BDLQ]). Finally, we prove in Theorem 8 that minimal surfaces with the symmetries of the Lawson genus 2 surface can be reconstructed uniquely from spectral data satisfying the conditions of Theorem 5. Moreover, we give an energy formula for those minimal surfaces in terms of their spectral data.
In the appendix, we shortly recall the gauge theoretic reformulation of the minimal surface equations in S^3 due to Hitchin [H] which leads to the associated family of flat connections. We also describe the construction of the Lawson minimal surface of genus 2.

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2. The moduli space of Lawson symmetric holomorphic structures

Before studying the associated family of flat $\mathrm{SL}(2,\mathbb{C})$-connections

$$\lambda \mapsto \nabla^\lambda$$

for a given compact oriented minimal or CMC surface in S^3 (see Appendix A or chapter 7 in the case of CMC surfaces), we need to understand the moduli space of gauge equivalence classes of flat $\mathrm{SL}(2,\mathbb{C})$-connections on the surface. We consider it as an affine bundle over the moduli space of isomorphism classes of holomorphic structures $(V,\bar{\partial})$ of rank 2 with trivial determinant over the Riemann surface. The complex structure is the one induced by the minimal immersion and the projection is given by taking the complex anti-linear part

$$\nabla'' := \frac{1}{2}(\nabla + i * \nabla)$$

of the flat connection ∇. The difference $\Psi = \nabla^2 - \nabla^1 \in \Gamma(M, K\,\mathrm{End}_0(V))$ between two flat $\mathrm{SL}(2,\mathbb{C})$-connections ∇^1 and ∇^2 with the same underlying holomorphic structure $\bar\partial = (\nabla^i)''$ satisfies

$$0 = F^{\nabla^2} = F^{\nabla^1} + d^{\nabla}\Psi = \bar\partial\,\Psi.$$

Therefore, the fiber of the affine bundle over a fixed isomorphism class of holomorphic structures (represented by the holomorphic structure $\bar\partial$) is given by the space of Higgs fields

$$H^0(M, K\,\mathrm{End}_0(V, \bar\partial)),$$

i.e., the space of holomorphic trace free endomorphism valued 1-forms on M. By Serre duality, this is naturally isomorphic to the cotangent space of the moduli space of holomorphic structures, at least at its smooth points.

In this paper we mainly focus on the Lawson minimal surface M of genus 2. Therefore we start by studying those holomorphic structures of rank 2 on M which can occur as the complex anti-linear parts of a connection ∇^λ in the associated family of M. As we will see, this simplifies the study of the moduli spaces and allows us to find explicit formulas for flat connections with a given underlying holomorphic structure.

The complex structure of the Lawson surface of genus 2 is given by (the compactification of) the complex curve

$$y^3 = \frac{z^2 - 1}{z^2 + 1}. \tag{2.1}$$

As a surface in S^3 it has a large group of extrinsic symmetries, see Appendix B. We will focus on the symmetries which are holomorphic on M and orientation preserving in S^3. The reason for this restriction relies on the fact that only those give rise to symmetries of the individual flat connections ∇^λ. As a group, they are generated by the following automorphisms, where the equations are written down with respect to the coordinates y and z of (2.1):

- the hyper-elliptic involution φ_2 of the surface of genus 2 which is given by
$$(y, z) \mapsto (y, -z);$$
- the automorphism φ_3 satisfying
$$\varphi_3(y, z) = (e^{\frac{2}{3}\pi i} y, z);$$
- the composition τ of the reflections at the spheres S_1 and S_2 is given by
$$(y, z) \mapsto (e^{\frac{1}{3}\pi i}\frac{1}{y}, \frac{i}{z}).$$

Every single connection ∇^λ is gauge equivalent to $\varphi_2^*\nabla^\lambda$, $\varphi_3^*\nabla^\lambda$ and $\tau^*\nabla^\lambda$. This can be deduced from the construction of the associated family of flat connections, see [He] for details.

Definition. *A* $\mathrm{SL}(2,\mathbb{C})$*-connection* ∇ *on* M *is called Lawson symmetric, if* ∇ *is gauge equivalent to* $\varphi_2^*\nabla$, $\varphi_3^*\nabla$ *and* $\tau^*\nabla$*. Similarly, a holomorphic structure* $\bar\partial$ *of rank* 2 *with trivial determinant on* M *is called Lawson symmetric if it is isomorphic to* $\varphi_2^*\bar\partial$, $\varphi_3^*\bar\partial$ *and* $\tau^*\bar\partial$.

We first determine which holomorphic structures occur in the family

$$\lambda \mapsto \bar\partial^\lambda := (\nabla^\lambda)'' = \frac{1}{2}(\nabla^\lambda + i * \nabla^\lambda)$$

associated to the Lawson surface. As ∇^λ is generically irreducible (see [He1]), and special unitary for $\lambda \in S^1$, $\bar\partial^\lambda$ is generically stable: A holomorphic bundle of rank 2 of degree 0 is (semi-)stable if every holomorphic line sub-bundle has negative (non-positive) degree, see [NS] or [NR]. On a compact Riemann surface of genus 2 the moduli space of stable holomorphic structures with trivial determinant on a vector bundle of rank $r = 2$ can be identified with an open dense subset of a projective 3-dimensional space, see [NR]: the set of those holomorphic line bundles, which are dual to a holomorphic line subbundle of degree -1 in the holomorphic rank 2 bundle, is given by the support of a divisor which is linear equivalent to twice the Θ-divisor in the Picard variety $Pic_1(M)$ of holomorphic line bundles of degree 1. This divisor uniquely determines the rank 2 bundle up to isomorphism if the bundle is stable. Therefore the moduli space of stable holomorphic structures of rank 2 with trivial determinant can be considered as a subset of the projective space of the 4-dimensional space $H^0(Jac(M), L(2\Theta))$ of Θ functions of rank 2 on the Jacobian of M. The complement of this subset in the projective space is the Kummer surface associated to the Riemann surface of genus 2. The points on the Kummer surface can be identified with the S-equivalence classes of strictly semi-stable holomorphic structures. Recall that the S-equivalence class of a stable holomorphic structure is just its isomorphism class but that S-equivalence identifies the strictly semi-stable holomorphic direct sum bundles $V = L \oplus L^*$ (where $deg(L) = 0$) with nontrivial extensions $0 \to L \to V \to L^* \to 0$. An extension $0 \to L \to V \to L^* \to 0$ (where L is allowed to have arbitrary degree) is given by a holomorphic structure of the form

$$\bar\partial = \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix},$$

where $\gamma \in \Gamma(M, \bar K\,\mathrm{Hom}(L^*, L))$. It is called non-trivial if the holomorphic structure is not isomorphic to the holomorphic direct sum $L \oplus L^*$. This is measured by the extension class $[\gamma] \in H^1(M, \mathrm{Hom}(L^*, L))$. Note that the isomorphism class of the holomorphic bundle V given by an extension $0 \to L \to V \to L^* \to 0$ with extension class $[\gamma]$ is already determined by L and $\mathbb{C}[\gamma] \in \mathbb{P}H^1(M, \mathrm{Hom}(L^*, L))$.

Proposition 2.1. *Let* $\mathcal{M} \subset \mathbb{P}^3 = \mathbb{P}H^0(Jac(M), L(2\Theta))$ *be the space of S-equivalence classes of semi-stable Lawson symmetric holomorphic structures over the Lawson surface* M*. Then the connected component* $\mathcal{S}$ *of* $\mathcal{M}$ *containing the trivial holomorphic structure* $(\underline{\mathbb{C}}^2, d'')$ *is given by a projective line in* $\mathbb{P}^3$.

Proof. The fix point set of any of these three symmetries is given by the union of projective subspaces of $\mathbb{P}^3$. Clearly, the common fix point set contains a projective subspace of dimension ≥ 1, as $\lambda \mapsto \bar\partial^\lambda$ is a non-constant holomorphic map into this space.

The space of S-equivalence classes of semi-stable non-stable bundles is the Kummer surface of M in $\mathbb{P}^3$. It has degree 4, and 16 double points. These double points are given by extensions of self-dual line bundles L by itself. In order to see that $\mathcal{S}$ is a projective line it is enough to show that the only strictly semi-stable bundles V, whose isomorphism classes are invariant under φ_2, φ_3 and τ, are the trivial rank two bundle $\underline{\mathbb{C}}^2$ (which is a double point in the Kummer surface) and the direct sum bundles

$$L(P_1 - P_2) \oplus L(P_2 - P_1), \quad L(P_1 - P_4) \oplus L(P_4 - P_1),$$

where $P_1, .., P_4 \in M$ are the zeros of the Hopf differential of M. So let L be a holomorphic line sub-bundle of V of degree 0. Because M has genus 2 there exists two points $P, Q \in M$ such that L is given as the line bundle $L(P-Q)$ associated to the divisor $P-Q$. If $P = Q$ then V is in the S-equivalence class of $\underline{\mathbb{C}}^2$. If $P \neq Q$ then $\varphi_2^* L(P-Q)$ is either isomorphic to $L(P-Q)$ or $L(Q-P)$, as $\varphi_2^* V$ and V are S-equivalent. Clearly, the same holds for τ, and as φ_3 is of order 3 we even get that $\varphi_3^* L(P-Q) = L(P-Q)$. From these observations we deduce that the points P and Q are fixed points of φ_3, and as a consequence V is S-equivalent to one of the above mentioned direct sum bundles. □

The next proposition shows that we do not need to care about S-equivalence of holomorphic bundles.

Proposition 2.2. *Every Lawson symmetric strictly semi-stable holomorphic rank 2 bundle $V \to M$ is isomorphic to the direct sum of two holomorphic line bundles.*

Proof. As we have seen in the proof of the previous theorem V is S-equivalent to one of the holomorphic rank 2 bundles $\underline{\mathbb{C}}^2$, $L(P_1 - P_2) \oplus L(P_2 - P_1)$ and $L(P_1 - P_4) \oplus L(P_4 - P_1)$. As

$$\varphi_2^* L(P_i - P_j) = L(P_j - P_i) \neq L(P_i - P_j)$$

for $i \neq j$ we see that V cannot be a non-trivial extension of $L(P_i - P_j)$ by its dual $L(P_j - P_i)$. It remains to consider the case where V is S-equivalent to $\underline{\mathbb{C}}^2$. Then the holomorphic structure of V is given by

$$\begin{pmatrix} \bar\partial^{\mathbb{C}} & \gamma \\ 0 & \bar\partial^{\mathbb{C}} \end{pmatrix}.$$

Here $\gamma \in \Gamma(M, \bar K)$ and the projective line of its cohomology class in $H^1(M, \mathbb{C})$ is an invariant of the isomorphism class of V. This projective line is determined by its annihilator in $H^0(M, K) = H^1(M, \mathbb{C})^*$. The annihilator of $[\gamma]$ is $H^0(M, K)$ exactly in the case where V is (isomorphic to) the holomorphic direct sum $\mathbb{C}^2 \to M$, and otherwise it is a line in $H^0(M, K)$. Since V is isomorphic to $\varphi_2^* V$, $\varphi_3^* V$ and $\tau^* V$ this line would be invariant under φ_2, φ_3 and τ which leads to a contradiction. □

2.1. **Non semi-stable holomorphic structures.** It was shown in [He1] that for a generic $\lambda \in \mathbb{C}^*$ the holomorphic structure $\bar\partial^\lambda$ is stable. Nevertheless there can exist special $\lambda \in \mathbb{C}^*$ such that $\bar\partial^\lambda$ is neither stable nor semi-stable. We now study which non semi-stable holomorphic structures admit Lawson symmetric flat connections.

Let ∇ be a flat, Lawson symmetric $\mathrm{SL}(2, \mathbb{C})$-connection on a complex rank 2 bundle over M such that $\nabla'' = \bar\partial$ is not semi-stable. By assumption, there exists a holomorphic line subbundle L of $(V, \bar\partial)$ of degree ≥ 1. The second fundamental form

$$\beta = \pi^{V/L} \circ \nabla_{|L} \in \Gamma(KHom(L, V/L)) = \Gamma(KL^{-2})$$

of L with respect to ∇ is holomorphic by flatness of ∇. As $deg(L) \geq 1$, L cannot be a parallel subbundle because in that case it would inherit a flat connection. This implies

$\beta \neq 0$. Therefore $L^{-2} = K^{-1}$ which means that L is a spin bundle of M. The only spin bundle S of M which is isomorphic to $\varphi_2^* S$, $\varphi_3^* S$ and $\tau^* S$ is given by

$$S = L(Q_1 + Q_3 - Q_5),$$

see [He]. As there exists a flat connection with underlying holomorphic structure $\bar\partial$, the bundle $(V, \bar\partial)$ cannot be isomorphic to the holomorphic direct sum $S \oplus S^* \to M$. Therefore it is given by a non-trivial extension $0 \to S \to V \to S^* \to 0$. As $H^1(M, S^2)$ is 1-dimensional, a non semi-stable holomorphic structure admitting a flat, Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connection is already unique up to isomorphism. A particular choice of such a flat connection ∇ is given by the uniformization connection, see [H2]: Consider the holomorphic direct sum $V = S \oplus S^* \to M$, where S is the spin bundle mentioned above. On M there exists a unique Riemannian metric of constant curvature -4 in the conformal class of the Riemann surface M. This Riemannian metric induces spin connections and unitary metrics on S and S^*. Let $\Phi = 1 \in H^0(M, K\,\mathrm{Hom}(S, S^*))$ and $\Phi^* = \mathrm{vol}$ be its dual with respect to the metric. Then

$$\nabla^u = \nabla = \begin{pmatrix} \nabla^{spin} & \mathrm{vol} \\ 1 & \nabla^{spin^*} \end{pmatrix}, \tag{2.2}$$

is flat. Moreover, it is also Lawson symmetric. This can easily be deduced from the uniqueness of the conformal Riemannian metric of constant curvature -4. The holomorphic structure ∇'' is clearly given by the non-trivial extension $0 \to S \to V \to S^* \to 0$.

Proposition 2.3. *Every flat, Lawson symmetric connection ∇ on M, whose underlying holomorphic structure ∇'' is not semi-stable, is gauge equivalent to*

$$\begin{pmatrix} \nabla^{spin} & C\,Q + \mathrm{vol} \\ 1 & \nabla^{spin^*} \end{pmatrix},$$

where ∇^{spin} and vol *are the spin connection and the volume form of the conformal metric of constant curvature -4 on M, $C \in \mathbb{C}$ and Q is the Hopf differential of the Lawson surface.*

Proof. Every other flat $\mathrm{SL}(2,\mathbb{C})$-connection ∇, whose underlying holomorphic structure is $\bar\partial$, is given by $\nabla = \nabla^u + \Psi$ where

$$\Psi \in H^0(M, K\,\mathrm{End}_0(V, \bar\partial))$$

is a Higgs field. An arbitrary section $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V, \bar\partial))$ is given by

$$\Psi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with respect to the decomposition $V = S \oplus S^*$ and the matrix entries are thus sections $a \in \Gamma(M, K)$, $b \in \Gamma(M, K^2)$ and $c \in \Gamma(M, \mathbb{C})$. Then

$$\bar\partial \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \bar\partial^K a + c\,\mathrm{vol} & \bar\partial^{K^2} b + 2a\,\mathrm{vol} \\ \bar\partial^{\mathbb{C}} c & -\bar\partial^K a - c\,\mathrm{vol} \end{pmatrix}.$$

This shows that $c = 0$ if Ψ is holomorphic. Moreover, for a holomorphic 1-form $\alpha \in H^0(M, K)$ the gauge

$$g := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

is holomorphic with respect to $\bar\partial$ and satisfies

$$g^{-1}\nabla^u g - \nabla^u = \begin{pmatrix} -\alpha & \partial^K \alpha \\ 0 & \alpha \end{pmatrix} \in H^0(M, K\,\mathrm{End}_0(V,\bar\partial)).$$

Therefore, we can restrict our attention to the case of Higgs fields which have the following form

$$\Psi := \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

where b is a holomorphic quadratic differential by holomorphicity of Ψ. The Hopf differential Q of the Lawson surface is (up to constant multiples) the only holomorphic quadratic differential on M which is invariant under φ_2, φ_3 and τ. From this it easily follows that if the gauge equivalence class of $\nabla^u + \Psi$ is invariant under φ_2, φ_3 and τ then b must be a constant multiple of Q. □

Remark 2.1. The orbits under the group of gauge transformations of the above mentioned non semi-stable holomorphic structure and $\bar\partial^0$ get arbitrarily close to each other: Consider the families of holomorphic structures

$$\bar\partial_t = \begin{pmatrix} \bar\partial^S & t\,vol \\ Q^* & \bar\partial^{S^*} \end{pmatrix} \text{ and } \tilde{\bar\partial}_t = \begin{pmatrix} \bar\partial^S & vol \\ t\,Q^* & \bar\partial^{S^*} \end{pmatrix}$$

on $V = S \oplus S^*$. Clearly, $\bar\partial_t$ and $\tilde{\bar\partial}_t$ are gauge equivalent for $t \neq 0$. On the other hand $\bar\partial_0 = \bar\partial^0$ and $\tilde{\bar\partial}_0 = \bar\partial$ which are clearly not isomorphic. We will see later how to distinguish such families of isomorphism classes of holomorphic structures if they are equipped with corresponding families of gauge equivalence classes of flat connections.

3. Hitchin's abelianization

A very useful construction for the study of a moduli space of holomorphic (Higgs) bundles is given by Hitchin's integrable system [H1, H2]. We do not describe this integrable system in detail but apply some of the methods in order to construct the moduli space $\mathcal{S}$, which was studied in the previous chapter, explicitly. The main idea is the following: A holomorphic structure of rank 2 equipped with a Higgs field is already determined by the eigenlines of the Higgs field (which are in general only well-defined on a double covering of the Riemann surface). In fact, the rank 2 bundle is the push forward of the dual of an eigenline bundle. In our situation, appropriate Higgs fields of a Lawson symmetric holomorphic structure are basically unique up to a multiplicative constant by Lemma 3.1 and its proof. In general the two eigenlines are given by points in a Prym variety which are dual to each other. This Prym variety turns out to be the Jacobian of a 1-dimensional square torus in the case of Lawson symmetric holomorphic structures with symmetric Higgs fields, see Lemma 3.2 and 3.3. Moreover, this Jacobian double covers the moduli space $\mathcal{S}$ in a natural way (Proposition 3.1).

Lemma 3.1. *Let $\bar\partial$ be a Lawson symmetric, semi-stable holomorphic structure on a rank 2 bundle over M which is not isomorphic to $\bar\partial^0$. Then there exists a Higgs field $\Psi \in H^0(M, K\,\mathrm{End}_0(V,\bar\partial))$ with*

$$\det\Psi = Q \in H^0(M, K^2)$$

which satisfies $\varphi^*\Psi = g^{-1}\Psi g$ for every Lawson symmetry φ, where g is the isomorphism between the holomorphic structures $\bar\partial$ and $\varphi^*\bar\partial$. This Higgs fields is unique up to sign.

Proof. By Proposition 2.2 every Lawson symmetric, semi-stable and non-stable holomorphic structure is the holomorphic direct sum of two line bundles. For these bundles, it is easy to construct a Higgs field Ψ with $\det\Psi = Q$. Moreover, this Higgs field Ψ can be constructed such that its pull-back $\varphi^*\Psi$ for a Lawson symmetry φ is conjugated to Ψ.

All stable holomorphic structures give rise to smooth points in the moduli space of holomorphic structures. Let $[\mu] \in H^1(M, \mathrm{End}_0(V))$ be a non-zero tangent vector of the isomorphism class of the stable holomorphic structure $\bar\partial$ in $\mathcal{S}$. By the non-abelian Hodge theory (see for example [AB]) and the Theorem of Narasimhan-Seshadri, the class $[\mu]$ can be represented by a endomorphism-valued complex anti-linear 1-form $\mu \in \Gamma(M, \bar K\,\mathrm{End}_0(V))$ which is parallel with respect to the (unique) unitary flat connection ∇ with $\nabla'' = \bar\partial$. Let φ be one of the symmetries φ_2, φ_3 or τ and g be a gauge, i.e., a smooth isomorphism, of V such that $\varphi^*\,\bar\partial = g^{-1}\,\bar\partial\, g$. As $\bar\partial$ is stable g is unique up to multiplication with a constant multiple of the identity. We claim that $\varphi^*\mu = g^{-1}\mu g$. To see this note that $g^{-1}\mu g$ represents (with respect to $g^{-1}\,\bar\partial\, g$) the same tangent vector in $T_{[\bar\partial]}\mathcal{S}$ as μ (with respect to $\bar\partial$) and as $\varphi^*\mu$ (with respect to $\varphi^*\,\bar\partial = g^{-1}\,\bar\partial\, g$). Therefore $[g^{-1}\mu g - \varphi^*\mu] = 0 \in T_{[\bar\partial]}\mathcal{S}$, and by non-abelian Hodge theory $g^{-1}\mu g - \varphi^*\mu$ is in the image of $g^{-1}\nabla g$. Moreover the unitary flat connections $g^{-1}\nabla g$ and $\varphi^*\nabla$ coincide by the uniqueness of Narasimhan and Seshadri Theorem. Hence the difference $g^{-1}\mu g - \varphi^*\mu$ is parallel. This is only possible if $g^{-1}\mu g - \varphi^*\mu = 0$ as claimed.

Consider the (non-zero) adjoint $\Psi = \mu^* \in H^0(M, K\,\mathrm{End}_0(V))$ which clearly satisfies $\varphi^*\Psi = g^{-1}\Psi g$ for φ and g as above. Therefore the holomorphic quadratic differential $\det\Psi \in H^0(M; K^2)$ is invariant under φ_2, φ_3 and τ. If $\det\Psi \neq 0$ this implies that it is a constant non-zero multiple of the Hopf differential of the Lawson surface. If $\det\Psi = 0$ consider the holomorphic line bundle $L = \ker\Psi \subset V$. As Ψ is trace-free it defines $0 \neq \tilde\Psi \in H^0(M, K\,\mathrm{Hom}(V/L, L)) = H^0(M, KL^2)$. Because $\deg(L) \leq -1$ as $\bar\partial$ is stable, L must be dual to a spin bundle, and because $\varphi^*\Psi = g^{-1}\Psi g$ it is even the dual of the holomorphic spin bundle $S = L(Q_1 + Q_3 - Q_5)$. This easily implies that $\bar\partial$ is isomorphic to $\bar\partial^0$ in the case of $\det\Psi = 0$. □

Definition. *The Higgs fields of Lemma 3.1 are called symmetric Higgs fields.*

3.1. **The eigenlines of symmetric Higgs fields.** The zeros of the Hopf differential Q of the Lawson surface M are simple. As a Higgs field is trace free by definition, the eigenlines of a symmetric Higgs field Ψ (for a Lawson symmetric holomorphic structure $\bar\partial$) with $\det\Psi = Q$ are not well-defined on the Riemann surface M. Following Hitchin [H1] we define a (branched) double covering of M on which the square root of Q is well-defined:

$$\pi\colon \tilde M := \{\omega_x \in K_x | x \in M, \omega_x^2 = Q_x\} \to M.$$

We denote the involution $\omega_x \mapsto -\omega_X$ by $\sigma\colon \tilde M \to \tilde M$. There exists a tautological section

$$\omega \in H^0(\tilde M, \pi^* K_M)$$

satisfying

$$\omega^2 = \pi^* Q \text{ and } \sigma^*\omega = -\omega.$$

As the Hopf differential is invariant under φ_2, φ_3 and τ these symmetries of M lift to symmetries of $\tilde M$ denoted by the same symbols. The tautological section is invariant under these symmetries

$$\varphi_2^*\omega = \omega,\ \varphi_3^*\omega = \omega,\ \tau^*\omega = \omega,$$

where we have naturally identified $\varphi_2^*\pi^*K_M = \pi^*\varphi_2^*K_M = \pi^*K_M$ and analogous for φ_3 and τ. On $\tilde{M}$ the eigenlines of $\pi^*\Psi$ are well-defined:

$$L_\pm := \ker \pi^*\Psi \mp \omega \,\mathrm{Id}\,.$$

Clearly, $\sigma^*L_\pm = L_\mp$. As the zeros of $Q = \det \Psi$ are simple, Ψ has a one-dimensional kernel at these zeros. Therefore, the eigenline bundles $L_\pm$ intersect each other of order 1 in π^*V at the branch points of π. Otherwise said, there is a holomorphic section

$$\wedge \in H^0(\tilde{M}, \mathrm{Hom}(L_+ \otimes L_-, \Lambda^2\pi^*V))$$

which has zeros of order 1 at the branch points of π. Thus, $\wedge$ can be considered as a constant multiple of $\omega \in H^0(\tilde{M}, \pi^*K_M)$ which has also simple zeros exactly at the branch points of π by construction. Because Λ^2V is the trivial holomorphic line bundle, the eigenline bundles satisfy

$$L_+ \otimes L_- = L_+ \otimes \sigma(L_+) = \pi^*K_M^*, \tag{3.1}$$

which means that $L_\pm$ lie in an affine Prym variety for π. Recall that the Prym variety of $\pi\colon \tilde{M} \to M$ is by definition

$$Prym(\pi) = \{L \in Jac(\tilde{M}) \mid \sigma^*L = L^*\}.$$

After fixing the line bundle $L = \pi^*S^*$, which clearly satisfies (3.1), every other line bundle L^+ satisfying (3.1) is given by $L^+ = \pi^*S^* \otimes E$ for some holomorphic line bundle $E \in Prym(\pi)$.

