Two-Sided Ideals and Congruences in the Ring of Bounded Operators in Hilbert Space

J. W. Calkin


Stable URL: http://links.jstor.org/sici?sici=0003-486X%28194110%292%3A42%3A4%3C839%3ATIACIT%3E2.0.CO%3B2-0


Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/annals.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.
TWO-SIDED IDEALS AND CONGRUENCES IN THE RING OF BOUNDED OPERATORS IN HILBERT SPACE

J. W. Calkin
(Received August 29, 1940)

Introduction

The developments of the present paper center around the observation that the ring $\mathcal{B}$ of bounded everywhere defined operators in Hilbert space contains non-trivial two-sided ideals.¹ This fact, which has escaped all but oblique notice in the development of the theory of operators, is of course fundamental from the point of view of algebra and at the same time differentiates $\mathcal{B}$ sharply from the ring of all linear operators over a unitary space with finite dimension number. As examples of two-sided ideals in $\mathcal{B}$ we may mention here the class of all operators $A$ such that $\mathcal{R}(A)$, the range of $A$, has a finite dimension number, the class of all operators of Hilbert-Schmidt type,² and the class $\mathcal{I}$ of all totally continuous operators. Except for the ideal $(0)$, every two-sided ideal in $\mathcal{B}$ contains the first ideal mentioned, and except for the ideal $\mathcal{B}$ itself, every two-sided ideal in $\mathcal{B}$ is contained in the ideal $\mathcal{I}$. Moreover, on the basis of the special spectral properties of the self-adjoint members of $\mathcal{I}$, it is possible to characterize every two-sided ideal in $\mathcal{B}$ very simply in terms of the spectra of its nonnegative self-adjoint elements; for both the formulation and the proof of this result, which together with the facts mentioned above is discussed in §1, the author is indebted to J. v. Neumann.

The restriction of our attention to those ideals in $\mathcal{B}$ which are two-sided is basic for the points which we wish to develop; the two-sidedness compensates for the absence of commutativity in $\mathcal{B}$ in such a way as to permit the construction of quotient rings by the standard methods of abstract algebra.³ These rings, which are of course homomorphs of $\mathcal{B}$ with respect to addition and multiplication, are also homomorphs of $\mathcal{B}$ with respect to the operation $\ast$, and exhibit all of the formal properties of matrix algebras. This is established in §2, and there also various properties of the associated congruences in $\mathcal{B}$ are discussed.

The remainder of the paper deals solely with the quotient ring $\mathcal{B}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of totally continuous operators, and the associated congruence in $\mathcal{B}$. For essentially topological reasons, this is the only one of the quotient rings in

¹ An additive subset $\mathcal{I}$ of $\mathcal{B}$ is a left (right) ideal if it contains $AB$ ($BA$) for all $A$ in $\mathcal{I}$ and $B$ in $\mathcal{B}$. If $\mathcal{I}$ is both a left and right ideal, it is called two-sided. See references [1] and [19] at the end of the paper for the elementary properties of ideals.

² For a discussion in abstract terms of operators of this type (operators of "finite norm"), see reference [17], pp. 65-70.

³ See, for example, [1], pp. 252-253, [19], pp. 56-57.
question which at present appears susceptible of deep analysis. For while there is in general no apparent way of introducing a topology in quotient rings over $\mathcal{B}$, the ring $\mathcal{B}/\mathcal{I}$ is actually a complete metric space, the norm deriving very simply from the norm (i.e., bound) in $\mathcal{B}$ itself ($\S$3).

Moreover, it is even possible to interpret $\mathcal{B}/\mathcal{I}$ as an algebraic ring$^4$ of operators in a suitably defined complex Euclidean space $\mathcal{I}$ whose dimension number is the cardinal number of the continuum.$^5$ Or, to put it differently, there exists in the ring of bounded everywhere-defined operators over $\mathcal{I}$ a subset $\mathfrak{M}$ which is a $(+,-,\ast)$-isomorphism of $\mathcal{B}/\mathcal{I}$. Furthermore, this correspondence is even an isometry, the norm of an element of $\mathcal{B}/\mathcal{I}$ being the bound of the corresponding element of $\mathfrak{M}$. These facts are all established in $\S\S$4, 5, and in $\S$5, various properties of the algebraic ring $\mathfrak{M}$ are discussed.

Apart from its intrinsic interest, the analysis of the ring $\mathfrak{M}$ yields in a very simple way theorems of considerable depth concerning $\mathcal{B}$ ($\S$5). Of results of this sort, we shall here mention only a generalization of Weyl’s classical theorem comparing the spectra of two self-adjoint operators with totally continuous difference [20].$^6$

Before proceeding, we should like to point out various further developments which are suggested by the present paper and to which we propose to return at another time.

First, while the algebraic ring $\mathfrak{M}$ is not closed with respect to the weak topology for operators and so is not an operator ring in the sense of von Neumann [9, 11] one obtains such a ring by adjoining to $\mathfrak{M}$ its weak condensation points in the ring of bounded everywhere-defined operators in $\mathcal{I}$. This “closure” $\mathcal{R}(\mathfrak{M})$ is of considerable interest from the point of view of the Murray-von Neumann theory of factors, especially since various preliminary results concerning $\mathcal{R}(\mathfrak{M})$ and $\mathfrak{M}'$ suggest that they may be factors of class III.$^\ast$$^7$

Second, while the factors of class II$^1$ and III$^\ast$ of Murray and von Neumann are like those of class I$^n$, $n < \aleph_0$, simple rings, those of class II$^\omega$ are not. The construction of quotient rings over such a factor is therefore possible and gives rise to various questions analogous to those concerning $\mathcal{B}$ with which we deal here.$^8$

Finally, we should like to point out that the maximal property of the ideal $\mathcal{I}$ of totally continuous operators described above does not persist if one considers

$^4$ We use the term algebraic ring with reference to operators to denote a class closed with respect to the operations $+,\cdot,\ast$, and scalar multiplication. This is not a ring in the sense of von Neumann, [11], since no topological conditions are imposed.

$^5$ For the theory of complex Euclidean spaces of arbitrary dimension number, see references [6], [7], [13], [16].

$^6$ All numbers in brackets refer to the bibliography at the end of the paper.

$^7$ For the notion of factors, and their classification, see [9]; for a construction of factors of class III$^\omega$, see [15].

$^8$ The class of members of a II$^\omega$ factor which are “normed” in the sense of [15] is a two-sided ideal.
instead of Hilbert space, a space \( \mathcal{S} \) with dimension number greater than \( \aleph_0 \). For example, in such a space, the class of all operators \( A \) such that \( \mathcal{R}(A) \) contains no closed subspace with dimension number exceeding \( \aleph_0 \) is a non-trivial ideal. Moreover, if \( m \) is the dimension number of \( \mathcal{S} \), \( m \geq \aleph_0 \), the class of all operators \( A \) which have the property that \( \mathcal{R}(A) \) contains no closed linear subspace of \( \mathcal{S} \) with dimension number \( m \) is a two-sided ideal different from \( \mathcal{B} \) and is identical with \( \mathcal{T} \) if \( m = \aleph_0 \). Furthermore, in every case, this ideal is maximal in the same sense in which \( \mathcal{T} \) is for \( m = \aleph_0 \), and as von Neumann has pointed out to the author, the spectral characterization of ideals can be extended in a satisfactory way to describe the general case. Finally, investigations which are still incomplete suggest that a considerable portion of the analysis of the present paper has a counterpart in every case.

1. Two-sided ideals in \( \mathcal{B} \)

We proceed now to the characterization of all two-sided ideals in \( \mathcal{B} \).

**Theorem 1.1.** If \( \mathcal{I} \) is a left (right) ideal, the set \( \mathcal{I}^* \) of all adjoints \( A^* \) of elements \( A \) of \( \mathcal{I} \) is a right (left) ideal.

Since \( \mathcal{I}^* \) is evidently closed with respect to addition, it is necessary only to show—when \( \mathcal{I} \) is a left ideal—that \( \mathcal{I}^* \) contains \( BA^* \) for all \( B \) in \( \mathcal{B} \) and \( A \) in \( \mathcal{I} \). But this follows at once from the relation \( BA^* = (AB^*)^* \) and the fact that \( AB^* \) belongs to \( \mathcal{I} \) along with \( A \). When \( \mathcal{I} \) is a right ideal, an analogous argument is valid.

**Theorem 1.2.** A necessary and sufficient condition that a left (right) ideal \( \mathcal{I} \) be two-sided is that \( \mathcal{I} = \mathcal{I}^* \).

The sufficiency of the condition is an immediate consequence of Theorem 1.1. Now suppose \( \mathcal{I} \) is two-sided, \( A \) an arbitrary member of \( \mathcal{I} \), \( A = WB \) its canonical decomposition.\(^9\) Then \( A^* = BW^* = W^*AW^* \) is clearly in \( \mathcal{I} \) and \( \mathcal{I} = \mathcal{I}^* \).

Since we now have no further direct concern with left or right ideals we shall refer to two-sided ideals merely as ideals.