3.2. **Reconstruction of holomorphic rank 2 bundles.** We shortly describe how to reconstruct the bundle V from an eigenline bundle $L_+ \to \tilde{M}$ of a symmetric Higgs field $\Psi \in H^0(M, K\,\mathrm{End}_0(V))$ with non-vanishing determinant $\det \Psi \neq 0$. This construction will be used later to study Lawson symmetric holomorphic connections on $\tilde{M}$. First consider an open subset $U \subset M$ which does not contain a branch value of π. The preimage $\pi^{-1}(U) \subset \tilde{M}$ consists of two disjoint copies $U_1 \cup U_2 \subset \tilde{M}$ of U. Because

$$\pi^*V_{|U_i} = (L_+ \oplus L_-)_{|U_i}$$

and $\sigma(L_\pm) = L_\mp$, we obtain a basis of holomorphic sections of V over U which is given by the non-vanishing sections

$$s_1 \in H^0(U_1, L_+) \;\text{ and }\; s_2 \in H^0(U_1, L_-) \cong H^0(U_2, L_+).$$

This local basis of holomorphic sections in V is special linear if and only if

$$\wedge(s_1 \otimes s_2) = 1 \in H^0(U_1, \pi^*K_M \otimes L_+ \otimes L_-) = H^0(U_1, \mathbb{C})$$

in U_1.

Next we consider the case of a branch point p of π: Let $z\colon U \subset \tilde{M} \to \mathbb{C}$ be a local coordinate centered at p such that $\sigma(z) = -z$ and $\sigma(U) = U$. A local coordinate on $\pi(U)$ around $\pi(p) \in M$ is given by y with $y = z^2$. We may choose z in such a way that

$$\wedge = zdy + \textit{higher order terms} \in H^0(U, \pi^*K_M),$$

where $dy \in H^0(U, \pi^*K_M)$ is the pull-back as a section and not as a 1-form. Let $t_1 \in H^0(U, L_+), t_2 = \sigma(t_1) \in H^0(U, L_-)$ be holomorphic sections without zeros such that

$$\wedge(t_1 \otimes t_2) = z \in H^0(U, \mathbb{C}).$$

Then there are local holomorphic basis fields s_1, s_2 of $V \to M$ with $s_1 \wedge s_2 = 1$ such that $\pi^* s_1(p) = t_1(p) = t_2(p)$ and

$$t_1 = \pi^* s_1 - \frac{z}{2}\pi^* s_2,\ t_2 = \pi^* s_1 + \frac{z}{2}\pi^* s_2 \tag{3.2}$$

in $\pi^* V$, or equivalently

$$\pi^* s_1 = \frac{1}{2}t_1 + \frac{1}{2}t_2,\ \pi^* s_2 = \frac{1}{z}t_2 - \frac{1}{z}t_1.$$

As the last equation is invariant under σ this gives us a well-defined special linear holomorphic frame $\pi^* s_1, \pi^* s_2$ of V over $\pi(U) \subset M$.

By going through the above construction carefully without a priori knowing the existence of a rank 2 bundle one can construct a holomorphic rank 2 bundle $V \to M$ for any line bundle L in the affine Prym variety. Then one can show that this rank 2 bundle has trivial determinant and that there exists a Higgs field on V whose determinant is Q. See for example [H1] for details on this.

Remark 3.1. The above reconstruction is the differential geometric formulation of the sheaf theoretic push-forward construction $\pi_* L_\pm^*$.

Remark 3.2. Because of Lemma 3.1 a generic Lawson symmetric stable bundle $V \to M$ corresponds via the above construction to exactly two different line bundles L_+ and $L_- = \sigma(L_+)$.

3.3. **The torus parametrizing holomorphic structures.** The Prym variety of the double covering $\pi\colon \tilde{M} \to M$ is complex 3-dimensional and the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic structures is only 1-dimensional. We now determine which line bundles L_+ in the affine Prym variety correspond to Lawson symmetric holomorphic structures.

Let $\bar\partial$ be a Lawson symmetric holomorphic structure which admits a symmetric Higgs field Ψ whose determinant is the Hopf differential Q of the Lawson surface. By the definition of symmetric Higgs fields the eigenlines $L_\pm$ of Ψ are isomorphic to $\varphi_2^* L_\pm$, $\varphi_3^* L_\pm$ and $\tau^* L_\pm$. Recall that the same is true for our base point $\pi^* S^*$ in the affine Prym variety. Therefore, it remains to determine the connected component of those holomorphic line bundles $\tilde{E}$ of degree 0 on $\tilde{M}$ whose isomorphism class is invariant under φ_2, φ_3 and τ. The quotient

$$\tilde{\pi}\colon \tilde{M} \to \tilde{M}/\mathbb{Z}_3$$

of the $\mathbb{Z}_3$-action induced by φ_3 is a square torus. Moreover, φ_2 and τ induce fix point free holomorphic involutions on $\tilde{M}/\mathbb{Z}_3$ (denoted by the same symbols). They are given by translations. Therefore, the pull-back $\tilde{E} = \tilde{\pi}^* E$ of every line bundle $E \in Jac(\tilde{M}/\mathbb{Z}_3)$ is invariant under φ_2, φ_3 and τ.

In general one has to distinguish between those bundles which are pull-backs of bundles on the quotient of some automorphism on a Riemann surface and bundles whose isomorphism class is invariant under the automorphism. In our situation they turn out to be the same:

Lemma 3.2. *Let $\tilde{E}$ be a holomorphic line bundle of degree 0 on $\tilde{M}$. If its isomorphism class is invariant under φ_2, φ_3 and τ then $\tilde{E}$ is isomorphic to the pull-back $\tilde{\pi}^* E$ for some $E \in Jac(\tilde{M}/\mathbb{Z}_3)$.*

Proof. We only sketch the proof of the lemma: Consider the corresponding flat unitary connection ∇ on $\tilde{E}$. As the isomorphism class of $\tilde{E}$ is invariant under φ_2, φ_3 and τ the gauge equivalence class of ∇ is also invariant under φ_2, φ_3 and τ. This gauge equivalence class is determined by its (abelian) monodromy representation

$$\pi_1(\tilde{M}) \to U(1) = S^1 \subset \mathbb{C}.$$

Using the symmetries φ_2, φ_3 and τ one can easily deduce that the connection is (gauge equivalent) to the pull-back of a flat connection on the torus $\tilde{M}/\mathbb{Z}_3$. $\square$

Lemma 3.3. *The connected component of the space of $\mathbb{Z}_3-$invariant line bundles in the Prym variety of $\pi\colon \tilde{M} \to M$ containing the trivial holomorphic line bundle is given by the (pull-back of the) Jacobian of the torus $\tilde{M}/\mathbb{Z}_3$.*

Proof. Any line bundle on the torus is given by $E = L(x - p)$, where x is a suitable point on the torus and p is the image of the branch point $P_1 \in \tilde{M}$. The involution σ descends to an involution on $\tilde{M}/\mathbb{Z}_3$ with four fix points which are exactly the images of the branch points of $\tilde{\pi}$. Therefore, the quotient of $\tilde{M}/\mathbb{Z}_3$ by σ is the projective line $\mathbb{P}^1$ and

$$E \otimes \sigma^* E = L(x - p + \sigma(x) - p) = \underline{\mathbb{C}}$$

which implies $\tilde{\pi}^* E \otimes \sigma^*(\tilde{\pi}^* E) = \tilde{\pi}^*(E \otimes \sigma^* E) = \underline{\mathbb{C}}$. $\square$

These two lemmas enable us to define a double covering $\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1$: Take a line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ and consider

$$L_+ := \pi^* S^* \otimes \tilde{\pi}^* L \to \tilde{M}.$$

The isomorphism class of this line bundle is invariant under φ_2, φ_3 and τ and it satisfies

$$L_+ \otimes \sigma(L_+) = \pi^* K_M$$

by Lemma 3.3. As we have seen in Section 3.2, L_+ is an eigenline bundle of a symmetric Higgs field of the pullback $\pi^* V \to \tilde{M}$ of a holomorphic rank two bundle $V \to M$ with trivial determinant.

Proposition 3.1. *There exists an even holomorphic map*

$$\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1 \tag{3.3}$$

of degree 2 to the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic bundles. This map is determined by $\Pi(L) = [\bar{\partial}]$ for $L \neq \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ such that $\pi^* S^* \otimes \tilde{\pi}^* L$ is isomorphic to an eigenline bundle of a symmetric Higgs field of the Lawson symmetric holomorphic rank two bundle $(V, \bar{\partial})$, and by $\Pi(\underline{\mathbb{C}}) = [\bar{\partial}^0] \in \mathcal{S}$ (see Lemma 3.1). The branch points are the spin bundles of $\tilde{M}/\mathbb{Z}_3$ and the branch images of the non-trivial spin bundles are exactly the isomorphism classes of the semi-stable non-stable holomorphic bundles.

Proof. First we show that for $L \neq \underline{\mathbb{C}}$, the corresponding rank two bundle is semi-stable: Assume that E is a holomorphic line subbundle of a Lawson symmetric holomorphic bundle $V \to M$ of degree greater than 0. If E is not a spin bundle of the genus 2 surface M the rank two bundle V would be isomorphic to the holomorphic direct sum $E \oplus E^*$. In this case one easily sees that there do not exists a Higgs field whose determinant has simple zeros. If E is a spin bundle it must be isomorphic to the spin bundle S of the Lawson immersion because of the symmetries. Let the rank two holomorphic structure be given by

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}^S & \alpha \\ 0 & \bar{\partial}^{S^*} \end{pmatrix}$$

on the topological direct sum $V = S \oplus S^*$ for some $\alpha \in \Gamma(M, \bar{K}K)$. The eigenline bundle $\pi^*S^* \otimes \tilde{\pi}^*L \subset V$ would be given by a map

$$\begin{pmatrix} a \\ b \end{pmatrix} : \pi^*S^* \otimes \tilde{\pi}^*L \to \pi^*S \oplus \pi^*S^*$$

satisfying $\bar{\partial}\, a + \alpha b = 0$ and $\bar{\partial}\, b = 0$. For $L \neq \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ there does not exists a holomorphic map from $\pi^*S^* \otimes \tilde{\pi}^*L$ to π^*S^*. Therefore the eigenline bundle $\pi^*S^* \otimes \tilde{\pi}^*L$ would be π^*S, which is impossible because of the degree. Moreover one easily sees that the corresponding holomorphic rank two bundle for $L = \underline{\mathbb{C}}$ must be isomorphic to the holomorphic direct sum $V = \pi^*S \oplus \pi^*S^*$. The orbit of this holomorphic structure under the gauge group is infinitesimal near to the one of the holomorphic structure $\bar{\partial}^0$ of Lemma 3.1. Therefore we can map $\underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ to the equivalence class of the stable holomorphic structure $\bar{\partial}^0$ in $\mathcal{S} = \mathbb{P}^1$ in order to obtain a well-defined holomorphic map $\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S}$.

Because of Lemma 3.1 and remark 3.2 the degree of the map Π is 2. Clearly $\Pi(L) = \Pi(L^*)$ for all $L \in Jac(\tilde{M}/\mathbb{Z}_3)$. Therefore the spin bundles of $\tilde{M}/\mathbb{Z}_3$ are the only branch points of Π. It remains to show that the non-trivial spin bundles in $Jac(\tilde{M}/\mathbb{Z}_3)$ correspond to the strictly semi-stable bundles $V \to M$. This can either be seen by analogous methods as in [H1] used for the computation of the unstable locus in the Prym variety, or more directly as follows: Consider for example the non stable semi-stable bundle $V = \underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$. Then, a symmetric Higgs field is given by

$$\Psi = \begin{pmatrix} 0 & \omega_1 \\ \omega_2 & 0 \end{pmatrix},$$

where ω_1 and ω_2 are holomorphic differentials with simple zeros at P_1 and P_3 respectively P_2 and P_4 such that $Q = \omega_1\omega_2$. Then the eigenlines $\ker(\Psi \pm \alpha)$ are both isomorphic to $L(-P_1 - P_3) = \pi^*S^* \otimes L(3P_1 - 3P_3)$. Clearly, $L(3P_1 - 3P_3) = \tilde{\pi}^*L(\tilde{\pi}(P_3) - \tilde{\pi}(P_1))$, and $L(\tilde{\pi}(P_3) - \tilde{\pi}(P_1))$ is a non-trivial spin bundle of $\tilde{M}/\mathbb{Z}_3$. Therefore, the gauge orbit of the trivial holomorphic rank 2 bundle $\underline{\mathbb{C}}^2 \to M$ is a branch image of Π, and similarly one can show that the same is true for the remaining two semi-stable non-stable holomorphic structures. □

Remark 3.3. This double covering of the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic rank two bundles is very similar to the one of the moduli space of holomorphic rank two bundles with trivial determinant on a Riemann surface Σ of genus 1. The later space consist of all bundles of the form $L \oplus L^*$ where $L \in Jac(\Sigma)$ together with the non-trivial extensions of the spin bundles of Σ with itself, see [A].

4. Flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connections

We use the results of the previous chapter to study the moduli space of flat Lawson symmetric connections on M as an affine bundle over the moduli space of Lawson symmetric holomorphic structures. A similar approach was used by Donagi and Pantev [DP] in their study of the geometric Langlands correspondence.

The underlying holomorphic structure ∇'' of a flat Lawson symmetric connection ∇ is determined by a holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ (Proposition 3.1). Conversely, for all non-trivial holomorphic line bundles $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ there exists a Lawson symmetric holomorphic structure which is semi-stable. Because of the Theorem of Narasimhan and Seshadri [NS], these holomorphic structures admit flat unitary connections, and, because of the uniqueness part in [NS], the gauge equivalence class of the flat unitary connection is also invariant under φ_2, φ_3 and τ. In order to obtain all flat Lawson symmetric connections we only need to add symmetric Higgs fields to the unitary connections. We will see in Theorem 1 that flat Lawson symmetric connections on M are uniquely and explicitly determined by a flat connection on the corresponding line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ as long as L is not isomorphic the trivial holomorphic bundle $\underline{\mathbb{C}}$. Adding a symmetric Higgs field on the Lawson symmetric connection on M is equivalent to adding a holomorphic 1-form to the line bundle connection on $L \to \tilde{M}/\mathbb{Z}_3$. Therefore the affine bundle structure of the space of gauge equivalence classes of flat Lawson symmetric connections on M is determined by the affine bundle structure of the moduli space of flat line bundle connections over the Jacobian of the torus $\tilde{M}/\mathbb{Z}_3$. The case of the remaining flat Lawson symmetric connections (corresponding to the holomorphic structures which are either isomorphic to $\bar{\partial}^0$ or to the non-trivial extension $0 \to S \to V \to S^* \to 0$) is dealt with in the next chapter. We will see that they occur as special limits as L converges to the trivial holomorphic line bundle.

Let ∇ be a flat Lawson symmetric connection such that its underlying holomorphic structure ∇'' admits a symmetric Higgs field $\Psi \in H^0(K, \mathrm{End}_0(V))$ with $\det \Psi = Q$. Equivalently, there is a non-trivial holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ with $\Pi(L) = [\nabla'']$. Consider the pull-back connection $\pi^*\nabla$ on $\pi^*V \to \tilde{M}$, where $\pi\colon \tilde{M} \to M$ is as in the previous chapter. As the eigenline bundles $L_\pm \to \tilde{M}$ of $\pi^*\Psi$ are holomorphic subbundles of π^*V, which only intersect at the branch points of π, there exists a holomorphic homomorphism

$$f\colon L_+ \oplus L_- \to \pi^*V$$

which is an isomorphism away from the branch points of π. Therefore there exists a unique meromorphic flat connection $\tilde{\nabla}$ on $L_+ \oplus L_- \to \tilde{M}$ such that f is parallel. The poles of $\tilde{\nabla}$ are at the branch points of π. Let z be a holomorphic coordinate on $\tilde{M}$ centered at a branch point P_i of π such that $\sigma(z) = -z$. Let s_1, s_2 be a special linear frame of V and let t_1 and $t_2 = \sigma(t_1)$ be local holomorphic sections in L_+ and L_- satisfying (3.2). The connection ∇ on $V \to M$ is determined locally by

$$\nabla s_j = \omega_{1,j} s_1 + \omega_{2,j} s_2$$

for $j = 1, 2$, where $\omega_{i,j}$ are the locally defined holomorphic 1−forms. As ∇ and the frame are special linear $\omega_{1,1} = -\omega_{2,2}$ holds. Because π has a branch point at P_i, the connection 1-forms $\pi^*\omega_{i,j}$ (of π^∇ with respect to π^*s_1, π^*s_2) have zeros at P_i. Using (3.2) one can compute the connection 1−forms of $\tilde{\nabla}$ with respect to the frame $t_1, t_2 = \sigma(t_1)$ of $L_+ \oplus L_-$. It turns out that they have first order poles at P_i. Moreover, the residue of $\tilde{\nabla}$ at P_i is given by

$$res_{P_i}\tilde{\nabla} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4.1}$$

with respect to the frame t_1, t_2. We need to interpret this formula more invariantly. With respect to the direct sum decomposition $L_+ \oplus L_-$ the connection $\tilde{\nabla}$ splits

$$\tilde{\nabla} = \begin{pmatrix} \nabla^+ & \beta^- \\ \beta^+ & \nabla^- \end{pmatrix}. \tag{4.2}$$

Here, $\nabla^\pm$ are meromorphic connections on $L_\pm$ with simple poles at the branch points of π, and $\beta^\pm \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} \operatorname{Hom}(L_\pm, L_\mp))$ are the meromorphic second fundamental forms of

$L_\pm$ which also have simple poles at the branch points. Recall that the eigenline bundles are given by

$$L_\pm = \pi^* S^* \otimes \tilde{\pi}^* L^{\pm 1} \tag{4.3}$$

for holomorphic line bundles $L^{\pm 1} \in Jac(\tilde{M}/\mathbb{Z}_3)$ and $\tilde{\pi}\colon \tilde{M} \to \tilde{M}/\mathbb{Z}^3$. Consider the holomorphic section $\wedge \in H^0(\tilde{M}, \pi^* K_M)$ which has simple zeros at the branch points of π. There exists an unique meromorphic connection ∇^{K_M} on $\pi^* K_M$ such that $\wedge$ is parallel. Then $res_{P_i} \nabla^{K_M} = -1$ at the branch points $P_1, .., P_4$. As $\pi^* S^2 = \pi^* K_M$ there exists a unique meromorphic connection ∇^{S^*} on $\pi^* S^*$ which has simple poles at the branch points of π with residue $\frac{1}{2}$. Using (4.3), the description of ∇^{S^*} and (4.1) we obtain holomorphic connections $\tilde{\nabla}^\pm$ on $\tilde{\pi}^* L^+$ and $\tilde{\pi}^* L^{-1}$ satisfying the formula

$$\nabla^\pm = \nabla^{S^*} \otimes \tilde{\nabla}^\pm.$$

Moreover, $\tilde{\nabla}^\pm$ are dual to each other. As in the proof of Lemma 3.2 one can show that $\tilde{\nabla}^\pm$ are invariant under φ_2, φ_3 and τ and that there exists holomorphic connections $\nabla^{L^\pm}$ on $L^\pm \to \tilde{M}/\mathbb{Z}^3$ such that

$$\tilde{\nabla}^\pm = \tilde{\pi}^* \nabla^{L^\pm}.$$

Then, all holomorphic connections on $L = L^+ \to \tilde{M}/\mathbb{Z}^3$ with its fixed holomorphic structure are given by $\nabla^{L^+} + \alpha$ for a holomorphic $1-$form $\alpha \in H^0(\tilde{M}/\mathbb{Z}^3, K_{\tilde{M}/\mathbb{Z}^3})$. Clearly, the corresponding effect on the connection ∇ on $V \to M$ is given by the addition of (a multiple of) the symmetric Higgs field Ψ which diagonalizes on $\tilde{M}$ with eigenlines $L_\pm$.

4.1. **The second fundamental forms.** Next, we compute the second fundamental forms $\beta^\pm \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} \operatorname{Hom}(L_\pm, L_\mp))$ of the eigenlines of the symmetric Higgs field. We fix some notations first: The symmetries φ_2 and τ of $\tilde{M}$ yield fix point free symmetries on the torus $\tilde{M}/\mathbb{Z}_3$ denoted by the same symbols. The quotient by these actions is again a square torus, denoted by T^2, which is fourfold covered by $\tilde{M}/\mathbb{Z}_3$ and the corresponding map is denoted by

$$\pi^T\colon \tilde{M}/\mathbb{Z}_3 \to T^2.$$

Each $L \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the pull-back of a line bundle $\hat{L}$ of degree 0 on T^2. This line bundle is not unique. Actually, the pullback map defines a fourfold covering

$$Jac(T^2) \to Jac(\tilde{M}/\mathbb{Z}_3).$$

In particular, there are four different line bundles on T^2 which pull-back to the trivial one on $\tilde{M}/\mathbb{Z}_3$. These are exactly the spin bundles on T^2, so their square is the trivial holomorphic bundle. This implies, that $\hat{L}^{\pm 2}$ is independent of a choice $\hat{L} \in Jac(T^2)$ which pulls back to a given $L \in Jac(\tilde{M}/\mathbb{Z}_3)$. Let $0 \in T^2$ be the (common) image of the branch points P_i of π. Then every holomorphic line bundle $E \to T^2$ of degree 0 is (isomorphic to) the line bundle $L(y - 0)$ associated to an divisor of the form $D = y - 0$ for some $y \in T^2$. In particular, for $y \neq 0$ there exists a meromorphic section $s_{y-0} \in \mathcal{M}(T^2, E)$ with divisor $(s_{y-0}) = D$. Moreover, y is uniquely determined by E and s_{y-0} is unique up to a multiplicative constant.

Proposition 4.1. *Let ∇ be a flat Lawson symmetric connection on M. Let $L_+ = L \in Jac(\tilde{M}/\mathbb{Z}_3)$ be a non-trivial holomorphic line bundle which is given by $L = (\pi^T)^* L(x - 0)$ for some $x \in T^2$ such that* $\Pi(L) = [\nabla'']$. *Then the point $y := -2x \in T^2$ is not* 0 *and the second fundamental form of L_+ is*

$$\beta^+ = \tilde{\pi}^* (\pi^T)^* s_{y-0} \in \mathcal{M}(\tilde{M}, K_{\tilde{M}} \operatorname{Hom}(L_\pm, L_\mp)) = \mathcal{M}(\tilde{M}, K_{\tilde{M}} \tilde{\pi}^* L^{\mp 2})$$

where for

$$s_{y-0} \in \mathcal{M}(T^2, L(y-0)) = \mathcal{M}(T^2, K_{T^2}L(y-0))$$

the multiplicative constant is chosen appropriately and the pullbacks are considered as pullbacks of (bundle-valued) 1-forms. If we denote $y^- = -y = 2x \in T^2$*, then the second fundamental form* β^- *is given by* $\beta^- = \tilde{\pi}^*(\pi^T)^* s_{y^- - 0}$.

Proof. By assumption $L = (\pi^T)^* L(x-0)$ is not the trivial holomorphic line bundle. Therefore, $L(x-0)$ cannot be a spin bundle of T^2. Equivalently, $L(x-0)^{-2} = L(y-0)$ is not the trivial holomorphic line bundle which implies that $y \neq 0$.

The gauge equivalence class of the connection ∇ is invariant under the symmetries. Therefore, the set of poles and the set of zeros of the second fundamental forms $\beta^\pm$ of the eigenlines of the symmetric Higgs field are fixed under the symmetries, too. There are exactly 4 simple poles of β^+ and because $\tilde{M}$ has genus 5 and the degree of $\mathrm{Hom}(L_\pm, L_\mp)) = \tilde{\pi}^* L^{\mp 2}$ is 0 there are 12 zeros of β^+ counted with multiplicity. The only fix points of φ_3 are the branch points P_i of π and φ_2 and τ are fix point free on $\tilde{M}$. Therefore, the orbit of a zero of β^+ under the actions of φ_2, φ_3 and τ consists of exactly 12 points. This implies that the zeros of β^+ are simple. Moreover, these 12 points are mapped via $\pi^T \circ \tilde{\pi}$ to a single point $\tilde{y}$ in T^2. We claim that $\tilde{y} = y \in T^2$. To see this, we consider the (bundle-valued) meromorphic 1-form $\tilde{\pi}^*(\pi^T)^* s_{\tilde{y}-0}$ on $\tilde{M}$, which has simple poles exactly at the branch points P_i of π and simple zeros at the preimages of $\tilde{y}$. Therefore, $\tilde{\pi}^*(\pi^T)^* s_{\tilde{y}-0}$ is (up to a multiplicative constant) the second fundamental form β^+. As the bundle $L(y-0)$ is uniquely determined by L_+ we also get $\tilde{y} = y$. ☐

Remark 4.1. In the case of $y = 0 \in T^2$ there is no meromorphic section in the trivial line bundle $L(y-0) = \underline{\mathbb{C}}$ with a simple pole at 0. But $y = 0$ holds exactly for the trivial bundle $\underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$. This line bundle corresponds to the non-stable holomorphic direct sum bundle $S^* \oplus S \to M$, see the proof of Proposition 3.1. As we have seen in Section 2.1 there does not exists a holomorphic connection on $S^* \oplus S \to M$.

By now, we have determined the second fundamental forms up to a constant. It remains to determine the exact multiplicative constant of

$$\hat{\gamma}^\pm := s_{y^\pm - 0}.$$

Note that the involution σ on $\tilde{M}$ gives rise to involutions on $\tilde{M}/\mathbb{Z}_3$ and T^2, denoted by the same symbol. Then, $\sigma(\beta^\pm) = \beta^\mp$ and $\sigma(\hat{\gamma}^\pm) = \hat{\gamma}^\mp$. From Equations 4.1 and 4.2 one sees that

$$\beta^+\beta^- \in \mathcal{M}(\tilde{M}, K^2_{\tilde{M}})$$

is a well-defined meromorphic quadratic differential with double poles at the branch points $P_1, .., P_4$ and with residue

$$res_{P_i}(\beta^+\beta^-) = \frac{1}{4}.$$

As the branch order of $\tilde{\pi}$ at P_i is 2 we have

$$res_0(\hat{\gamma}^+\hat{\gamma}^-) = \frac{1}{36}. \tag{4.4}$$

Together with $\sigma(\hat{\gamma}^\pm) = \hat{\gamma}^\mp$ this completely determines $\hat{\gamma}^\pm$ and therefore also $\beta^\pm$ up to sign. Note that the sign has no invariant meaning as the sign of the off-diagonal terms of the connection can be changed by applying a diagonal gauge with entries i and $-i$.

4.2. **Explicit formulas.** We are now going to write down explicit formulas for a flat Lawson symmetric connection ∇ whose underlying holomorphic structure admits a symmetric Higgs field Ψ with $\det\Psi = Q$. To be precise, we compute the connection 1-form of $\pi^*\nabla \otimes \nabla^S$ with respect to some frame, where ∇^S is defined as above by the equation $(\nabla^S \otimes \nabla^S)\omega = 0$ for the tautological section $\omega \in H^0(\tilde{M}, \pi^* K_M)$. Then $\pi^*\nabla \otimes \nabla^S$ is a meromorphic connection on $\tilde{\pi}^* L^+ \oplus \tilde{\pi}^* L^- \to \tilde{M}$ with simple, off-diagonal poles at the branch points $P_1, .., P_4$ of π, where L^+ and L^- are holomorphic line bundles of degree 0 on the torus $\tilde{M}/\mathbb{Z}_3$ which are dual to each other and correspond to the eigenlines of the symmetric Higgs field via Proposition 3.1.

Recall that $\tilde{M}/\mathbb{Z}_3$ is a square torus, and we identify it as

$$\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z}).$$

We may assume without loss of generality that the half lattice points are exactly the images of the branch points P_i. The fourfold (unbranched) covering map π^T gets into the natural quotient map

$$\pi^T : \tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \cong T^2.$$

Let E be one choice of a holomorphic line bundle on T^2 which pulls back to $L^+ \to \tilde{M}/\mathbb{Z}_3$. As before, it is given by $E = L([x] - [0])$ for some $[x] \in T^2$, where $[0] \in \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \cong T^2$ is the common image of the points P_i.

The following lemma is of course well-known. We include it as it produces the trivializing sections which we use to write down the connection 1-form.

Lemma 4.1. *Consider the square torus $T^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ and the holomorphic line bundle $E = L([x] - [0])$ for some $x \in \mathbb{C}$. Then there exists a smooth section $\underline{1} \in \Gamma(T^2, E)$ such that the holomorphic structure $\bar{\partial}^E$ of E is given by*

$$\bar{\partial}^E \underline{1} = -\pi x d\bar{z} \underline{1}.$$

Proof. The proof is merely included to fix our notations about the Θ-function of $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, see [GH] for details. There exists an even entire function $\theta \colon \mathbb{C} \to \mathbb{C}$ which has simple zeros exactly at the lattice points $\mathbb{Z} + i\mathbb{Z}$ and which satisfies

$$\begin{aligned} &\theta(z+1) = \theta(z) \\ &\theta(z+i) = \theta(z) \exp(-2\pi i(z - \frac{1+i}{2}) + \pi). \end{aligned} \tag{4.5}$$

Then the function

$$s(z) := \frac{\theta(z-x)}{\theta(z)} \exp(\pi x(\bar{z} - z))$$

is doubly periodic and has simple poles at the lattice points $\mathbb{Z} + i\mathbb{Z}$ and simple zeros at $x+\mathbb{Z}+i\mathbb{Z}$. Moreover it satisfies $\bar{\partial} s = \pi x s$. Therefore, s can be considered as a meromorphic section with respect to the holomorphic structure $\bar{\partial} - \pi x d\bar{z}$ on $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with simple poles at $[0] \in T^2$ and simple zeros at $[x] \in T^2$. This implies that the holomorphic structure $\bar{\partial} - \pi x d\bar{z}$ is isomorphic to the holomorphic structure of $E = L([x] - [0])$. The image $\underline{1}$ of the constant function 1 under this isomorphism satisfies the required equation $\bar{\partial}^E \underline{1} = -\pi x d\bar{z} \underline{1}$. □

The second fundamental forms $\beta^{\pm} = \tilde{\pi}^* (\pi^T)^* \hat{\gamma}^{\pm}$ can be written down in terms of Θ-functions as follows: From Proposition 4.1 and the proof of Lemma 4.1 one obtains that (with respect to the smooth trivializing section $\underline{1}$ of $E = L([x]-[0])$ and its dual section $\underline{1}^* \in \Gamma(T^2, E^*)$) $\hat{\gamma}^{\pm}$ are given by

$$\begin{aligned} \hat{\gamma}^+(z)\underline{1} &= c\frac{\theta(z-y)}{\theta(z)}e^{-2\pi i y \operatorname{Im}(z)}\underline{1}^* dz \\ \hat{\gamma}^-(z)\underline{1}^* &= c\frac{\theta(z+y)}{\theta(z)}e^{2\pi i y \operatorname{Im}(z)}\underline{1} dz \end{aligned} \tag{4.6}$$

for some $c \in \mathbb{C}$, where θ is as in the proof of 4.1 and $y = -2x$. The constant $c \in \mathbb{C}$ is given by (4.4) as a choice of a square root

$$c = \frac{1}{6}\sqrt{\frac{\theta'(0)^2}{\theta(y)\theta(-y)}}, \tag{4.7}$$

where $'$ denotes the derivative with respect to z.

Remark 4.2. Note that c can be considered as a single-valued meromorphic function depending on $y \in \mathbb{C}$ with simple poles at the integer lattice points by choosing the sign of the square root at some given point $y \notin \mathbb{Z} + i\mathbb{Z}$.

Altogether, the connection $\pi^*\nabla \otimes \nabla^S$ is given on $T^2 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with respect to the frame $\underline{1}$, $\underline{1}^*$ by the connection 1-form

$$\begin{pmatrix} \pi a dz - \pi x d\bar{z} & c\frac{\theta(z+y)}{\theta(z)}e^{2\pi i y \operatorname{Im}(z)}dz \\ c\frac{\theta(z-y)}{\theta(z)}e^{-2\pi i y \operatorname{Im}(z)}dz & -\pi a dz + \pi x d\bar{z} \end{pmatrix} \tag{4.8}$$

for some $a \in \mathbb{C}$. The connection 1-form 4.8 is only meromorphic, but the corresponding connection ∇ on the rank 2 bundle over M has no singularities. Varying $a \in \mathbb{C}$ corresponds to adding a multiple of the symmetric Higgs field on the connection ∇.

Remark 4.3. In (4.8) we have written down the connection 1-forms on the torus $T^2 \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. But as the fourfold covering $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z}) \to T^2 \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is simply given by

$$z \mod 2\mathbb{Z} + 2i\mathbb{Z} \longmapsto z \mod \mathbb{Z} + i\mathbb{Z}$$

(4.8) gives also the connection 1-form for the connection $\pi^*\nabla \otimes \nabla^S$ on $\tilde{M}/\mathbb{Z}_3$ with respect to the frame $(\pi^T)^*\underline{1}$, $(\pi^T)^*\underline{1}^*$.

We summarize our discussion:

Theorem 1 (The abelianization of flat $\mathrm{SL}(2,\mathbb{C})$-connections)**.** *Let $\bar{\partial}$ be a Lawson symmetric semi-stable holomorphic structure on a rank 2 vector bundle over M. Assume that $\bar{\partial}$ is determined by the non-trivial holomorphic line bundle $L \in Jac(\tilde{M}/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar{\partial}]$. Then there is a 1:1 correspondence between holomorphic connections on $L \to \tilde{M}/\mathbb{Z}_3$ and flat Lawson symmetric connections ∇ with $\nabla'' = \bar{\partial}$. The correspondence is given explicitly by the connection 1-form (4.8).*

4.3. **Flat unitary connections.** A famous result due to Narasimhan and Seshadri ([NS]) states that for every stable holomorphic structure on a complex vector bundle over a compact Riemann surface there exists a unique flat connection which is unitary with respect to a suitable chosen metric and whose underlying holomorphic structure is the given one. From the uniqueness we observe the following: If the isomorphism class of a a stable holomorphic structure $\bar{\partial}$ is invariant under some automorphisms of the Riemann surface then the gauge equivalence class of the unitary flat connection ∇ with $\bar{\partial} = \nabla''$ is also invariant under the same automorphisms. We apply this to the situation of Theorem 1.

Theorem 2. *Consider a Lawson symmetric holomorphic structure $\bar\partial$ of rank 2 on M whose isomorphism class is given by a non-trivial holomorphic line bundle $L \in Jac(\tilde M/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar\partial]$. Let $x \in \mathbb{C} \setminus (\frac{1}{2}\mathbb{Z} + \frac{1}{2}i\mathbb{Z})$ such that the holomorphic structure of E is given by*

$$\bar\partial^E = \bar\partial^0 - \pi x d\bar z$$

on $\mathbb{C} \to \tilde M/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$. Then there exists a unique $a^u = a^u(x) \in \mathbb{C}$ such that the flat Lawson symmetric connection ∇ on M which is given by the connection 1-form 4.8 is unitary with respect to a suitable chosen metric. The function

$$x \mapsto a^u(x)$$

is real analytic and odd in x. It satisfies

$$a^u(x + \frac{1}{2}) = a^u(x) + \frac{1}{2}$$

and

$$a^u(x + \frac{i}{2}) = a^u(x) - \frac{i}{2}$$

which means that it gives rise to a well-defined real analytic section $\mathcal{U}$ of the affine bundle of (the moduli space of) flat $\mathbb{C}^*$-connections over the Jacobian of $\tilde M/\mathbb{Z}_3$ away from the origin.

Remark 4.4. We show in Theorem 3 below that the section $\mathcal{U}$ has a first order pole at the origin.

Proof. As the unitary flat connections depend (real) analytic on the underlying holomorphic structure, the function $x \mapsto a^u(x)$ is also real analytic. Moreover, it must be odd in x as the flat connection induced on $L^+ \to \tilde M/\mathbb{Z}_3$ is dual to the one induced on $L^- \to \tilde M/\mathbb{Z}_3$. The functional equations are simply a consequence of the gauge invariance of our discussion: On $\tilde M/\mathbb{Z}_3 = \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$ the flat connections $d + \pi a dz - \pi x d\bar z$, $d + \pi(a - \frac{1}{2})dz - \pi(x - \frac{1}{2})d\bar z$ and $d + \pi(a + \frac{i}{2})dz - \pi(x - \frac{i}{2})d\bar z$ are gauge equivalent as well as the corresponding flat $\mathrm{SL}(2,\mathbb{C})$-connections on M. □

Remark 4.5. The Narasimhan-Seshadri section which maps an isomorphism class of stable holomorphic structures to its corresponding gauge class of unitary flat connections is a real analytic section in the holomorphic affine bundle of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections to the moduli space of stable holomorphic structures. The later space is equipped with a natural symplectic structure. Then, the natural (complex anti-linear) derivative of the Narasimhan-Seshadri section can be interpreted as the symplectic form, see for example [BR].

5. The exceptional flat $\mathrm{SL}(2,\mathbb{C})$-connections

In the previous chapter we have studied all flat Lawson symmetric connections on M whose underlying holomorphic structures admit symmetric Higgs fields Ψ such that $\det\Psi = Q$. The holomorphic structures are determined by non-trivial holomorphic line bundles $L \in Jac(\tilde M/\mathbb{Z}_3)$, see Proposition 3.1. The construction of a connection 1-form as in (4.8) breaks down for the trivial holomorphic line bundle $\underline{\mathbb{C}} \to \tilde M/\mathbb{Z}_3$, because the trivial line bundle corresponds to the holomorphic direct sum bundle $S \oplus S^* \to M$ which does not admit a holomorphic connection. But as we have already mentioned above, the gauge orbits of the remaining holomorphic structures which admit Lawson symmetric holomorphic connections are infinitesimal near to the gauge orbit of $S \oplus S^* \to M$ (see for example the proof of Proposition 3.1). We use this observation to construct the remaining flat Lawson symmetric connections as limits of the connections studied in Theorem 1 when L tends to the trivial holomorphic line bundle. Even more important for our purpose, we exactly determine for which meromorphic family of flat line bundle connections on $\tilde{M}/\mathbb{Z}_3$ the corresponding family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M extends holomorphically through the points where the holomorphic line bundle is the trivial one, see Theorem 3 and Theorem 4 below.

5.1. **The case of the stable holomorphic structure.** We start our discussion with the case of a Lawson symmetric stable holomorphic structure which does not admit a symmetric Higgs field with non-trivial determinant. As we have seen, this holomorphic structure is isomorphic to $\bar{\partial}^0$.

Let ∇ be a flat unitary Lawson symmetric connection such that $(\nabla)'' = \bar{\partial}^0$. As we have seen in the proof of Lemma 3.1, $\bar{\partial}^0$ admits a nowhere vanishing symmetric Higgs field $\Psi \in H^0(M, K\,\mathrm{End}_0(V, \bar{\partial}^0))$ with $\det \Psi = 0$. The kernel of Ψ is the dual of the spin bundle S of the Lawson surface. We split the connection

$$\nabla = \begin{pmatrix} \bar{\partial}^{S^*} & \bar{q} \\ 0 & \bar{\partial}^S \end{pmatrix} + \begin{pmatrix} \partial^{S^*} & 0 \\ -q & \partial^S \end{pmatrix}$$

with respect to the unitary decomposition $V = S^* \oplus S \to M$. Note that q is a multiple of the Hopf differential Q of the Lawson surface and that $\bar{q} \in \Gamma(M, \bar{K}K^{-1})$ is its adjoint with respect to the unitary metric. As explained above, we want to study $\nabla = \nabla^0$ as a limit of a family of flat Lawson symmetric connections

$$t \mapsto \nabla^t,$$

such that the holomorphic structures vary non-trivially in t. We restrict to the case where a choice of a corresponding line bundle $L_t^+ \in Jac(\tilde{M}/\mathbb{Z}_3)$ with $\Pi(L_t) = [(\nabla^t)'']$ is given by the holomorphic structure

$$\bar{\partial}_0 + t d\bar{z},$$

where $\bar{\partial}_0 = d''$ is the trivial holomorphic structure on $\underline{\mathbb{C}} \to \tilde{M}/\mathbb{Z}_3$. As Π branches at $\underline{\mathbb{C}}$ (Proposition 3.1) this can always be achieved by rescaling the family as long as the map $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ has a branch point of order 1 at 0. Pulling the family of connections back to $\tilde{M}$ (and applying gauge transformations to them which depend holomorphically on t on a disc containing $t = 0$) the holomorphic structures of the connections take the following form

$$(\pi^*\nabla^t)'' = \begin{pmatrix} \bar{\partial}^{S^*} + t\bar{\eta} & \bar{q} \\ 0 & \bar{\partial}^S - t\bar{\eta} \end{pmatrix},$$

where $\bar{\eta} = \tilde{\pi}^* d\bar{z}$. A family of symmetric Higgs fields $\Psi_t \in H^0(M, K_M\,\mathrm{End}_0(V, (\nabla^t)'')$ is given by

$$\pi^*\Psi_t = \begin{pmatrix} tc\eta & \omega + t\beta(t) \\ 0 & -tc\eta \end{pmatrix} \tag{5.1}$$

after pulling them back as 1-forms to $\tilde{M}$. Here $\beta(t)$ is a t-dependent section of $\pi^*K_M = K_{\tilde{M}}\,\mathrm{Hom}(\pi^*S, \pi^*S^*)$, and $\omega \in H^0(\tilde{M}, K_{\tilde{M}}\,\mathrm{Hom}(\pi^*S, \pi^*S^*))$ is the canonical section which has zeros at the branch points of π and c is a some non-zero constant. Note that ω can be considered as the pull-back of the bundle-valued 1-form $1 \in H^0(M, K\,\mathrm{Hom}(S, S^*))$, or as a square root of η. With respect to the fixed (non-holomorphic) background decomposition $\pi^*V = \pi^*S^* \oplus \pi^*S$ the eigenlines $L^t_\pm$ of $\pi^*\Psi_t$ on $\tilde{M}$ are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ -2c\omega t + t^2(...) \end{pmatrix}.$$

Therefore the expansion in t of the singular gauge transformation $f_t \colon L^t_+ \oplus L^t_- \to \pi^*V = \pi^*S^* \oplus \pi^*S$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & -2c\omega t + t^2(...) \end{pmatrix}.$$

The expansion of $\pi^*\nabla^t$ is of the form

$$\pi^*\nabla^t = \pi^*\nabla + t\begin{pmatrix} \bar{\eta} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} + t\Gamma(t),$$

where $\Gamma(t) \in \Gamma(\tilde{M}, K_{\tilde{M}} \operatorname{End}_0(V))$ depends holomorphically on t. Applying the gauge f_t we obtain the following asymptotic behavior

$$\nabla^t \cdot f_t = \frac{1}{t}\begin{pmatrix} -\frac{\pi^*q}{2c\omega} & 0 \\ 0 & \frac{\pi^*q}{2c\omega} \end{pmatrix} + \ldots$$

The pullback $\pi^*q \in H^0(K_{\tilde{M}}K_M)$ has zeros of order 3 at the branch points of π and therefore it is a constant multiple of $\eta\omega$. Hence, the holomorphic line bundle connections on E^t given by the $1:1$ correspondence in Theorem 1 have the following expansion

$$\nabla^{E^t} = d + t d\bar{z} + \frac{\tilde{c}}{t} dz + \hat{e}(t) dz \tag{5.2}$$

for some holomorphic function $\hat{e}(t)$. In order to determine $\tilde{c}$, we expand the family of equations $(\nabla^t)''\Psi_t = 0$ as follows:

$$0 = (\pi^*\nabla^t)''\pi^*\Psi_t = t\begin{pmatrix} 0 & -2\pi^*\bar{q}c\eta + 2\omega\bar{\eta} + \bar{\partial}^{\pi^*K_M}\beta(0) \\ 0 & 0 \end{pmatrix} + t^2(...).$$

As we have fixed $\omega \in H^0(\tilde{M}, K_{\tilde{M}} \operatorname{Hom}(S, S^*)) = H^0(\tilde{M}, \pi^*K_M)$ up to sign by $\omega^2 = \eta = \tilde{\pi}^*dz$ we obtain from Serre duality applied to the bundle π^*K_M

$$\int_{\tilde{M}} \pi^*\bar{q}c\eta\omega = \int_{\tilde{M}} \bar{\eta}\omega^2 = 3\int_{\tilde{M}/\mathbb{Z}_3} d\bar{z} \wedge dz = 24i. \tag{5.3}$$

Recall that we have identified $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$ and dz is the corresponding differential. The degree of $\pi^*S^* \to \tilde{M}$ is -2 and we obtain from the flatness of ∇ that

$$4\pi i = \int_{\tilde{M}} \pi^*\bar{q} \wedge \pi^*q. \tag{5.4}$$

Combining (5.3) and (5.4) we obtain

$$-\frac{\pi^*q}{2c\omega} = -\frac{\pi}{12}\eta, \tag{5.5}$$

which exactly tells us the asymptotic of the family 5.2.

Theorem 3. *Let ∇^t be a holomorphic family of flat Lawson symmetric connections on M such that $(\nabla^0)''$ is isomorphic to $\bar{\partial}^0$. If $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ branches of order 1 at $t = 0$, then, after reparametrization the family, ∇^t induces by means of Theorem 1 and (4.8) a meromorphic family of flat connections of the form*

$$\tilde{\nabla}^t = d + td\bar{z} - \frac{\pi}{12t}dz + te(t)dz \tag{5.6}$$

on $\underline{\mathbb{C}} \to \tilde{M}/\mathbb{Z}_3$, where $e(t)$ is a holomorphic function in t.

Conversely, let $\tilde{\nabla}^t$ be a meromorphic family of flat connections on $\underline{\mathbb{C}} \to \tilde{M}/\mathbb{Z}_3$ of the form 5.6. Then the induced family of flat Lawson symmetric connections ∇^t on the complex rank 2 bundle $V \to M$ extends (after a suitable $t-$dependent gauge) holomorphically to $t = 0$ such that ∇^0 is a flat Lawson symmetric connection and $(\nabla^0)''$ is isomorphic to $\bar{\partial}^0$.

Proof. Our primarily discussion was restricted to the case where ∇^0 is unitary. In that case it remains to show that the function $\hat{e}(t)$ in (5.2) has a zero at $t = 0$. This follows from the fact that the function a^u in Theorem 2 is odd. For the general case we need to study the effect of adding a holomorphic family of Lawson symmetric Higgs fields

$$\Psi(t) \in H^0(M; K\,\mathrm{End}_0(V, (\nabla^t)'')).$$

Such a holomorphic family of Higgs fields is given by

$$h(t)\begin{pmatrix} tc\eta & \omega + t\beta(t) \\ 0 & -tc\eta \end{pmatrix}$$

for some function $h(t)$ which is holomorphic in t, see (5.1). From this the first part easily follows. Moreover, by reversing the arguments one also obtains a proof of the converse direction. □

Corollary 5.1. *The unitarizing function $a^u \colon \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} \to \mathbb{C}$ in Theorem 2 is given by*

$$a^u(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x} + b(x),$$

where θ is the Θ-function as in (4.5), θ' is its derivative and $b(x) \colon \mathbb{C} \to \mathbb{C}$ is an odd smooth function which is doubly periodic with respect to the lattice $\frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z}$.

Proof. The function $\tilde{a} \colon \mathbb{C} \setminus \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} \to \mathbb{C}$ defined by

$$\tilde{a}(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x}$$

is an odd function in x which satisfies the same functional equations (see Theorem 2) as a^u. Note that the parametrization of the family of holomorphic rank 1 structures in Theorem 2 and in Theorem 3 differ by the multiplicative factor $-\pi$. Therefore, $\tilde{a}$ has the right asymptotic behavior at the lattice points $\frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z}$. So the difference $b = a^u - \tilde{a}$ is an odd, smooth and doubly periodic function. □

5.2. **The case of the non-stable holomorphic structure.** We have already seen in Section 2.1 that every flat Lawson symmetric connection on M whose holomorphic structure is not semi-stable is gauge equivalent to

$$\nabla = \begin{pmatrix} \nabla^{spin*} & 1 \\ \mathrm{vol} + cQ & \nabla^{spin} \end{pmatrix}$$

with respect to $V = S^* \oplus S \to M$. In this formula ∇^{spin} and vol are induced by the Riemannian metric of constant curvature -4, $c \in \mathbb{C}$ and Q is the Hopf differential of the Lawson surface. The gauge orbit of the holomorphic structure ∇'' is infinitesimal close to the gauge orbits of the holomorphic structures $\bar\partial^0$ and $\bar\partial^S \oplus \bar\partial^{S^*}$. As in Section 5.1, we approximate ∇ by a holomorphic family of flat Lawson symmetric connections $t \mapsto \nabla^t$ such that the isomorphism classes of the holomorphic structures $(\nabla^t)''$ vary in t. We obtain a similar result as Theorem 3.

Theorem 4. *Let ∇^t be a holomorphic family of flat Lawson symmetric connections on M such that $(\nabla^0)''$ is isomorphic to the non-trivial extension $S \to V \to S^*$ and such that $t \mapsto [(\nabla^t)''] \in \mathcal{S}$ branches of order 1 at $t = 0$. After reparametrization the family, ∇^t corresponds (via Theorem 1 and (4.8)) to a meromorphic family of flat connections $\tilde\nabla^t$ on $\underline{\mathbb{C}} \to \tilde M/\mathbb{Z}_3$ of the form*

$$\tilde\nabla^t = d + td\bar z + \frac{\pi}{12t}dz + te(t)dz, \tag{5.7}$$

where $e(t)$ is holomorphic in t.

Conversely, let $\tilde\nabla^t$ be a meromorphic family of flat connections on $\underline{\mathbb{C}} \to \tilde M/\mathbb{Z}_3$ of the form 5.7. Then the induced family of flat Lawson symmetric connections ∇^t on the complex rank 2 bundle $V \to M$ extends (after a suitable $t-$dependent gauge) holomorphically to $t = 0$ such that $(\nabla^0)''$ is isomorphic to the non-trivial extension $0 \to S \to V \to S^* \to 0$. Moreover, ∇^0 is gauge equivalent to the uniformization connection (see (2.2)) if the function e has a zero at $t = 0$.

Proof. Consider a holomorphic family of flat Lawson symmetric connections $\hat\nabla^t$ such that $(\nabla^t)''$ is isomorphic to $(\hat\nabla^t)''$ for all t and such that $\hat\nabla^0$ is unitary. In particular, $(\hat\nabla^0)''$ is isomorphic to $\bar\partial^0$. Then, after applying the t-dependent gauge g_t the difference

$$\Psi_t := \hat\nabla^t - g_t^{-1}\nabla^t g \in H^0(M, K\,\mathrm{End}_0(V, (\hat\nabla^t)''))$$

satisfies

$$\det \Psi_t = \frac{q}{t} + \text{higher order terms},$$

where q is a non-zero multiple of the Hopf differential. This implies, that the line bundle connections $\tilde\nabla^t$ have an expansion like

$$\tilde\nabla^t = d + td\bar z + \frac{c}{t}dz + \text{higher order terms}$$

for some non-zero $c \in \mathbb{C}$. Then, analogous to the computation in Section 5.1, one obtains $c = \frac{1}{12\pi}$. Note that the reason for the different signs is because of the last sign in the degree formula for S^* :

$$-2\pi i \deg(S^*) = \int_M \bar q \wedge q = -\int_M 1 \wedge vol.$$

To show that the 0.-order term in the expansion of $\tilde\nabla^t$ vanishes we first observe that there exists an additional (holomorphic) symmetry $\tilde\tau \colon M \to M$ which induces the symmetry $z \mapsto iz$ on $\tilde M/\mathbb{Z}_3$. Note that $\tilde\tau^* Q = -Q$. Because the gauge equivalence class of the uniformization connection ((2.2)) is also invariant under $\tilde\tau$, one easily gets (as in the proof of Theorem 5.2) that the 0.-order term vanishes. Moreover one obtains that in the case of the uniformization connection also the first order term vanishes. □

6. The spectral data

So far we have seen that the generic Lawson symmetric flat connection is determined (up to gauge equivalence), after the choice of one eigenline bundle of a symmetric Higgs field, by a flat line bundle connection on a square torus. Moreover, the remaining flat connections are explicitly given as limiting cases of the above construction. We now apply these results to the case of the family of flat connections ∇^λ associated to a minimal surface. We assume that the minimal surface is of genus 2 and has the conformal type and the symmetries φ_2, φ_3 and τ of the Lawson surface. The family of flat connections induces a family of Lawson symmetric holomorphic structures $\bar\partial^\lambda = (\nabla^\lambda)''$ which extends to $\lambda = 0$. As it is impossible to make a consistent choice of the eigenline bundles of symmetric Higgs fields with respect to $\bar\partial^\lambda$ for all $\lambda \in \mathbb{C}^*$ (see Proposition 6.1) we need to introduce a so-called spectral curve which double covers the spectral plane $\mathbb{C}^*$ and enables us to parametrize the eigenline bundles. Then, the family of flat connections ∇^λ is determined (up to a λ-dependent gauge) by the corresponding family of flat line bundles over the torus. The behavior of this family of flat line bundles is very similar (at least around $\lambda = 0$) to the family of flat line bundles parametrized by the spectral curve of a minimal or CMC torus, compare with [H]. The main difference is that we have some kind of symmetry breaking between $\lambda = 0$ and $\lambda = \infty$: We do not treat the holomorphic and anti-holomorphic structures of a flat connection in the same way but consider the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures. As a consequence, we do not have an explicitly known reality condition, which seems to be the missing ingredient to explicitly determine the Lawson surface.

By taking the gauge equivalence classes of the associated family of holomorphic structures $\bar\partial^\lambda$ we obtain a holomorphic map

$$\mathcal{H}\colon \mathbb{C} \to \mathcal{S} \cong \mathbb{P}^1$$

to the moduli space of semi-stable Lawson symmetric holomorphic structures, see Proposition 2.1. This map is given by $\mathcal{H}(\lambda) = [\bar\partial^\lambda]$ for those λ where $\bar\partial^\lambda$ is semi-stable. By remark 2.1 it extends holomorphically to the points λ where $\bar\partial^\lambda$ is not semi-stable.

Proposition 6.1 (The definition of the spectral curve)**.** *There exists a holomorphic double covering $p\colon \Sigma \to \mathbb{C}$ defined on a Riemann surface Σ, the spectral curve, together with a holomorphic map $\mathcal{L}\colon \Sigma \to Jac(\tilde M/\mathbb{Z}_3)$ such that*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \mathcal{L}\ } & Jac(\tilde M/\mathbb{Z}_3) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Pi} \\ \mathbb{C} & \xrightarrow[\ \mathcal{H}\ ]{} & \mathcal{S} \end{array}$$

commutes, where $\Pi\colon Jac(\tilde M/\mathbb{Z}_3) \to \mathcal{S}$ is as in Proposition 3.1. The map p branches over $0 \in \mathbb{C}$.

Proof. We first define

$$\underline{\Sigma} = \{(\lambda, L) \in \mathbb{C} \times Jac(\tilde M/\mathbb{Z}_3) \mid \Pi(L) = \mathcal{H}(\lambda)\}$$

which is clearly a non-empty analytic subset of $\mathbb{C} \times Jac(\tilde M/\mathbb{Z}_3)$. Then, the spectral curve is given by the normalization

$$\Sigma \to \underline{\Sigma},$$

and $\mathcal{L}$ is the composition of the normalization with the projection onto the second factor. It remains to prove that Σ branches over 0. Because Π branches over $[\bar\partial^0]$ this follows if we can show that the map $\mathcal{H}$ is immersed at $\lambda = 0$. As $\bar\partial^0$ is stable, the tangent space at $[\bar\partial^0]$ of the moduli space of (stable) holomorphic structures with trivial determinant is given by $H^1(M, K\,\mathrm{End}_0(V, \bar\partial^0))$. The cotangent space is given via trace and integration by $H^0(M, KEnd_0(V, \bar\partial^0))$. With

$$\frac{\partial}{\partial\lambda}\bar\partial^\lambda = \begin{pmatrix} 0 & 0 \\ \mathrm{vol} & 0 \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in H^0(M, K\,\mathrm{End}_0(V, \bar\partial^0))$$

we see that

$$\Phi(\frac{\partial}{\partial\lambda}\bar\partial^\lambda_{|\lambda=0}) = \int_M \mathrm{vol} \neq 0$$

which implies that $\mathcal{H}$ is immersed at $\lambda = 0$. □

In order to study the family of gauge equivalence classes $[\nabla^\lambda]$ we consider the moduli space of flat $\mathbb{C}^*$-connection on $\tilde{M}/\mathbb{Z}_3$ as an affine holomorphic bundle

$$\mathcal{A}^f \to Jac(\tilde{M}/\mathbb{Z}_3)$$

over the Jacobian by taking the complex anti-linear part of a connection. Then, as a consequence of Theorem 1 together with our discussion in Section 5, we obtain a meromorphic lift

$$\begin{array}{ccc} & \overset{\mathcal{D}}{\nearrow} & \mathcal{A}^f \\ & & \downarrow {''} \\ \Sigma & \xrightarrow[\mathcal{L}]{} & Jac(\tilde{M}/\mathbb{Z}_3) \end{array}$$

of the map $\mathcal{L}$ which parametrizes the gauge equivalence classes $[\nabla^\lambda]$. The map $\mathcal{D}$ has poles at those points $p \in \Sigma$ where $\mathcal{L}(p) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the trivial holomorphic bundle. Note that the (unique) preimage $0 \in \Sigma$ of $\lambda = 0$ always satisfies $\mathcal{L}(p) = \underline{\mathbb{C}}$. The poles at $p \neq 0$ are generically simple, and the exact asymptotic behavior of $\mathcal{D}$ around p is determined by the results of Section 5.

Definition. *The triple $(\Sigma, \mathcal{L}, \mathcal{D})$, which is determined by the associated family of flat connections of a compact minimal surface in S^3 with the symmetries φ_2, φ_3 and τ of the Lawson surface of genus 2, is called spectral data of the surface.*

6.1. **Asymptotic behavior of the family of flat connections.** We have already seen that the spectral curve Σ of a compact minimal surface in S^3 with the symmetries of the Lawson surface branches over $\lambda = 0$ and that the map $\mathcal{L}$ is holomorphic. We claim that the asymptotic behavior of $\mathcal{D}$ around the preimage of $\lambda = 0$ is analogous to the case of minimal tori in S^3 [H].

In order to show this we consider a holomorphic family of flat Lawson symmetric $\mathrm{SL}(2, \mathbb{C})$-connections

$$\lambda \mapsto \hat\nabla^\lambda$$

defined on an open neighborhood of $\lambda = 0$ such that $(\hat{\nabla}^\lambda)'' = \bar{\partial}^\lambda$. This implies that for small $\lambda \neq 0$ the difference

$$\nabla^\lambda - \hat{\nabla}^\lambda$$

is a symmetric Higgs field $\Psi \in H^0(M, K\,\mathrm{End}_0(V, \bar{\partial}^\lambda))$ whose determinant is a non-zero multiple of Q. An expansion of $\hat{\nabla}^\lambda$ around $\lambda = 0$ is given by

$$\hat{\nabla}^\lambda = \begin{pmatrix} \nabla^{spin^*} + \omega_0 & -\frac{i}{2}Q^* + \alpha \\ -\frac{i}{2}q & \nabla^{spin} - \omega_0 \end{pmatrix} + \lambda \begin{pmatrix} \omega_1 & \alpha_1 \\ -\mathrm{vol} + \beta_1 & -\omega_1 \end{pmatrix} + \dots, \tag{6.1}$$

where q is a constant multiple of the Hopf differential, $\alpha_i \in \Gamma(M, KK^{-1})$, $\omega_i \in \Gamma(M, K)$ and $\beta_1 \in \Gamma(M, K^2)$. We claim that

$$Q - q \neq 0$$

is a non-zero constant multiple of the Hopf differential. To see this note that

$$\begin{aligned} F^{\nabla^{spin^*}} &= \frac{1}{4}Q^* \wedge Q + \mathrm{tr}(\Phi \wedge \Phi^*) \\ &= \frac{1}{4}Q^* \wedge q - \bar{\partial}\,\omega_0 \end{aligned}$$

as a consequence of the flatness of ∇^λ as well as of $\hat{\nabla}^0$. The claim then follows from $\int_M \bar{\partial}\,\omega_0 = 0$ and $\int_M \mathrm{tr}(\Phi \wedge \Phi^*) \neq 0$. Comparing (6.1) with Proposition A.1 in appendix A we obtain

$$\det(\nabla^\lambda - \hat{\nabla}^\lambda) = -\frac{i}{4}\lambda^{-1}(Q - q) + \dots. \tag{6.2}$$

This leads to the following theorem.

Theorem 5. *Let $(\Sigma, \mathcal{L}, \mathcal{D})$ be the spectral data associated to a compact minimal surface in S^3 with the symmetries φ_2, φ_3 and τ of the Lawson surface of genus 2. Let t be a coordinate of Σ around $p^{-1}(\{0\})$ such that locally*

$$\mathcal{L}(t) = \bar{\partial}_0 + t d\bar{z},$$

where z is the affine coordinate on $\tilde{M}/\mathbb{Z}_3 \cong \mathbb{C}/(2\mathbb{Z} + 2i\mathbb{Z})$. The asymptotic of the map $\mathcal{D}$ at 0 is given by

$$\mathcal{D}(t) = d + t d\bar{z} + (c_{-1}\frac{1}{t} + c_1 t + ..)dz$$

for some $c_{-1} \neq \pm\frac{\pi}{12}$ and with respect to the natural local trivialization of the affine bundle $\mathcal{A}^f \to Jac(\tilde{M}/\mathbb{Z}_3)$.

The covering $p\colon \Sigma \to \mathbb{C}$ branches at most over those points $\lambda \in \mathbb{C}$ where $\bar{\partial}^\lambda$ is one of the exceptional holomorphic structures, i.e., $\mathcal{L}(\mu) = \underline{\mathbb{C}}$ for $p(\mu) = \lambda$. Moreover $\mathcal{D}$ satisfies the reality condition

$$\mathcal{D}(\mu) = \mathcal{U}(\mathcal{L}(\mu))$$

for all $\mu \in p^{-1}(S^1) \subset \Sigma$ where $\mathcal{U}$ is the section given by Theorem 2, and the closing condition

$$\mathcal{D}(\mu) = [d + \frac{-1+i}{4}\pi dz + \frac{1+i}{4}\pi d\bar{z}]$$

for all $\mu \in p^{-1}(\{\pm 1\}) \subset \Sigma$.

Proof. As in the proof of Theorem 3 we see that the effect of adding a family of Higgs fields $\nabla^\lambda - \hat{\nabla}^\lambda$ with asymptotic as in (6.2) on the corresponding $\mathbb{C}^*$-connections over $\tilde{M}/\mathbb{Z}_3$ is given by adding

$$(\frac{c_{-1}}{t} + c_0 + c_1 t + ...)dz$$

with $0 \neq c_1, c_0, c_1 \in \mathbb{C}$. As $\det(\nabla^\lambda - \hat{\nabla}^\lambda)$ is even in t by the definition of t, the constant c_0 vanishes. Together with Theorem 3 this implies the first statement.

The reality condition is a consequence of the fact that the connections ∇^λ are unitary for $\lambda \in S^1$ and of Theorem 2. The closing condition follows from the observation that the trivial connection of rank 2 on M corresponds to the connection

$$d + \frac{-1+i}{4}\pi dz + \frac{1+i}{4}\pi d\bar{z}$$

on $\tilde{M}/\mathbb{Z}_3$.

It remains to prove that the spectral curve cannot branch over the points $\lambda \in \mathbb{C}$ where $\bar{\partial}^\lambda$ is semi-stable and not stable. For this we consider the holomorphic line bundle $L \to \mathbb{C}$ whose fiber is at a generic point λ spanned by the 1-dimensional space symmetric Higgs fields of $\bar{\partial}^\lambda$. But the space of symmetric Higgs fields at the semi-stable points is also 1-dimensional, and the determinant of a non-zero symmetric Higgs field is a non-zero multiple of the Hopf differential Q. Therefore, the eigenlines of the Higgs fields can be parametrized in λ as long as $\bar{\partial}^\lambda$ is not an exceptional holomorphic structure.

□

6.2. **Reconstruction.** Conversely, a hyper-elliptic Riemann surface $\Sigma \to \mathbb{C}$ together with a map $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ and a lift $\mathcal{D}$ into the affine bundle of line bundle connections which satisfy the asymptotic condition, the reality and closing conditions of Theorem 5 give rise to a compact minimal surface of genus 2 in S^3. To prove this we first need some preparation:

Theorem 6. *Let $\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

- *the asymptotic at $\lambda = 0$ is given by*
$$\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla} + ...$$
where $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ is nowhere vanishing and nilpotent;
- *for all $\lambda \in S^1 \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^\lambda$ is unitary with respect to this metric;*
- *$\tilde{\nabla}^\lambda$ is trivial for $\lambda = \pm 1$.*

Then there exists a unique (up to spherical isometries) minimal surface $f\colon M \to S^3$ such that its associated family of flat connections ∇^λ and the family $\tilde{\nabla}^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$.

Proof. It is a consequence of the asymptotic of $\tilde{\nabla}^\lambda$ that $(\tilde{\nabla}^\lambda)''$ is stable for generic $\lambda \in \mathbb{C}^*$, for more details see [He1]. This implies that the generic connection $\tilde{\nabla}^\lambda$ is irreducible. Therefore the hermitian metric for which $\tilde{\nabla}^\lambda$ is unitary is unique up to constant multiples for generic $\lambda \in S^1 \subset \mathbb{C}^*$. For those $\lambda \in S^1$ the hermitian metric $(,)^\lambda$ is unique if we impose that it is compatible with the determinant on V, i.e., the determinant of an orthonormal basis is unimodular. The metric $(\,,\,)^\lambda$ depends real-analytically on $\lambda \in S^1 \backslash S$, where $S \subset S^1$ is the set of points where ∇^λ is not irreducible, and can be extended through the set S.

From now on we identify $V = M \times \mathbb{C}^2$ and fix a unitary metric $(\,,\,)$ on it. Therefore, $(\,,\,)^\lambda$ can be identified with a section $[h] \in \Gamma(S^1 \times M, \mathrm{SL}(2, C)/\,\mathrm{SU}(2))$ which itself can be lifted to a section $h \in \Gamma(S^1 \times M, \mathrm{SL}(2, C))$. Clearly, h is real analytic in λ and satisfies

$$h^*_\lambda(\,,\,) = (\,,\,)^\lambda.$$

We now apply the loop group Iwasawa decomposition to $g = h^{-1}$, i.e.,

$$g = BF,$$

where $B \in \Gamma(D^1 \times M, \mathrm{SL}(2, \mathbb{C}))$ is holomorphic in λ on $D^1 = \{\lambda \in \mathbb{C} \mid \bar{\lambda}\lambda \leq 1\}$ and $F \in \Gamma(S^1 \times M, \mathrm{SU}(2))$ is unitary, see [PS] for details. Gauging

$$\nabla^\lambda = \tilde{\nabla}^\lambda \cdot B$$

we obtain a holomorphic family of flat connections ∇^λ on $D^1 \setminus \{0\}$ which is unitary with respect to $(\,,\,)$ on S^1 by construction. Applying the Schwarz reflection principle yields a holomorphic family of flat connection $\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda$ which is unitary on S^1 and trivial for $\lambda = \pm 1$. Moreover, as B extends holomorphically to $\lambda = 0$, ∇^λ has the following asymptotic

$$\nabla^\lambda \sim \lambda^{-1}\Phi + \nabla + ..$$

where $\Phi = B_0^{-1}\Psi B_0$ is complex linear, nowhere vanishing and nilpotent. Using the Schwarzian reflection again, we obtain

$$\nabla^\lambda = \lambda^{-1}\Phi + \nabla - \lambda\Phi^*$$

for a unitary connection ∇. This proves the existence of an associated minimal surface $f\colon M \to S^3$.

Let $f_1, f_2\colon M \to S^3$ be two minimal surface such that their associated families of flat connections ∇^λ_1 and ∇^λ_2 are gauge equivalent to $\tilde{\nabla}^\lambda$, where both families of gauge transformations extend holomorphically to $\lambda = 0$. Let $g \in \Gamma(\mathbb{C} \times M, \mathrm{SL}(2, \mathbb{C}))$ be the gauge between these two families which, by assumption, also extends to $\lambda = 0$. We may assume that for all $\lambda \in S^1$ the connections ∇^λ_1 and ∇^λ_2 are unitary with respect to the same hermitian metric. As the connections are generically irreducible the gauge g is unitary along the unit circle. By the Schwarz reflection principle g extends to $\lambda = \infty$, and therefore g is constant in λ. Hence, the corresponding minimal surfaces f_1 and f_2 are the same up to spherical isometries. □

Remark. There exists similar results as Theorem 6 and Theorem 8 for the DPW approach to minimal surface, see [SKKR] and [DW].

Remark 6.1. The above theorem is still true if the individual connections $\tilde{\nabla}^\lambda$ are only of class C^k for $k \geq 3$ and not necessarily smooth.

Similar to the case of tori, the knowledge of the gauge equivalence class of the associated family of flat connections $[\nabla^\lambda]$ for all λ is in general not enough to determine the minimal immersion uniquely. The freedom is given by λ-dependent meromorphic gauge transformations g which is unitary along the unit circle. Applying such a gauge transformation is known in the literature as dressing, see for example [BDLQ] or [TU]. For tori, dressing is closely related to the isospectral deformations induced by changing the eigenline bundle of a minimal immersion. In fact, simple factor dressing with respect to special eigenlines of the connections ∇^λ (those which correspond to the eigenline bundle) generate the abelian group of isospectral deformations. The remaining eigenlines, which only occur at values of λ where the monodromy takes values in $\{\pm \mathrm{Id}\}$, produce singularities in the spectral curve and therefore do not correspond to isospectral deformations in the sense of Hitchin. Due to the fact that for minimal surfaces of higher genus the generic connection ∇^λ is irreducible there are in general no continuous families of dressing deformations:

Theorem 7. *Let $f, \tilde{f} \colon M \to S^3$ be two conformal minimal immersions from a compact Riemann surface of genus $g \geq 2$ together with their associated families of flat connections ∇^λ and $\hat{\nabla}^\lambda$. Assume that ∇^λ is gauge equivalent to $\tilde{\nabla}^\lambda$ for generic $\lambda \in \mathbb{C}^*$. Then there exists a meromorphic map*

$$g \colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$$

such that $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$. This map g is holomorphic and takes values in the invertible endomorphisms away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or equivalently $\tilde{\nabla}^{\lambda_0}$ is reducible.

The space of such dressing deformations of surfaces $f \mapsto \tilde{f}$ is generated by simple factor dressing, i.e., by maps $d \colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$ of the form

$$d(\lambda) = \pi^L + \frac{1 - \bar{\lambda}_0^{-1}}{1 - \lambda_0} \frac{\lambda - \lambda_0}{\lambda - \bar{\lambda}_0^{-1}} \pi^{L^\perp},$$

where L is an eigenline bundle of the connection ∇^{λ_0} and $L^\perp$ is its orthogonal complement.

Proof. We first show that ∇^λ and $\tilde{\nabla}^\lambda$ are gauge equivalent away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or $\tilde{\nabla}^{\lambda_0}$ is reducible. The gauge between two irreducible gauge equivalent connections ∇^λ and $\tilde{\nabla}^\lambda$ is unique up to a constant multiple of the identity. Moreover, multiples of this gauge are the only parallel endomorphisms with respect to the connection $\nabla^\lambda \otimes (\tilde{\nabla}^\lambda)^*$. As the connections depend holomorphic on λ there exists a holomorphic line bundle $\mathcal{G} \to \mathbb{C}^*$ whose line at $\lambda \in \mathbb{C}^*$ is a subset of the parallel endomorphisms (and coincides with it whenever ∇^λ or equivalently $\tilde{\nabla}^\lambda$ is irreducible). A non-vanishing section $g \in \Gamma(U, \mathcal{G})$ around $\lambda \in U \subset \mathbb{C}^*$ gives rise to the gauge between ∇^λ and $\tilde{\nabla}^\lambda$ as long as g^λ is an isomorphism. This can fail only in the case where ∇^λ or equivalently $\tilde{\nabla}^\lambda$ is reducible. We need to prove that g extends holomorphically to an isomorphism at $\lambda = 0$. Then, as g is unitary along the unit circle, g also extends holomorphically to an isomorphism at $\lambda = \infty$ by the Schwarz reflection principle. Note that locally around $\lambda = 0$ all connections are irreducible and all holomorphic structures are stable away from $\lambda = 0$. Then, as above, there exists a family of gauge transformations g^λ which extend to a holomorphic endomorphism g^0 with respect to $\bar{\partial}^0 \otimes (\tilde{\bar{\partial}}^0)^*$. From the fact that the connections around $\lambda = 0$ are gauge equivalent and the expansions of the two families one deduces that g^0 is a holomorphic endomorphism between the stable pairs $(\tilde{\bar{\partial}}, \tilde{\Phi})$ and $(\bar{\partial}, \Phi)$, i.e., $\Phi \circ g^0 = g^0 \circ \tilde{\Phi}$. Therefore Proposition (3.15) of [H1] implies that g^0 is an isomorphism.

In order to find the globally defined dressing $g \colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$ we first investigate the bundle $\mathcal{G} \to \mathbb{C}^*$. We have seen that it extends to $\lambda = 0$ holomorphically and by switching to anti-holomorphic structures, one can also show that it extends holomorphically to $\lambda = \infty$. Therefore there exists a meromorphic section $\tilde{g} \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ whose only poles are at $\lambda = \infty$. As $\mathcal{G}$ is a holomorphic subbundle of $\mathbb{C} \times \Gamma(M; \mathrm{End}(V))$ the determinant is a holomorphic map

$$\det \colon \mathcal{G} \to \mathbb{C}.$$

Consider the holomorphic function

$$h \colon \mathbb{C} \to \mathbb{C},\ h(\lambda) = \det(\tilde{g}^\lambda).$$

Note that we may assume that h is non-vanishing along the unit circle as $\tilde{g}$ is a complex multiple of a unitary gauge there. The Iwasawa decomposition $h = h_+ h_u$ determines functions $h_+ : \{\lambda \mid \lambda\bar{\lambda} \leq 1\} \to \mathbb{C}^*$ and $h_u : \mathbb{C}^* \to \mathbb{C}$ which satisfies $\| h_u(\lambda) \| = 1$ for $\lambda \in S^1$. These are unique up to unimodular constants. The square root $\sqrt{h_+}$ is then well-defined on $\{\lambda \mid \lambda\bar{\lambda} \leq 1\}$ and we define

$$g = \frac{1}{\sqrt{h_+}} \tilde{g} \in H^0(\{\lambda \mid \lambda\bar{\lambda} \leq 1\}, \mathcal{G}).$$

The determinant $\det g$ is unimodular along the unit circle, and therefore, g is unitary along the unit circle. By the Schwarz reflection principle, we obtain a meromorphic map $g \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ which satisfies $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$ by construction.

It is shown in [BDLQ] that a simple factor dressing $\nabla^\lambda \mapsto \nabla^\lambda \cdot d$ makes the associated family of a new minimal surface. We want to show by induction that any g as above is the product of simple factor dressings. Note that $\det g : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational function. If its degree is 0, then $\det g$ is a non-zero constant, and g is constant in λ. As it is unitary on the unit circle, g acts as a spherical isometry on the surface. Assume that $\det g$ has a zero at λ_0. As we have seen $\lambda_0 \in \mathbb{C}^* \setminus S^1$. By multiplying with (a power of) $\frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0} \frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}} \mathrm{Id}$ we can also assume that $g^{\lambda_0} \neq 0$. As g^{λ_0} is a non-zero parallel endomorphism with respect to $\nabla^{\lambda_0} \otimes (\tilde{\nabla}^{\lambda_0})$ we see that the line bundle $L \to M$, which is given by $L_p = \ker g_p^{\lambda_0}$ at generic points $p \in M$, is an eigenline bundle of $\tilde{\nabla}^{\lambda_0}$. As a consequence of the unitarity of $\tilde{\nabla}^\lambda$ along the unit circle, $L^\perp$ is an eigenline bundle of $\tilde{\nabla}^{\bar{\lambda}^{-1}}$. We can apply the simple factor dressing

$$d(\lambda) = \pi^{L^\perp} + \frac{1-\lambda_0}{1-\bar{\lambda}_0^{-1}} \frac{\lambda - \bar{\lambda}_0^{-1}}{\lambda - \lambda_0} \pi^L$$

to $\tilde{\nabla}^\lambda$. Then, the product gd is again a meromorphic family of gauge transformations which extends holomorphically through λ_0, and the degree of the rational function $\det(gd)$ is the degree of the rational function $\det(g)$ minus 1. □

Lemma 6.1. *Let $f : M \to S^3$ be an immersed minimal surface of genus 2 having the symmetries φ_2, φ_3 and τ. Then a (non-trivial) dressing transformation of such a minimal immersion does not admit all symmetries φ_2, φ_3 and τ.*

Proof. We have seen in the proof of Theorem 5 that there exists a local holomorphic family of Higgs fields $\Psi^\lambda \in H^0(M, K\,\mathrm{End}_0(V, \bar{\partial}^\lambda))$ around every point λ_0 where ∇^{λ_0} is reducible such that $\det \Psi^\lambda$ is nowhere vanishing. Let $g : \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$ be a meromorphic family of gauges as in Theorem 7. Assume that $g(\lambda_0)$ exists but is a non-zero endomorphism which is not invertible. It is easy to see that the family of Higgs fields

$$\hat{\Psi}^\lambda = g(\lambda)^{-1} \Psi^\lambda g(\lambda)$$

with respect to $\tilde{\nabla}^\lambda = \nabla^\lambda \cdot g$ has a pole at λ_0. Then, by resolving the pole by multiplying with an appropriate power of $(\lambda - \lambda_0)$, the (local) holomorphic nowhere vanishing family of Higgs fields

$$\tilde{\Psi}^\lambda = (\lambda - \lambda_0)^k \hat{\Psi}^\lambda$$

satisfies $\det(\tilde{\Psi}^{\lambda_0}) = 0$. This is not possible for a surface $\tilde{f}$ which has all three symmetries φ_2, φ_3 and τ. □

Theorem 8. *Let Σ be a Riemann surface and $p\colon \Sigma \to \mathbb{C}$ be a double covering induced by the involution $\sigma\colon \Sigma \to \Sigma$ such that p branches over 0. Let $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ be a non-constant holomorphic map which is odd with respect to σ and satisfies $\mathcal{L}(0) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$. Let $\mathcal{D}\colon \Sigma \setminus p^{-1}(0) \to \mathcal{A}^f$ be a meromorphic lift of $\mathcal{L}$ to the moduli space of flat $\mathbb{C}^*$- connections on $\tilde{M}/\mathbb{Z}_3$ which is odd with respect to σ and which satisfies the conditions of Theorem 3 and Theorem 4 at its poles, i.e., $\mathcal{D}$ defines a holomorphic map from $\mathbb{C}^*$ to the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M. If $\mathcal{D}$ has a first order pole at 0 and satisfies the reality condition*

$$\mathcal{D}(\mu) = \mathcal{U}(\mathcal{L}(\mu))$$

for all $\mu \in p^{-1}(S^1) \subset \Sigma$, where $\mathcal{U}$ is the section given by Theorem 2, and the closing condition

$$\mathcal{D}(\mu) = [d + \frac{-1+i}{4}\pi dz + \frac{1+i}{4}\pi d\bar{z}]$$

for all $\mu \in p^{-1}(\{\pm 1\}) \subset \Sigma$ then there exists an immersed minimal surface $f\colon M \to S^3$ such that $(\Sigma, \mathcal{L}, \mathcal{D})$ are the spectral data of f. Let t be a holomorphic coordinate of Σ around $p^{-1}(0)$ such that $t^2 = \lambda$, and consider the expansion

$$\mathcal{D} \sim d - (x_1 t + ..)\pi d\bar{z} + (a_{-1}\frac{1}{t} + ...)\pi dz.$$

Then the area of f is given by

$$Area(f) = -12\pi(\frac{1}{6} - 2\pi x_1 a_{-1}).$$

If p only branches at those $\mu \in \Sigma$ where $\mathcal{L}(\mu) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ then there is a unique f which has the symmetries φ_2, φ_3 and τ.

Proof. We first show that the spectral data give rise to a holomorphic $\mathbb{C}^*$-family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M satisfying the conditions of Theorem 6. By assumption we obtain a holomorphic map into the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M. Reversing the arguments of Section 4 and 5 we obtain locally on open subsets of $\mathbb{C}^*$ holomorphic families of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M which are lifts of the map to the moduli space. We cover $\mathbb{C}^*$ by these open sets U_i, $i \in \mathbb{N}$, such that for every U_i there exists at most one point where the corresponding connection is reducible. There also exist an open set U_0 containing 0 such that on $U_0 \setminus \{0\}$ there exists a lift ∇_0^λ of the map to the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on M which has at most a first order pole at $\lambda = 0$. Moreover, as $\mathcal{L}(0) = \underline{\mathbb{C}}$, the residuum at 0 must be a complex linear 1-form $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ which is nilpotent. We now fix such families of flat $\mathrm{SL}(2,\mathbb{C})$-connections ∇_i^λ on every set U_i. Let $\mathcal{G}$ be the complex Banach Lie group of C^k gauges

$$\mathcal{G} = \{g\colon M \to \mathrm{SL}(2,\mathbb{C}) \mid g \text{ is of class } C^k\},$$

where we have fixed a trivialization of the rank 2 bundle $V = M \times \mathbb{C}^2$ and $k \geq 4$. On $U_i \cap U_j$ we define a map $g_{i,j}\colon U_i \cap U_j \to \mathcal{G}$ by

$$\nabla_j^\lambda = \nabla_i^\lambda . g_{i,j}.$$

Clearly, the maps $g_{i,j}$ are well-defined, and give rise to a 1-cocycle of $\mathbb{C} = \cup_{i \in \mathbb{N}_0} U_i$ with values in $\mathcal{G}$. As $\mathbb{C}$ is a Stein space the generalized Grauert theorem as proven in [Bu] shows the existence of maps $f_i\colon U_i \to \mathcal{G}$ satisfying $f_i f_j^{-1} = g_{i,j}$. Then

$$\nabla_i^\lambda . f_i = \nabla_j^\lambda . f_j$$

on $U_i \cap U_j$ and we obtain a well-defined $\mathbb{C}^*$-family of flat $\mathrm{SL}(2,\mathbb{C})$-connections $\tilde{\nabla}^\lambda$ which satisfies the reality condition and the closing condition of Theorem 6. Applying the proof of Theorem 6 we see that the holomorphic structure $(\tilde{\nabla}^0)''$ is stable and therefore the residuum of $\tilde{\nabla}^\lambda$ at $\lambda = 0$ is a nowhere vanishing nilpotent complex linear 1-form. By Theorem 6 we obtain an immersed minimal surface $f\colon M \to S^3$. The formula for the energy of f can be computed by similar methods as used in Section 5.

Now assume that p only branches at those $\mu \in \Sigma$ where $\mathcal{L}(\mu) = \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$. Then the map $\mathcal{D}$ into the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections can be locally lifted (denoted by ∇_i^λ) to the space of flat connections in such a way that a corresponding nowhere vanishing family of Higgs fields Ψ_λ^i has non-zero determinant whenever $\mathcal{L}(\mu) \neq \underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ for $p(\mu) = \lambda$. Arguing in the same lines as in the proof of Lemma 6.1 one sees that all families of connections ∇_i^λ are gauge equivalent to $\varphi^*\nabla_i^\lambda$ by holomorphic families of gauges for all symmetries $\varphi = \varphi_2, \varphi_3, \tau$. Then the uniqueness part of Theorem 6 proves that the corresponding minimal surface has the symmetries φ_2, φ_3 and τ. Moreover, Theorem 7 and Lemma 6.1 show the uniqueness of this minimal surface.

□

Remark 6.2. Computer experiments in [HS] suggest that the spectral curve of the Lawson surface of genus 2 is not branched over the punctured unit disc $D = \{\lambda \in \mathbb{C} \mid 0 < \| \lambda \| \leq 1\}$. With these numerical spectral data the Lawson surface of genus 2 can be visualized as a conformal immersion from the Riemann surface M into S^3 by an implementation of Theorem 8 in the xlab software of Nicholas Schmitt (see Figure 1).

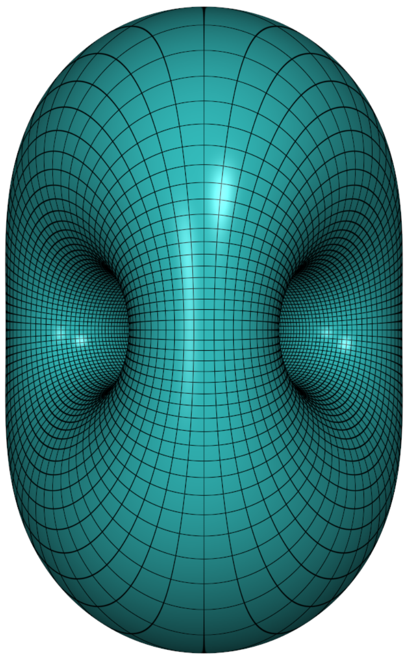

FIGURE 1. Lawson genus 2 surface, picture by Nicholas Schmitt.

7. Lawson symmetric CMC surface of genus 2

In [HS] we found numerical evidence that there exist a deformation of the Lawson surface of genus 2 through compact CMC surface $f\colon M\to S^3$ of genus 2 which preserves the extrinsic symmetries φ_2, φ_3 and τ. We call these surfaces Lawson symmetric CMC surfaces. We shortly explain how to generalize our theory to Lawson symmetric CMC surfaces.

Due to the Lawson correspondence, one can treat CMC surfaces in S^3 in the same way as minimal surfaces, see for example [B]. Consequently, there also exists an associated family of flat $\mathrm{SL}(2,\mathbb{C})$-connections $\lambda\in\mathbb{C}^*\mapsto\nabla^\lambda$ which are unitary along the unit circle. In contrast to the minimal case the Sym points $\lambda_1\neq\lambda_2\in S^1$, at which the connections ∇^{λ_i} are trivial, must not be the negative of each other. Then, the CMC surface is obtained as the gauge between these two flat connections, but the mean curvature is now given by $H=i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. For $\lambda_1=-\lambda_2$ we get a minimal surface.

As the extrinsic symmetries φ_2, φ_3 and τ are (assumed to be) holomorphic on the surface, the Riemann surface structure is almost fixed: It is given by the algebraic equation

$$y^3=\frac{z^2-a}{z^2+a}$$

for some $a\in\mathbb{C}^*$. The Lawson Riemann surface structure is then given by $a=1$. Moreover, the every individual connection ∇^λ of the associated family is equivariant with respect to the Lawson symmetries. All the theory developed for flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connections on the Lawson surface carries over to flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connections on M : The moduli space of Lawson symmetric holomorphic structures is double covered by the Jacobian of a complex 1-dimensional torus. This torus itself is given by the equation $y^2=\frac{z^2-a}{z^2+a}$. There is a $2:1$ correspondence between gauge equivalence classes of flat line bundle connections on the above mentioned torus and gauge equivalence classes of flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connections on M away from divisors in the corresponding moduli spaces. The correspondence extends to these divisors in the sense of Theorem 3 and Theorem 4. The concrete formulas are analogous to the case of the Lawson surface.

From the observation that the moduli spaces of the flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connections can be described analogously to the case of the Lawson surface itself, it is clear that the definition and the basic properties of the spectral curve carries over to Lawson symmetric CMC surfaces of genus 2. Of course, the extrinsic closing condition changes, as well as the precise form of the energy formula.

Appendix A. The associated family of flat connections

In this appendix we shortly recall the gauge theoretic description of minimal surfaces in S^3 which is due to Hitchin [H]. For more details, one can also consult [He].

The Levi-Civita connection of the round S^3 is given with respect to the left trivialization $TS^3=S^3\times\operatorname{im}\mathbb{H}$ as

$$\nabla=d+\frac{1}{2}\omega,$$

where ω is the Maurer-Cartan form of S^3 which acts via adjoint representation.

The hermitian complex rank 2 bundle $V=S^3\times\mathbb{H}$ with complex structure given by right multiplication with $i\in\mathbb{H}$ is a spin bundle for S^3 : The Clifford multiplication is given by $TS^3\times V\to V;(\lambda,v)\mapsto\lambda v$ where $\lambda\in\operatorname{Im}\mathbb{H}$ and $v\in\mathbb{H}$, and this identifies TS^3 as the skew symmetric trace-free complex linear endomorphisms of V. There is an unique complex unitary connection on V which induces on $TS^3 \subset \mathrm{End}(V)$ the Levi-Civita connection. It is given by

$$\nabla = \nabla^{spin} = d + \frac{1}{2}\omega,$$

where the $\mathrm{Im}\,\mathbb{H}-$valued Maurer-Cartan form acts by left multiplication in the quaternions.

Let M be a Riemann surface and $f\colon M \to S^3$ be a conformal immersion. Then the pullback $\phi = f^*\omega$ of the Maurer-Cartan form satisfies the structural equations

$$d^\nabla \phi = 0, \tag{A.1}$$

where $\nabla = f^*\nabla = d + \frac{1}{2}\phi$. The conformal map f is minimal if and only if it is harmonic, i.e., if

$$d^\nabla * \phi = 0. \tag{A.2}$$

holds. Let

$$\frac{1}{2}\phi = \Phi - \Phi^*$$

be the decomposition of $\phi \in \Omega^1(M; f^*TS^3) \subset \Omega^1(M; \mathrm{End}_0(V))$ into the complex linear and complex anti-linear parts. As f is conformal

$$\det \Phi = 0.$$

Note that f is an immersion if and only if Φ is nowhere vanishing. In that case $\ker \Phi = S^*$ is the dual to the holomorphic spin bundle S associated to the immersion. The Equations A.1 and A.2 are equivalent to

$$\nabla''\Phi = 0, \tag{A.3}$$

where $\nabla'' = \frac{1}{2}(d^\nabla + i * d^\nabla)$ is the underlying holomorphic structure of the pull-back of the spin connection on V. Of course (A.3) does not contain the property that $\nabla - \frac{1}{2}\phi = d$ is trivial. Locally this is equivalent to

$$F^\nabla = [\Phi \wedge \Phi^*] \tag{A.4}$$

as one easily computes.

From (A.3) and A.4 one sees that the associated family of connections

$$\nabla^\lambda := \nabla + \lambda^{-1}\Phi - \lambda\Phi^* \tag{A.5}$$

is flat for all $\lambda \in \mathbb{C}^*$, unitary along $S^1 \subset \mathbb{C}^*$ and trivial for $\lambda = \pm 1$. This family contains all the informations about the surface, i.e., given such a family of flat connections one can reconstruct the surface as the gauge between ∇^1 and ∇^{-1}. Using Sym-Bobenko formulas one can also make CMC surfaces in S^3 and $\mathbb{R}^3$ out of the family of flat connections. These CMC surfaces do not close in general.

The family of flat connections can be written down in terms of the well-known geometric data associated to a minimal surface:

Proposition A.1. *Let $f\colon M \to S^3$ be a conformal minimal immersion with associated complex unitary rank 2 bundle (V, ∇) and with induced spin bundle S. Let $V = S^{-1} \oplus S$ be the unitary decomposition, where $S^{-1} = \ker \Phi \subset V$ and Φ is the $K-$part of the differential of f. The Higgs field $\Phi \in H^0(M, K\,\mathrm{End}_0(V))$ can be identified with*

$$\Phi = \frac{1}{2} \in H^0(M; K\,\mathrm{Hom}(S, S^{-1})),$$

and its adjoint Φ^* *is given by* $i\,\mathrm{vol}$ *where* vol *is the volume form of the induced Riemannian metric. The family of flat connections is given by*

$$\nabla^\lambda = \begin{pmatrix} \nabla^{spin^*} & -\frac{i}{2}Q^* \\ -\frac{i}{2}Q & \nabla^{spin} \end{pmatrix} + \lambda^{-1}\Phi - \lambda\Phi^*,$$

where ∇^{spin} *is the spin connection corresponding to the Levi-Civita connection on* M *and* Q *is the Hopf differential of* f.

Appendix B. Lawson's genus 2 surface

We recall the construction of Lawson's minimal surfaces of genus 2 in S^3, see [L]. Consider the round 3−sphere

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C} \oplus \mathbb{C}$$

and the geodesic circles $C_1 = S^3 \cap (\mathbb{C} \oplus \{0\})$ and $C_2 = S^3 \cap (\{0\} \oplus \mathbb{C})$ on it. Take the six points

$$Q_k = (e^{i\frac{\pi}{3}(k-1)}, 0) \in C_1$$

in equidistance on C_1, and the four points

$$P_k = (0, e^{i\frac{\pi}{2}(k-1)}) \in C_2$$

in equidistance on C_2. A fundamental piece of the Lawson surface is the solution to the Plateau problem for the closed geodesic convex polygon $\Gamma = P_1Q_2P_2Q_1$ in S^3. This means that it is the smooth minimal surface which is area minimizing under all surfaces with boundary Γ. To obtain the Lawson surface one reflects the fundamental piece along the geodesic through P_1 and Q_1, then one rotates everything around the geodesic C_2 by $\frac{2}{3}\pi$ two times, and in the end one reflects the resulting surface across the geodesic C_1. Lawson has shown that the surface obtained in this way is smooth at all points. It is embedded, orientable and has genus 2. The umbilics, i.e., the zeros of the Hopf differential Q are exactly at the points $P_1, .., P_4$ of order 1.

A generating system of the symmetry group of the Lawson surface is given by

- the $\mathbb{Z}^2$−action generated by φ_2 with $(a,b) \mapsto (a,-b)$; it is orientation preserving on the surface and its fix points are $Q_1, ..Q_6$;
- the $\mathbb{Z}_3$−action generated by the rotation φ_3 around P_1P_2 by $\frac{2}{3}\pi$, i.e., $(a,b) \mapsto (e^{i\frac{2}{3}\pi}a, b)$, which is holomorphic on M with fix points $P_1, .., P_4$;
- the reflection at P_1Q_1, which is antiholomorphic; it is given by $\gamma_{P_1Q_1}(a,b) = (\bar{a}, \bar{b})$;
- the reflection at the sphere S_1 corresponding to the real hyperplane spanned by $(0,1), (0,i), (e^{\frac{1}{6}\pi i}, 0)$, with $\gamma_{S_1}(a,b) = (e^{\frac{\pi}{3}i}\bar{a}, b)$; it is antiholomorphic on the surface,
- the reflection at the sphere S_2 corresponding to the real hyperplane spanned by $(1,0), (i,0), (0, e^{\frac{1}{4}\pi i})$, which is antiholomorphic on the surface and satisfies $\gamma_{S_2}(a,b) = (a, i\bar{b})$.

Note that all these actions commute with the $\mathbb{Z}_2$−action. The last two fix the polygon Γ. They and the first two map the oriented normal to itself. The third one maps the oriented normal to its negative.

Using the symmetries, one can determine the Riemann surface structure of the Lawson surface $f\colon M \to S^3$ as well as the other holomorphic data associated to it:

Proposition B.1. *The Riemann surface M associated to the Lawson genus 2 surface is the three-fold covering $\pi\colon M\to\mathbb{CP}^1$ of the Riemann sphere with branch points of order 2 over $\pm1,\pm i\in\mathbb{CP}^1$, i.e., the compactification of the algebraic curve*

$$y^3=\frac{z^2-1}{z^2+1}.$$

The hyper-elliptic involution is given by $(y,z)\mapsto(y,-z)$ and the Weierstrass points are $Q_1,..,Q_6$. The Hopf differential of the Lawson genus 2 surface is given by

$$Q=\pi^*\frac{ir}{z^4-1}(dz)^2$$

for a nonzero real constant $r\in\mathbb{R}$ and the spin bundle S of the immersion is

$$S=L(Q_1+Q_3-Q_5).$$

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APPENDIX D

Deformations of symmetric CMC surfaces in the 3-sphere

by Sebastian Heller and Nicholas Schmitt
submitted to *Experimental Mathematics*
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In dieser Arbeit habe ich die den Experimenten zugrundeliegende Theorie entwickelt und das initiale Experiment bezüglich der Lawsonfläche durchgeführt. Darüber hinaus habe ich die daraus gewonnenen Daten in das DPW Setup übertragen um die weiteren Experimente und die Visualisierung zu ermöglichen. Außerdem habe ich mit meinem theoretischen Wissen zum Überwinden numerischer Probleme beigetragen.

DEFORMATIONS OF SYMMETRIC CMC SURFACES IN THE 3-SPHERE

SEBASTIAN HELLER AND NICHOLAS SCHMITT

Abstract. In this paper we numerically construct CMC deformations of the Lawson minimal surfaces $\xi_{g,1}$ using a spectral curve and a DPW approach to CMC surfaces in spaceforms.

1. Introduction

The moduli spaces of CMC (constant mean curvature) spheres and embedded CMC tori in the 3-sphere are well understood by now. The only CMC spheres are totally umbilic due to the vanishing of their Hopf differential. Brendle [3] and Andrews and Li [1] have classified the embedded minimal and embedded CMC tori. Additionally, all CMC immersions from a torus into 3-dimensional space forms are given rather explicitly in terms of algebro-geometric data on their associated spectral curves [22, 12, 2]. These integrable system methods are also applied to study the moduli space of all CMC tori, see for example [18, 20].

In contrast, higher genus CMC surfaces in S^3 are not very well understood. There are examples like the Lawson minimal surfaces [21] which exist for all genera. All known examples have been constructed by implicit methods from geometric analysis. However, there is no theory which describes the space of all CMC surfaces of higher genus, nor is there any classification of the embedded ones.

The study of CMC surfaces via integrable systems is based on the associated family

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a fixed hermitian rank 2 bundle [12]. For minimal surfaces in S^3 the flatness of this family of connections is just a gauge theoretic reformulation of the Gauss-Codazzi and harmonic map equations. For CMC surfaces, the family of flat connections comes from the Lawson correspondence together with the Sym-Bobenko formula. The connections ∇^λ are unitary for $\lambda \in S^1 \subset \mathbb{C}^*$ and trivial at two Sym points $\lambda_1 \neq \lambda_2 \in S^1$. The immersion can be obtained as the gauge between ∇^{λ_1} and ∇^{λ_2}, and its mean curvature is given by $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. By loop group factorization methods, CMC surfaces can also be constructed out of families of flat connections which have a certain asymptotic behavior at $\lambda = 0$ and are unitarizable along the unit circle, i.e., unitary with respect to a λ-dependent metric (see Theorem 2).

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These families of flat connections can be constructed by two different methods: the spectral curve approach and the DPW approach. The first describes the family via flat line bundles parametrized by a spectral curve, i.e., a double covering of the spectral plane, as in Theorem 4. The flat line bundles are defined on a double covering of our Riemann surface, and the moduli space of them is given by an affine bundle over the Prym variety. The second uses a so-called DPW potential [5], a loop of meromorphic $\mathfrak{sl}(2,\mathbb{C})$-valued 1-forms. The first method has the advantage that it is easier to deal with the unitarity condition, while the second can take advantage of the implementation of DPW in the XLab software suite.

The main difficulty in constructing higher genus CMC surfaces is that the generic connection ∇^λ is irreducible. Therefore, it is not understood by now how to make families of flat connections which are unitarizable along the unit circle. A flat connection is unitarizable if and only if its monodromy representation is unitary modulo conjugation. This is a condition which can be tackled numerically: using numerical ODE solvers one can compute the monodromy representation, and then apply basic results like Proposition 2 to determine whether a connection is unitarizable.

In the case of the spectral curve approach one also has theoretical support: as a consequence of the Narasimhan-Seshadri theorem it is known that for every holomorphic line bundle there exists exactly one flat compatible connection such that the corresponding flat $\mathrm{SL}(2,\mathbb{C})$-connection is unitary. This enables us to numerically determine the space of unitary connections. With this knowledge we can numerically search for families of flat connections which are unitarizable along the unit circle. With this spectral curve approach we reconstructed the Lawson surface $\xi_{2,1}$.

In the DPW approach, on the other hand, we combine these two steps, directly computing the families of unitarizable DPW potentials. The explicit translation from the spectral curve to the DPW theory provided initial data and elucidated the conditions at the sym points. We have carried out the DPW experiments for a special class of CMC surfaces, namely Lawson symmetric ones. They are equipped with a large group of extrinsic orientation preserving symmetries, which are holomorphic automorphisms on the Riemann surface. Due to this symmetry group, the moduli space of the possible Riemann surface structures is complex 1-dimensional. Its cotangent space is spanned by a quadratic differential which is the Hopf differential of a possible Lawson symmetric CMC immersion. A nice feature of such an immersion is that its curvature lines are closed (see Figure 1).

Our experiments give strong evidence to the existence of real 1-dimensional families of Lawson symmetric CMC surfaces passing through the Lawson surfaces $\xi_{g,1}$ themselves (see Figure 4). In the case of $g = 1$ this family is known from the spectral theory of CMC tori. We reconstructed this 1-parameter family numerically as a test of our procedure, bifurcating into the 2-lobed Delaunay tori of spectral genus 1, or continuing along the homogeneous tori of spectral genus 0. For higher genus Lawson symmetric CMC surfaces such bifurcations into higher spectral genus did not appear; these families continue until they collapse into double coverings of minimal spheres (as the Delaunay tori do). In genus 2 we have also found a family of Lawson symmetric CMC surfaces, disjoint from the family passing through $\xi_{2,1}$, which seems to converge

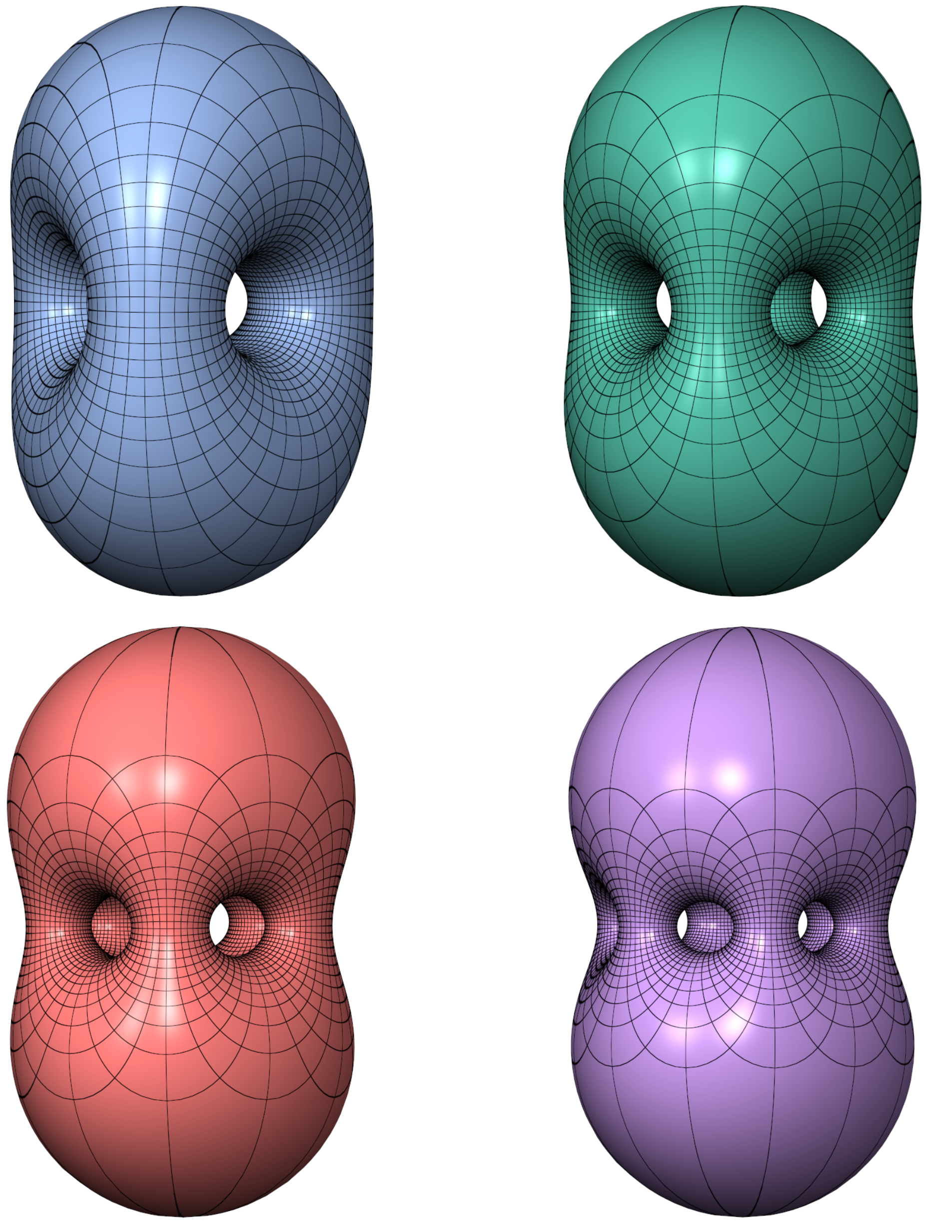


FIGURE 1. The Lawson surfaces $\xi_{g,1}$ of genus $g = 2, 3, 4, 5$.

to a threefold covering of a CMC sphere (see Figure 5). Altogether, our experiments begin to map out the moduli space of Lawson symmetric CMC surface of genus 2.

The paper is organized as follows: In chapter 2 we describe the necessary theory for our experiments. In chapter 3 we discuss the first experiments on the Lawson surface of genus 2 via the spectral curve approach. Chapter 4 concerns the numerical deformations of Lawson symmetric CMC surfaces of genus 2. Chapter 5 collects experiments with Lawson symmetric surfaces of higher genus. In the last chapter 6 we give a short outlook on the computational aspects of our studies.

2. Theoretical background

We shortly recall the well known description of conformal CMC immersions $f\colon M \to S^3$, where M is a Riemann surface and S^3 is equipped with its round metric [12, 2, 8]. Due to the Lawson correspondence, there is a unified treatment for all mean curvatures $H \in \mathbb{R}$:

Theorem 1. *Let $f\colon M \to S^3$ be a conformal CMC immersion. Then there exists an associated family of flat* $\mathrm{SL}(2,\mathbb{C})$*-connections*

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

on a hermitian rank 2 bundle $V \to M$ which is unitary along $S^1 \subset \mathbb{C}^*$ and trivial at $\lambda_1 \neq \lambda_2 \in S^1$. Here, Φ is a nowhere vanishing complex linear 1-form which is nilpotent and Φ^* is its adjoint. Conversely, the immersion f is given as the gauge between ∇^{λ_1} and ∇^{λ_2} where we identify $\mathrm{SU}(2) = S^3$, and its mean curvature is $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$. Therefore, every family of flat $\mathrm{SL}(2,\mathbb{C})$-connections satisfying the properties above determines a conformal CMC immersion.

Note that the complex linear part of the family of flat connections extends to $\lambda = \infty$ whereas the complex anti-linear part extends to $\lambda = 0$. It is well known [12], that for compact CMC surfaces which are not totally umblic, the generic connection ∇^λ of the associated family is not trivial. Moreover, for CMC immersions from a compact Riemann surface of genus $g \geq 2$, the generic connection ∇^λ of the associated family is irreducible [8].

An important observation is that it is often enough to work with a family connections which is only gauge equivalent (in a certain sense) to the associated family of a CMC surface. This enables us to use our preferred connections like meromorphic ones. In our situation we make use of the following theorem in order to construct compact CMC surfaces.

Theorem 2. *Let $U \subset \mathbb{C}$ be an open set containing the disc of radius $1+\epsilon$. Let $\lambda \in U \setminus \{0\} \mapsto \tilde\nabla^\lambda$ be a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

- *the asymptotic at $\lambda = 0$ is given by*
$$\tilde\nabla^\lambda \sim \lambda^{-1}\Psi + \tilde\nabla + \ldots$$
where $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ is nowhere vanishing and nilpotent;
- *for all $\lambda \in S^1 \subset U \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde\nabla^\lambda$ is unitary with respect to this metric;*
- *$\tilde\nabla^\lambda$ is trivial for $\lambda_1 \neq \lambda_2 \in S^1$.*

Then there exists a unique (up to spherical isometries) CMC surface $f\colon M \to S^3$ of mean curvature $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ such that its associated family of flat connections ∇^λ and the family $\tilde\nabla^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g(\lambda) = \tilde\nabla^\lambda$ for all λ.

In the above form, this theorem was proven in [10], but there are earlier variants adapted to the DPW approach to k-noids [23, 6].

Remark 1. The theorem remains true, if there exists λ-independent apparent singularities of the connections $\tilde{\nabla}^\lambda$. It also remains true, if there exists finitely many point on the unit circle, where the monodromy is not unitary. In both cases, the corresponding singularities (on the Riemann surface in the first case, and on the spectral plane in the second) are captured in the positive part of the Iwasawa decomposition (see the proof of this theorem in [10]). Therefore, the actual associated family of flat connections ∇^λ has no singularities anymore, and the CMC immersion is well-defined.

From now on we focus on CMC immersions from a compact Riemann surfaces of genus 2 which have the following (extrinsic, space orientation preserving) symmetries:

- an involution φ_2 with exactly 6 fix points which is holomorphic on the surface and commutes with the other symmetries;
- a $\mathbb{Z}_3$-symmetry generated by φ_3 with 4 fix points which is also holomorphic on the surface;
- another holomorphic involution τ with only 2 fix points.

These surfaces are called Lawson symmetric CMC surfaces (of genus 2). The symmetries already fix the Riemann surface structure up to one complex parameter. To be more precise, the underlying Riemann surface is given by the equation

$$y^3 = \frac{z^2 - z_0^2}{z^2 - z_1^2}.$$

Clearly, Lawson symmetric Riemann surfaces corresponding to tuples $(z_0, z_1, -z_0, -z_1)$ with the same cross-ratio are isomorphic. The Riemann surface structure of the Lawson surface $\xi_{2,1}$ is given by $z_0 = 1$, $z_1 = i$. In this picture the symmetries are given on the Riemann surface by $\varphi_2(y, z) = (y, -z)$, $\varphi_3(y, z) = (e^{\frac{2}{3}\pi i} y, z)$ and $\tau(y, z) = ((\frac{z_0}{z_1})^{\frac{2}{3}} \frac{1}{y}, \frac{z_0 z_1}{z})$.

There is a method called dressing which makes new CMC surfaces out of old, see for example [4]. The idea is that a CMC surface is in general not uniquely determined by the family of gauge equivalence classes of its associated family of flat connections. It was shown in [10] that a dressing deformation of a Lawson symmetric CMC surface is not Lawson symmetric anymore. Therefore, for Lawson symmetric CMC surfaces it is enough to know the family of gauge equivalence classes of its associated family of flat connections.

Altogether, in order to find CMC surfaces we need to find a holomorphic curve in the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections on M which may be lifted to a family of flat connections satisfying the properties of Theorem 2. Moreover for Lawson symmetric CMC surfaces, we do not need to consider the moduli space of all flat $\mathrm{SL}(2, \mathbb{C})$ connections but only those which are equivariant with respect to φ_2, φ_3 and τ. We call these connections flat Lawson symmetric connections.

2.1. **The spectral curve approach.** One way to construct families of (gauge equivalence classes of) flat connections is based on Hitchin's abelianization [11]. We will restrict our discussion to the case of flat Lawson symmetric connections ∇. On a Riemann surface, a connection can be decomposed into a holomorphic and an antiholomorphic structure

$$\nabla = \bar{\partial}^{\nabla} + \partial^{\nabla},$$

where $\bar{\partial}^{\nabla}$ maps to complex anti-linear 1-forms and ∂^{∇} maps to complex linear 1-forms. There are several reasons why it is useful to consider holomorphic structures in the discussion of flat connections on a (compact) Riemann surface: By the Narasimhan-Seshadri theorem there exists for a generic holomorphic structure on a degree 0 bundle a unique flat connection ∇ such that ∇ is unitary with respect to a suitable hermitian metric and such that $\bar{\partial}^{\nabla} = \bar{\partial}$. Second, if ∇ is already flat, and we add a (trace-free) complex linear 1-form $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ then $\nabla + \Psi$ is flat if and only if Ψ is holomorphic. Such 1-forms are called Higgs fields. This observation shows that the (moduli) space of flat connections is an affine bundle over the (moduli) space of holomorphic structures, where the fibers consist of the finite dimensional space of Higgs fields, at least at its smooth points. Moreover, in the generic fiber there is a unique point such that the corresponding flat connection is unitary for a suitable hermitian metric. And lastly, as we have already mentioned above, the family of holomorphic structures $\bar{\partial}^{\nabla^{\lambda}}$ extends to $\lambda = 0$. Therefore it seems to be very useful to discuss the moduli space of flat connections as an affine bundle over the moduli space of holomorphic structures in order to analyze the asymptotic behavior of ∇^{λ} for $\lambda \to 0$.

As we are only interested in Lawson symmetric connections, we only need to deal with Lawson symmetric Higgs fields, i.e., Higgs fields which are also equivariant with respect to φ_2, φ_3 and τ. It was shown in [10] that for a generic Lawson symmetric holomorphic structure $\bar{\partial}^{\lambda}$, the Lawson symmetric Higgs fields constitute a complex line. Their determinant is a holomorphic quadratic differential and invariant under the symmetries. Therefore, for a generic Lawson symmetric holomorphic structure and a non-zero Lawson symmetric Higgs field Ψ its determinant $\det \Psi$ is a non-zero multiple of the pull-back of $\frac{dz^2}{(z^2-z_0^2)(z^2-z_1^2)}$. Its zeros are simple, so the eigenlines of Ψ are only well defined on a double covering $\pi\colon \tilde{M} \to M$. Clearly, $\tilde{M}$ inherits the symmetries of M. Note that $\tilde{M}/\mathbb{Z}_3$ is a torus while $M/\mathbb{Z}_3$ is the projective line. The eigenlines $L_{\pm}$ of Lawson symmetric Higgs fields with non-zero determinant satisfy $L_+ \otimes L_- = \pi^* K_M$. Therefore, the eigenlines for all those Lawson symmetric Higgs fields constitute an affine Prym variety for $\tilde{M} \to M$. As a base point of this affine Prym variety we fix the pull-back of the dual of the (unique) Lawson symmetric spin bundle $S^* = L(-Q_1 - Q_3 + Q_5) \to M$, where the points Q_1, Q_3 and Q_5 are Weierstrass points which make an orbit under the $\mathbb{Z}_3$-action. This enables us to understand the moduli space of Lawson symmetric holomorphic structures $\bar{\partial}^{\lambda}$.

Proposition 1. [10] *There exists an even holomorphic map*

$$\Pi\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathcal{S} = \mathbb{P}^1 \tag{2.1}$$

of degree 2 to the moduli space $\mathcal{S}$ of Lawson symmetric holomorphic bundles. This map is determined by $\Pi(L) = [\bar\partial]$ for $L \neq \underline{\mathbb{C}} \in Jac(\tilde M/\mathbb{Z}_3)$ such that $\pi^* S^* \otimes \tilde\pi^* L$ is isomorphic to an eigenline bundle of a symmetric Higgs field of the Lawson symmetric holomorphic rank two bundle $(V, \bar\partial)$. The branch points of Π are the spin bundles of $\tilde M/\mathbb{Z}_3$ and the branch images of the non-trivial spin bundles are exactly the isomorphism classes of the strictly semi-stable holomorphic bundles, i.e., the corresponding unitary flat connections are reducible.

Away from the zeros of $\det \Psi$, the eigenlines of a Lawson symmetric Higgs field Ψ with respect to a Lawson symmetric holomorphic structure $\bar\partial$ span the holomorphic rank 2 bundle $\pi^* V$, i.e., there is a holomorphic map $\phi\colon L_+ \oplus L_- \to \pi^* V$ which is an isomorphism away from the zeros. A flat connection ∇ with $\bar\partial^\nabla = \bar\partial$ can be pulled back to $L_+ \oplus L_- \to \tilde M$ in order to yield a meromorphic connection also denoted by ∇. The second fundamental forms of ∇ with respect to the eigenlines are meromorphic line bundle valued 1-forms, and the residuum of ∇ at the zeros of $\det \Psi$ can be easily computed. Adding a multiple of the Higgs field Ψ to ∇ on V corresponds to adding a diagonal 1-form to ∇ on $L_+ \oplus L_-$. In our Lawson symmetric situation, the connection ∇ on $L_+ \oplus L_-$ is given explicitly in terms of theta-functions on the torus $\tilde M/\mathbb{Z}_3$. But it is even easier to work on the quotient of $\tilde M/\mathbb{Z}_3$ by the symmetries φ_2 and τ which a again is a torus, denoted by T^2. We will only state the formulas in the case of the Lawson Riemann surface structure. In this case $\tilde M/\mathbb{Z}_3$ as well as T^2 are square tori. If we identify $T^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ then a Lawson flat symmetric connections corresponds to the connection 1-form

$$\omega = \omega_{x,a} = \begin{pmatrix} \pi a dz - \pi x d\bar z & c\frac{\theta(z-2x)}{\theta(z)} e^{-4\pi i x \operatorname{Im}(z)} dz \\ c\frac{\theta(z+2x)}{\theta(z)} e^{4\pi i x \operatorname{Im}(z)} dz & -\pi a dz + \pi x d\bar z \end{pmatrix} \tag{2.2}$$

where θ is the theta-function of T^2 which has a simple zero at 0 and

$$c = \frac{1}{6}\sqrt{\frac{\theta'(0)^2}{\theta(2x)\theta(-2x)}}. \tag{2.3}$$

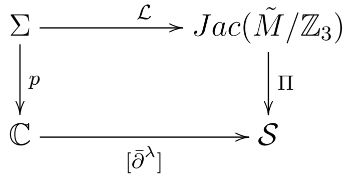
The corresponding holomorphic structure $\bar\partial^\nabla$ on the rank 2 bundle is determined by $\Pi(\bar\partial^0 \pm \pi x d\bar z)$, and adding a multiple of the Higgs fields on ∇ is equivalent to adding a multiple of the diagonal matrix with entries dz and $-dz$ on ω. This discussion also leads to a full understanding of the moduli space of Lawson symmetric flat connections:

Theorem 3. [10] *Let $\bar\partial$ be a Lawson symmetric semi-stable holomorphic structure on a rank 2 vector bundle over M. Assume that $\bar\partial$ is determined by the non-trivial holomorphic line bundle $L \in Jac(\tilde M/\mathbb{Z}_3)$, i.e., $\Pi(L) = [\bar\partial]$. Then there is a 1:1 correspondence between holomorphic connections on $L \to \tilde M/\mathbb{Z}_3$ and flat Lawson symmetric connections ∇ with $\nabla'' = \bar\partial$. The correspondence is given explicitly by the connection 1-form (2.2).*

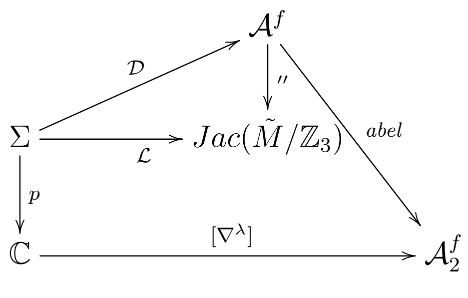
The remaining flat Lawson symmetric connections are given by two lines lying over the point $\underline{\mathbb{C}} \in Jac(\tilde M/\mathbb{Z}_3)$. For this case $x = 0$, and formula 2.2 breaks down. This is not surprising, since the holomorphic structure corresponding to $\underline{\mathbb{C}} \in Jac(\tilde{M}/\mathbb{Z}_3)$ is the holomorphic direct sum $S^* \oplus S \to M$ which does not provide a flat connection. Nevertheless, the gauge orbit of this holomorphic structure is infinitesimal close to the gauge orbits of two other holomorphic structures, namely the holomorphic structure corresponding to the uniformization of the Riemann surface (which does not provide a unitary flat connection) and the holomorphic structure $\bar{\partial}^{\nabla^0}$ given by the (well-defined) limit of $\bar{\partial}^{\nabla^\lambda}$ for $\lambda \to 0$ of the associated family. Both holomorphic structures admit an affine line of Lawson symmetric flat connections, and they are given as special limits of (2.2), see [10] for details.

Theorem 3 shows that flat Lawson symmetric connections are uniquely determined by a flat line bundle connection on $\tilde{M}/\mathbb{Z}_3$. But this flat line bundle is not unique as its dual gives rise to the same flat $\mathrm{SL}(2, \mathbb{C})$-connection. Therefore, in order to parametrize families of flat connections $\lambda \in \mathbb{C}^* \to \nabla^\lambda$, one needs in general a double covering $\Sigma \to \mathbb{C}^*$ in order to parametrize the corresponding family of flat line bundles. This leads to the following picture:

Theorem 4. [10] *Let $\lambda \mapsto \nabla^\lambda$ be the associated family of a conformal Lawson symmetric CMC immersion of a compact Riemann surface of genus 2. Then there exists a Riemann surface $p\colon \Sigma \to \mathbb{C}$ double covering the spectral plane $\mathbb{C}$ together with a map $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ such that*



commutes. The spectral curve Σ branches at 0. Moreover, there exists a meromorphic lift $\mathcal{D}$ with a first order pole over $\lambda = 0$ into the affine moduli space $\mathcal{A}^f$ of flat line bundles on $\tilde{M}/\mathbb{Z}_3$ such that



commutes, where $\mathcal{A}_2^f$ is the moduli space of flat Lawson symmetric connections on M and abel is the map discussed in Theorem 3.

Conversely, a triple $(\Sigma, \mathcal{L}, \mathcal{D})$ as above determines a family of Lawson symmetric flat connection on M which has the asymptotic behavior as in Theorem 2. In order to obtain a CMC immersion the family of flat connections has to satisfy the reality condition and the closing condition. The second condition is easy compared to the first one as one knows which flat line bundle on the torus $\tilde{M}/\mathbb{Z}_3$ determines the trivial connection on M: It is the flat unitary line bundle which has monodromy

-1 along both of the "standard" generators of the first fundamental group of the torus $\tilde{M}/\mathbb{Z}_3$. The main difficulty is to find spectral data $(\Sigma, \mathcal{L}, \mathcal{D})$ which satisfy the reality condition, i.e., the corresponding family of flat $\mathrm{SL}(2, \mathbb{C})$-connections must be unitarizable along the unit circle. We work out the necessary theory to attack this problem numerically. As we have discussed above, for each (Lawson symmetric) holomorphic structure, there exists a unique compatible flat (Lawson symmetric) $\mathrm{SL}(2, \mathbb{C})$-connection which is unitarizable, i.e., unitary with respect to a suitable chosen metric. Clearly, this property is equivalent to have unitarizable monodromy. From Theorem 3 we see that for each holomorphic line bundle on the torus $\tilde{M}/\mathbb{Z}_3$ there is a compatible flat connection such that the corresponding flat Lawson symmetric $\mathrm{SL}(2, \mathbb{C})$-connection is unitarizable. Therefore, we obtain a (real analytic) section

$$a^u \in \Gamma(Jac(\tilde{M}/\mathbb{Z}_3), \mathcal{A}^F)$$

of the affine moduli space of flat line bundles over the Jacobian. With the same notations as used in (2.2) this section is given in the case of the Lawson Riemann surface by

$$a^u(x) = -\frac{1}{12\pi}\frac{\theta'(-2x)}{\theta(-2x)} + \frac{1}{12\pi}\frac{\theta'(2x)}{\theta(2x)} + \frac{1}{3}x + \frac{2}{3}\bar{x} + b(x), \tag{2.4}$$

where $b\colon Jac(\tilde{M}/\mathbb{Z}_3) \to \mathbb{C}$ is a doubly periodic real-analytic function. This function can easily be approximated to arbitrary order, see section 3.

The reality condition can now be rephrased as follows: For all $\mu \in \Sigma$ with $p(\mu) \in S^1$ the spectral data have to satisfy

$$a^u(\mathcal{L}(\mu)) = \mathcal{D}(\mu). \tag{2.5}$$

We will use this equation later on to determine the spectral data of the Lawson surface of genus 2 numerically, see Figure 3.

2.2. **The DPW approach.** Another approach to CMC surfaces in S^3 was developed by Dorfmeister, Pedit and Wu [5]. The basic idea is to work with families of meromorphic connections with respect to the trivial holomorphic rank 2 bundle $\underline{\mathbb{C}}^2 \to M$ instead of using the varying holomorphic structures $\bar{\partial}^\lambda$. Clearly, one needs to allow poles in the connection 1-forms as the only holomorphic unitarizable connection on $\underline{\mathbb{C}}^2 \to M$ over a compact Riemann surface is the trivial one.

In order to construct CMC surfaces one tries to find a DPW potential

$$\eta = \eta(\lambda) = \lambda^{-1}\eta_{-1} + \eta_0 + \eta_1\lambda + ..,$$

i.e., a meromorphic λ-family of meromorphic $\mathfrak{sl}(2, \mathbb{C})$-valued 1-forms on M with first order pole in λ such that the corresponding family of flat connections $\nabla^\lambda = d + \eta(\lambda)$ satisfies the properties of Theorem 2. In general, the DPW potential η does not exist on the whole spectral plane $\mathbb{C}^*$ but only on a small punctured disk around $\lambda = 0$. Moreover, it is not clear in general how many (possibly varying) poles one needs to allow in order to obtain a potential which give rise to a closed CMC surface in S^3. In the case of the Lawson surface of genus 2, the existence and precise form up to two unknown functions in λ of such a potential was determined in [9]. In the more general situation of Lawson symmetric CMC surfaces on a Riemann surface given by the equation

$$y^3 = \frac{z^2 - z_0^2}{z^2 - z_1^2} \tag{2.6}$$

one can easily prove by the same methods that a DPW potential is given by

$$\eta = \eta_{A,B} = \pi^* \begin{pmatrix} -\frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} + \frac{A}{z} & \lambda^{-1} - \frac{(A+\frac{2}{3})(A-\frac{1}{3})}{B} z^2 \\ \frac{B}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{\lambda A(A+1) z_0^2 z_1^2}{z^2(z^2-z_0^2)(z^2-z_1^2)} & \frac{2}{3}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{A}{z} \end{pmatrix} dz. \tag{2.7}$$

Here, A, B are λ-dependent holomorphic functions on a neighborhood of $\lambda = 0$ and $\pi\colon M \to M/\mathbb{Z}_3 = \mathbb{CP}^1$. All poles are apparent on M, i.e. the local monodromy around every pole is trivial. On the quotient $M/\mathbb{Z}_3 = \mathbb{CP}^1$ the poles at $z = 0$ and $z = \infty$ are still apparent whereas the conjugacy class of the monodromy around the poles at the four branch points $\pm z_0$ and $\pm z_1$ is given by the third root of the identity.

The functions A and B need to be chosen in such a way that the closing condition and the reality condition is satisfied for the family of flat connections $d + \eta(\lambda)$. As was proven in [9] there do not exists finite values for A, B and λ such that the holonomy of $d + \eta_{A,B}$ is trivial. Nevertheless, there exists values for A and B such that the monodromy is upper triangular, and these values will guarantee our closing condition. The reason behind this is that the gauge from the associated family of flat connections to the connections $d + \xi_{A(\lambda),B(\lambda)}$ is singular at the Sym points. This can be deduced by comparing the spectral curve approach with the DPW approach: As we have two different ways to describe Lawson symmetric flat $\mathrm{SL}(2, \mathbb{C})$-connections there must exist a transformation between them. This transformation

$$(x, a) \mapsto (A(x, a), B(x, a)) \tag{2.8}$$

satisfies that the connections $d+\omega_{x,a}$ and $d+\eta_{A(x,a),B(x,a)}$ are gauge equivalent whence pulled back to $\tilde{M}$. It can be computed explicitly in terms of theta functions of the torus $\tilde{M}/\mathbb{Z}_3$. The gauge gets singular at the trivial connection but the transformation holomorphically extends through the corresponding values of x and a. As a consequence, the corresponding meromorphic connection $d + \eta_{A(x,a),B(x,a)}$ on the 4-punctured projective line has only upper triangular monodromy and not a diagonal one. Using this observation we obtain the following generalized extrinsic closing conditions at the Sym pointsfor Lawson symmetric CMC surfaces of genus 2 : The functions A and B are related at the Sym points λ_1 and λ_2 by

$$B(\lambda_k) = S_k(\lambda_k) \quad \text{and} \quad B'(\lambda_k) = S_k'(\lambda_k) \tag{2.9}$$

where

$$S_k(\lambda) = z_k^2 \lambda R(\lambda) \quad \text{with} \quad R(\lambda) = A(\lambda)(A(\lambda) - \frac{1}{3}). \tag{2.10}$$

One can easily verify by hand that for functions A and B satisfying the above equations the flat connections $d+\xi A(\lambda_k), B(\lambda_k)$ have upper triangular monodromy. Note that Theorem 2 can still be applied, see Remark 1 or [23, 6].

For our numerical computations, we do not work with the DPW potential $d + \eta_{A,B}$ as it has a singularity at $z = 0$. One can easily gauge this apparent singularity away by the gauge $\begin{pmatrix} 1 & 0 \\ -\frac{A\lambda}{z} & 1 \end{pmatrix}$ to obtain a meromorphic potential $d+\tilde{\eta}_{A,B}$ which is smooth at $z=0$. Moreover, it satisfies

$$\varphi_2^*(d+\tilde{\eta}_{A,B}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(d+\tilde{\eta}_{A,B})\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

where φ_2 on $\mathbb{CP}^1$ is given by $z \mapsto -z$. This implies that at $z=0$ the monodromy matrices M_1, M_2, M_3 and M_4 around the poles z_0, z_1 $-z_0$ and z_1 with respect to the standard basis of $\mathbb{C}^2$ are related as follows

$$M_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M_1 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \text{ and } M_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M_2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

All these matrices are in $\mathrm{SL}(2,\mathbb{C})$ and of trace -1 as the singularities are apparent when pulled back to the threefold covering $M \to \mathbb{CP}^1$. We denote the traces of the products by

$$2t_{i,j} = \mathrm{tr}(M_i M_j).$$

The following Proposition gives an easy characterization of unitarizable representations which we apply in our experiments below:

Proposition 2. *Let the four matrices M_k be given as above such that they have no common eigenline. Then they are simultaneously unitarizable if and only if $t_{k,l} \in [-1,1]$ for all $k,l \in \{1,..,4\}$. This condition already holds if $t_{1,2} \in (-1,1)$ and $t_{1,3} \in (-1,1)$. In this case, the four matrices are unitarizable by a diagonal matrix.*

3. Experiments: The Lawson surface $\xi_{2,1}$

As we have described in section 2.1 we need to find a family of flat line bundles on the torus $\tilde{M}/\mathbb{Z}_3$ parametrized on the spectral curve $\Sigma \to \mathbb{C}$ which satisfies the reality condition (2.5) in order to construct the Lawson surface $\xi_{2,1}$. To do so, we first determine the set of unitary connections numerically, i.e., we compute the doubly-periodic function b in (2.4): For each $x \in \mathbb{C}$ and the holomorphic line bundle $\bar{\partial}^0 - \pi x d\bar{z}$ on $\tilde{M}/\mathbb{Z}_3$ we searched for the unique $a^u(x)$ such that the monodromy of the corresponding flat Lawson symmetric $\mathrm{SL}(2,\mathbb{C})$-connection is unitarizable. An irreducible flat connection is unitarizable if and only if the traces of all its individual monodromies are contained in the interval $[-2,2] \subset \mathbb{R}$. This leads naturally to a functional depending on a which can be numerically minimized by using a numerical ODE solver as implemented for example in *Mathematica*. This procedure was done for all points x in the torus $Jac(\tilde{M}/\mathbb{Z}_3)$ lying on a grid. The doubly-periodic function b, which is the difference between a^u and an explicitly known expression (2.4), can then be approximated by Fourier series on the Jacobian. For the Lawson Riemann surface the real part of the function b is shown in Figure 2 whereas its imaginary part is given via the formula $b(ix) = -ib(x)$ due to a real symmetry of the Lawson surface. Equipped with these numerical data, we searched for the spectral data of the Lawson surface. We have started with the assumption that the spectral curve does not branch over the closed punctured unit disc $\{\lambda \in \mathbb{C} \mid 0 < \lambda\bar{\lambda} \neq 1\}$. This assumption seems to be natural in view of the assertion concerning the branch points in Theorem 5 in

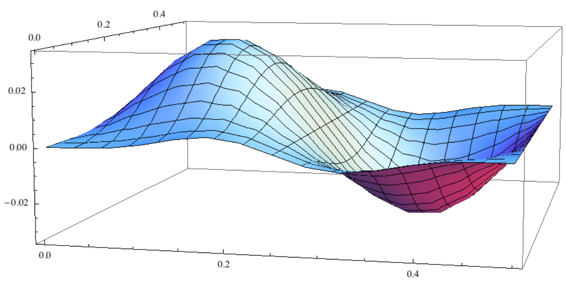


FIGURE 2. The real part of the function b

[10]. Then, an appropriate coordinate on Σ is given by t with $t^2 = \lambda$, and the maps $\mathcal{L}$ and $\mathcal{D}$ in Theorem 4 are given by holomorphic respectively meromorphic functions

$$x\colon \{t \in \mathbb{C} \mid t\bar{t} < 1\} \to \mathbb{C}$$

and

$$a\colon \{t \in \mathbb{C}^* \mid t\bar{t} < 1\} \to \mathbb{C},$$

where a has a first order pole at $t = 0$ and is holomorphic elsewhere. These functions can be approximated by their Taylor respectively Laurent series. Note that both functions are odd in t. Moreover, due to a symmetry of the Lawson surface covering $z \mapsto iz$ which is not space orientation preserving, see [9], the series coefficients x_k of the function x vanish if $k \mod 4 \neq 1$ and the series coefficients a_k of the function a vanish if $k \mod 4 \neq 3$. Moreover, the coefficients x_k are real multiples of $\frac{1+i}{4}$ and the coefficients a_k are real multiples of $\frac{1-i}{4}$ due to a anti-holomorphic symmetry of the Lawson surface covering $z \mapsto \bar{z}$.

The numerical search for the coefficients of x and a has been designed as follows: We have implemented the extrinsic closing condition from the beginning and searched for a finite number N of real coefficients of the numerical approximates x^N and a^N :

$$x^N(t) := \frac{1+i}{4}((1 - x_1 - x_2 - ... - x_N)t + x_1t^5 + x_2t^9 + .. + x_Nt^{4N+1})$$

and

$$a^N(t) := \frac{1-i}{4}((1 - a_1 - a_2 - ... - a_N)\frac{1}{t} + a_1t^3 + a_2t^7 + .. + a_Nt^{4N-1}).$$

Then we have chosen a finitely many $K >> 2N$ sample points t_k in equidistance on an arc with angle $\frac{\pi}{2}$ on the circle. Note that a quarter of the circle is enough due to the symmetries of the Lawson surface and of the functions. Then we numerically minimized the functional

$$\mathcal{F}\colon \mathbb{R}^{2N} \to \mathbb{R}; (x_1, .., x_N, a_1, ..a_N) \mapsto \sum_{k=1}^{K} \parallel a^u(x^n(t_k)) - a^n(t_k) \parallel^2$$

with the help of the *FindMinimum* routine in *Mathematica*. For example for $N = 10$ and $K = 120$ we have found a numerical root of this functional with an error of 10^{-12} which seems reasonable good compared with the expertise of earlier experiments on k-noids by the second author. The image of the unit circle of these functions is shown in Figure 3. Note that the (numerical computed) surface obtained out of the

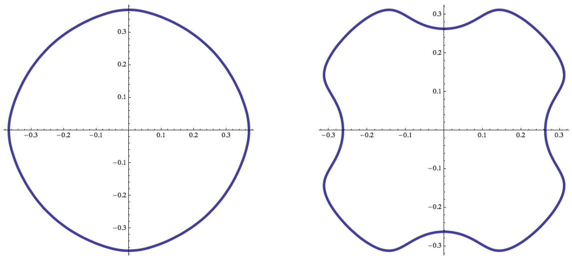


Figure 3. The spectral data (the complex anti-linear and the complex linear part of the flat connections on the eigenline bundles) of the Lawson genus 2 surface along the unit circle of the spectral plane. These data completely determine the associated family of flat connections of the Lawson surface $\xi_{2,1}$.

spectral data by means of Theorem 2 has the symmetries φ_2, φ_3 and τ. Moreover, it has the additional space orientation reversing and the anti-holomorphic symmetries discussed above. From this one can deduce that the so constructed minimal surface in S^3 must be the Lawson surface. In fact, the energy formula in [10] applied to our numerical spectral data yields an area of 21.91, a value which only slightly differs from what has been numerically computed in [13] using the Willmore flow.

The reconstruction of CMC surfaces as in Theorem 2 has been implemented in the software suite *Xlab* by the second author. However, the input data must be given as a DPW potential. Therefore, we applied the transformation (2.8) to obtain a DPW potential of the form (2.7) for $z_0 = 1$ and $z_1 = i$. Note that the functions $A(x(t), a(t))$ and $B(x(t), a(t))$ are automatically even in t which means that we have obtained holomorphic functions $A(\lambda)$ and $B(\lambda)$ depending on the spectral parameter $\lambda \in \{\lambda \in \mathbb{C} \mid \lambda\bar{\lambda} < 1 + \epsilon\}$. The symmetries imply that A and B have real coefficients and that they are also even with respect to λ. An image of the Lawson surface of genus 2 is shown in Figure 4. Note that the existence of such an image also serves as a positive test for our numerical experiments.

4. Experiments: Whitham Deformation of Lawson symmetric CMC surfaces of genus 2

The physical idea behind these experiments is the following: Starting with the Lawson surface of genus 2 and changing the pressure inside the Lawson surface slightly will make compact CMC surfaces in S^3. As these small deformations should be unique by physical reasoning the CMC surfaces should again be Lawson symmetric, so we can use the DPW potential in (2.7) to construct them. The main difference to the Lawson surface is that there are no space orientation reversing symmetries anymore as the pressure inside and outside the CMC surface differs. Therefore, the functions $A(\lambda)$ and $B(\lambda)$ are not even anymore. This can also be deduced from the Sym point condition (2.9) and (2.10).

As we have discussed in section 2 there is a complex one-dimensional family of Riemann surfaces of genus 2 which admit the holomorphic Lawson symmetries. But the

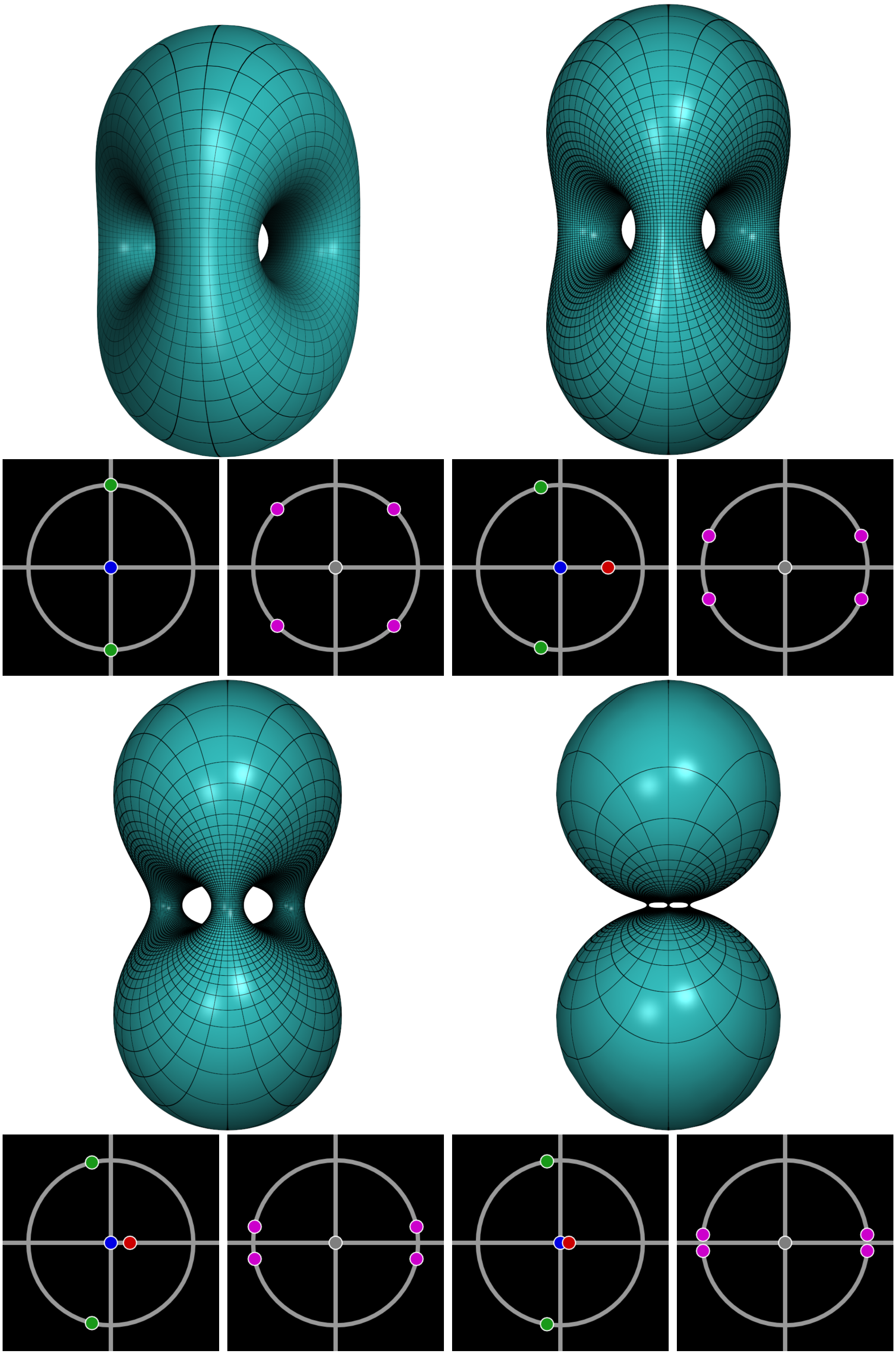


Figure 4. A family of CMC surfaces of genus 2, starting with the Lawson surface in the upper left corner, together with their spectral curves and their Riemann surface type.

physical insights only indicates a real one-dimensional family of Lawson symmetric CMC surfaces. By an analogy to tori we may expect that the real one-dimensional family of Riemann surfaces induced by Lawson symmetric CMC surfaces consists of those surfaces given by (2.6) with $\bar{z}_0 = z_1$, $z_0\bar{z}_0 = 1$, which we call rectangular Lawson symmetric surfaces from now on. Our experiments suggest that this is true, see Figure 4. In fact, we have not found any Lawson symmetric CMC surface whose Riemann surface is not rectangular.

We have designed our experiments as follows: Start with a rectangular Lawson symmetric Riemann surface and with the corresponding DPW potential (2.7). Write

$$A = \sum_{k=0}^{\infty} a_k\lambda^k \text{ and } B = \sum_{k=0}^{\infty} a_k\lambda^k$$

and approximate them by

$$A^n = \sum_{k=0}^{N} a_k\lambda^k \text{ and } B^n = \sum_{k=0}^{N} a_k\lambda^k.$$

Define the functional

$$\mathcal{F}_1\colon S^1 \times \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R};\ (\lambda, a_1, .., a_N, b_1, .., b_N) \mapsto \sum(\operatorname{Im} t_{i,j})^2 + \sum(\chi(\operatorname{Re} t_{i,j}))^2$$

where $t_{i,j} = \frac{1}{2}\operatorname{tr}(M_iM_j)$ for the monodromy matrices M_i of the connection

$$\nabla = d + \eta_{A(\lambda),B(\lambda)}$$

on the four-punctured sphere $\mathbb{CP}^1 \setminus \{\pm z_0, \pm z_1\}$ and

$$\chi\colon \mathbb{R} \to \mathbb{R}; x \mapsto \begin{cases} 0 & x \in [-1, 1] \\ \| x \| & \text{otherwise} \end{cases}.$$

Next, we impose the extrinsic closing conditions (2.9) and (2.10) in our search: Write B as

$$B = fR + hC \tag{4.1}$$

where f is the unique polynomial of degree ≤ 3 satisfying

$$f(\lambda_1) = z_0^2\lambda_1,\ f(\lambda_2) = z_1^2\lambda_2,\ f'(\lambda_1) = z_0^2,\ f'(\lambda_2) = z_1^2$$

and

$$h(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2^2)$$

for the Sym points $\lambda_1, \lambda_2 \in S^1$. We again approximate

$$C = \sum_{k=1}^{N-4} c_k\lambda^k.$$

There is no reason to assume that the anti-holomorphic symmetry of the Lawson surface is broken for the rectangular Lawson symmetric CMC surfaces. Therefore, after rotating the spectral plane such that $\bar{\lambda}_1 = \lambda_2$, we work with the assumption that A and B are real, i.e., the coefficients a_k, c_k are real numbers. We fix $\bar{\lambda}_1 = \lambda_2$ and

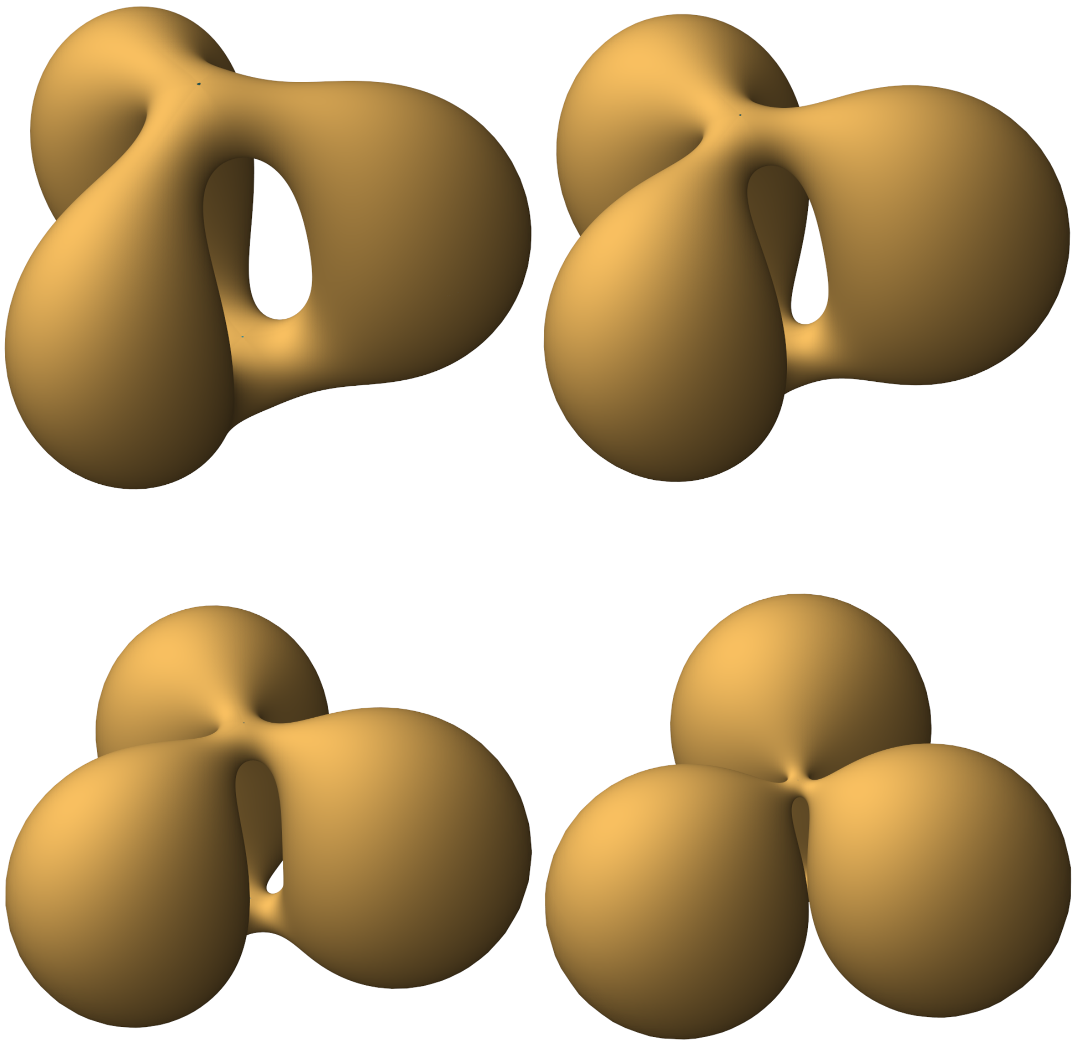


FIGURE 5. Unlike the CMC Lawson family in Figure 4, this family of genus two CMC surfaces in the 3-sphere is not connected to Lawson's minimal surface $\xi_{2,1}$. The family is conjectured to limit to a necklace of three CMC spheres as the conformal type degenerates (lower right).

define a functional as follows: Take a finite number of sample points $\lambda_3, .., \lambda_K \in S^1$ in equidistance and define

$$\mathcal{F}\colon \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R};\ (\lambda_1, a_1, .., a_N, c_1, .., c_N) \mapsto \sum_{k=3}^{K} \mathcal{F}_1(\lambda_k, a_1, .., b_N).$$

where the b_k are computed according to (4.1). Then, we searched numerically for minimizers of $\mathcal{F}$ starting with the initial data of the Lawson surface on a slightly deformed rectangular Lawson symmetric Riemann surface. We found evidence, i.e., the

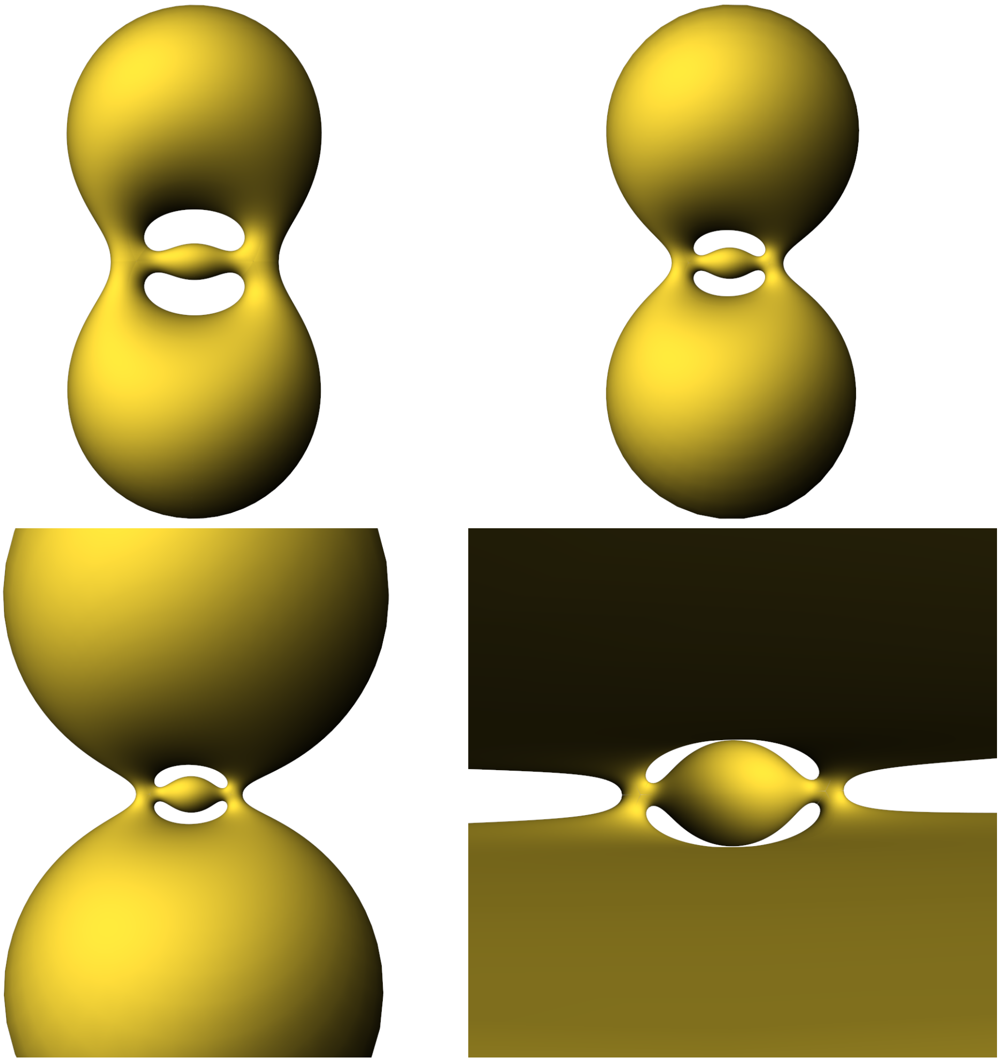


FIGURE 6. The Delaunay perspectives of the surfaces in Figure 5.

numerical search has found a minimum of order 10^{-12}, for the existence of a Lawson symmetric CMC surface nearby the Lawson surface. We have repeated this method in order to obtain a family of Lawson symmetric CMC surfaces through the Lawson surface itself. The family of Lawson symmetric CMC surfaces is shown in Figure 4 together with images of their DPW spectral curve and with the corresponding four-punctured sphere defining the Riemann surface structure. The meaning of the DPW spectral curve picture is as follows: The circle is the unit circle in the spectral plane, whereas the green points are the Sym points. The blue point is just the center $\lambda = 0$ of the spectral plane, while the red point coming inside the unit disc is a zero of the function B.

4.1. **Apparent singularities in the DPW potential.** Looking at the DPW potential (2.7) more carefully, one observes that a zero of B causes a pole in the upper right corner of $\eta_{A,B}$. For a DPW potential corresponding to a CMC immersion, this pole must be apparent in λ. By the reality condition, this holds automatically if all zeros of B are outside the unit disc. When a (simple) zero λ_0 of B is inside we have to ensure that the pole is apparent. There are in principal four possibilities (in the genus 2 case): $A(\lambda_0) = -\frac{2}{3}$ and $A(\lambda_0) = \frac{1}{3}$, so that the singularity in the upper right corner is removable, or $A(\lambda_0) = -1$ and $A(\lambda_0) = 0$, so that the lower left corner of the DPW potential has a zero at λ_0 and the first order pole in the upper right gets apparent by a diagonal gauge only depending on λ.

The family of CMC surfaces through the Lawson surface corresponds to the case $A(\lambda_0) = -\frac{2}{3}$ where λ_0 is the zero of B of smallest distance to $\lambda = 0$. The family of Lawson symmetric CMC surfaces converges against a doubly covered minimal sphere while the zero of B converges to $\lambda = 0$. The reason for this is that the Hopf differential is given by the pull-back of $B(0)\frac{dz^2}{(z^2-z_0^2)(z^2-z_1^2)}$ and hence vanishes in the limit, so the corresponding surface is totally umbilic and therefore a covering of the round sphere. Continuing this family through the double covered sphere produces the same Lawson symmetric CMC surfaces, but this time inside out, until we end at the Lawson surface again. This gives one component of the space of Lawson symmetric CMC surfaces in S^3, see Figure 7.

There also exits a distinct family of rectangular Lawson symmetric CMC surfaces. To find this family, we have implemented the condition that at the zero λ_0 of B inside the unit disc we have $A(\lambda_0) = -1$, and apart from that we have used the same methods as above. This family converges against a chain of three round CMC spheres in S^3, see Figure 5. In Figure 6 we show the same surfaces but this time after the stereographic projection preserving the symmetries φ_2 and τ. Basically, they are almost 2-lobed Delaunay tori where a piece of a Delaunay cylinder is glued in.

We have not been able to find any surfaces for the remaining two cases $A(\lambda_0) = 0$ and $A(\lambda_0) = \frac{1}{3}$. If such families would exists, they could not converge against a chain of spheres. This follows from the energy formula $E(f) = -12\pi A(0)$, which implies $A(0)$ must be negative. Moreover, we have not found any Lawson symmetric CMC surfaces which are not rectangular. There is no reason to believe that they could not exists. Nevertheless one could expect that they are only immersed not embedded, thus only exists at a "higher energy level".

5. Experiments: The Lawson surfaces $\xi_{g,1}$

A natural generalization of our experiments is given by looking at deformations of the Lawson surfaces $\xi_{g,1}$ of genus g. These are quite similar to the Lawson surface of genus 2 but now have a $g+1$-fold symmetry instead of the threefold one. By analogy, we used the following DPW potential

$$\eta = \eta_{A,B} = \pi^* \begin{pmatrix} -\frac{g}{g+1}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} + \frac{A}{z} & \lambda^{-1} - \frac{(A+\frac{2}{g+1})(A+\frac{1-g}{1+g})}{B} z^2 \\ \frac{B}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{\lambda A(A+1)z_0^2 z_1^2}{z^2(z^2-z_0^2)(z^2-z_1^2)} & \frac{g}{g+1}\frac{z(2z^2-z_0^2-z_1^2)}{(z^2-z_0^2)(z^2-z_1^2)} - \frac{A}{z} \end{pmatrix} dz \tag{5.1}$$

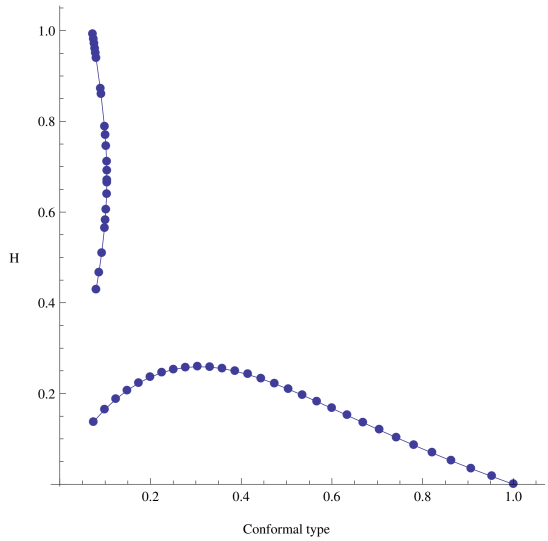


FIGURE 7. This graph represents two families of genus two CMC surfaces based on Lawson's minimal surface $\xi_{2,1}$ in the 3-sphere, plotting conformal type against mean curvature. The CMC Lawson family starts at Lawson's surface at the lower right and limits to a doubly covered minimal 2-sphere at the origin at the lower left (see Figure 4). The plot at the upper left represents a separate family conjectured to limit to a necklace of three CMC spheres as the conformal type degenerates (see Figure 5).

in order to perform experiments for Lawson symmetric CMC surfaces of genus g with Riemann surface structure given by $y^{g+1} = \frac{z^2-z_0^2}{z^2-z_1^2}$. The Sym point condition is almost the same as in the case $g = 2$ with the only difference that the function R in (2.10) is given by $R = A(A + \frac{1-g}{1+g})$. We have performed the experiments for $g = 1, .., 8$ totally analogous to the case of $g = 2$. In all cases we have obtained for $z_0 = 1$, $z_1 = i$ the Lawson surface $\xi_{g,1}$, see Figure 1, and for small rectangular variations of the Riemann surface structure we have obtained CMC deformations through Lawson symmetric surfaces. We thus have found numerical evidence that for all genera g there exists Lawson symmetric CMC deformations of the Lawson surface $\xi_{g,1}$. Especially, in the case of $g = 1$ we have recomputed the Clifford torus and CMC deformations of it which are of course the homogeneous tori of spectral genus 0 first and then bifurcate to the Delaunay tori of spectral genus 1. The bifurcation can be explained in our setup as follows. The zero $\lambda_0 > 1$ of B next to the origin is of order 2 for the Clifford torus. When it crosses the unit circle it can continue either as a double zero to the inside or bifurcate to two simple zeros reflected across the unit circle. When it continues as a double zero the CMC tori remain homogenous whereas in the second case one obtains unduloidal rotational Delaunay tori of spectral genus 1. We have done the corresponding experiments for these tori in the DPW approach independent to the well-established theory of spectral curves for CMC tori. We like to mention that the numerics worked in that case as good as in the case of genus $g \geq 2$ surfaces giving again evidence for our experimental setup.

6. Computational aspects

The surfaces were computed using XLab, a computer framework for surface theory, experimentation and visualization written in C++. XLab implements the DPW construction [5] of CMC surfaces in $\mathbb{S}^3$ in three steps:

- The holomorphic frame is computed as the numerical solution to an ordinary differential equation. Loops appearing in the DPW construction are infinite dimensional; for computation, they are represented finitely as Laurent polynomials about $\lambda = 0$ by chopping off the infinite Laurent series to heuristically determined powers, typically running from λ^{-40} to λ^{40}. This chopping is similar to the way real numbers are represented by rational numbers for numerical computation.
- The unitary frame is computed from the holomorphic frame via loop group Iwasawa factorization. This calculation applies linear methods to matrices of coefficients of the Laurent polynomials representing the holomorphic frame [17].
- The CMC immersion is computed by evaluating the unitary frame at the sym points.

The most difficult part of the construction of the CMC families was the search for the accessory parameters in the DPW potential for which its monodromy is unitarizable. As with the holomorphic frame, the infinite space of accessory parameters was made finite by chopping off its power series. The accessory parameters were compute by optimization (minimization) algorithms. The objective function for the search measured how far the monodromy of the DPW potential was from being simultaneously unitarizable. This measure was computed as the average over a set of equally spaced sample points on the unit circle in the λ-plane. To speed up these lengthy calculations, the objective function was computed in parallel over the sample points simultaneously. Once the accessory parameters in the DPW potential were found, the diagonal unitarizer, computed as in [19], was used as the initial value for the holomorphic frame.

Each Lawson CMC surface was built up by applying its symmetry group to one fundamental piece computed by the DPW construction. The completed surface was viewed, manipulated and rendered in the XLab $\mathbb{S}^3$ viewer.

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