**Theorem 1.3.** The class \( \mathcal{T} \) of all totally continuous operators in \( \mathcal{B} \) is an ideal.

That \( \mathcal{T} \) is an additive class is obvious from the definition of a totally continuous operator; \( \mathcal{T} \) is totally continuous if it takes every bounded set into a compact set. Moreover, by a well-known theorem, \( \mathcal{T} = \mathcal{T}^* \).\(^{10}\) Finally, since every member \( A \) of \( \mathcal{B} \) clearly takes bounded sets into bounded sets, \( \mathcal{T} \) is a left ideal. Hence by Theorem 2 it is an ideal.

Throughout the remainder of the paper \( \mathcal{T} \) has the same meaning as in Theorem 1.3.

**Theorem 1.4.** Let \( \mathcal{I} \) be an arbitrary ideal in \( \mathcal{B} \). Then either \( \mathcal{I} = \mathcal{B} \) or \( \mathcal{I} \subseteq \mathcal{T} \).

The proof of this theorem is based on a characteristic property of totally continuous operators which the writer has noted elsewhere.\(^{11}\)

\(^9\) For the notion of canonical decomposition, see [9] and [12].
\(^{10}\) [2], p. 100, Théorème 4.
\(^{11}\) [3], Lemma 3.1.
this result a member $T$ of $\mathcal{B}$ is totally continuous if and only if every closed linear manifold in its range has a finite dimension number. Hence, if $\mathcal{J}$ is an ideal which is not contained in $\mathcal{I}$, $\mathcal{J}$ contains an element $A$ such that $\mathcal{R}(A)$ contains a Hilbert space $\mathcal{M}$. We denote by $\mathcal{M}$ the manifold of zeros of $A$ and by $A_1$ the transformation induced on $\mathcal{S} \ominus \mathcal{M}$ by $A$.

The transformation $A_1$ evidently possesses an inverse and the same range as $A$; moreover the fact that $A_1$ is bounded assures us that the set $\mathcal{A} = A_1^{-1} \mathcal{M}$ is closed. Thus $\mathcal{A}$ is also a Hilbert space. Hence there exist in $\mathcal{B}$ partially isometric operators $X, Y$ both with initial sets $\mathcal{S}$, while $\mathcal{R}(X) = \mathcal{A}$, $\mathcal{R}(Y) = \mathcal{M}$. Therefore the operator $B = Y^*AX$ has domain and range identically $\mathcal{S}$ and belongs to $\mathcal{B}$. But $Bf = 0$ implies either $Xf = 0$, $Xf$ in $\mathcal{M}$, or $AXf$ in $\mathcal{S} \ominus \mathcal{M}$, and all of these are impossible in view of the definition of $X$ and $Y$. Hence $B^{-1}$ exists and since it is closed with domain $\mathcal{S}$, it belongs to $\mathcal{B}$. Therefore $I = B^{-1}B$ belongs to $\mathcal{J}$ and $\mathcal{J} = \mathcal{B}$.

**Theorem 1.5.** Let $\mathcal{J}$ be an arbitrary ideal in $\mathcal{B}$, $\mathcal{J}$ the class of nonnegative definite self-adjoint transformations in $\mathcal{J}$. Then $\mathcal{J}_0$ is the class of all operators $B = (A^*A)^{1/2}$ such that $A$ is an element of $\mathcal{J}$, and if $\mathcal{J}_1$ is an ideal containing $\mathcal{J}_0$, then $\mathcal{J}_1 \supseteq \mathcal{J}$.

If $B$ is in $\mathcal{J}_0$, it is obvious that $B$ is of the form described in the theorem. On the other hand, if $A$ is an arbitrary element of $\mathcal{J}$, $(A^*A)^{1/2} = B = W^*A$, where $W$ is partially isometric; hence $(A^*A)^{1/2}$ belongs to $\mathcal{J}$, and thus to $\mathcal{J}_0$. Finally the relations $A = WB$, $B = (A^*A)^{1/2}$ assure us that $\mathcal{J}$ is the smallest ideal containing $\mathcal{J}_0$.

If $\mathcal{J}$ is an ideal in $\mathcal{B}$, we call the subset $\mathcal{J}_0$ defined in Theorem 1.5 the positive part of $\mathcal{J}$. In order to characterize those subsets of the positive part of $\mathcal{B}$ which appear as the positive parts of ideals, we recall that every self-adjoint operator $T$ in $\mathcal{I}$ can be reduced to diagonal form; that is, for each such $T$ there exists a complete orthonormal set $\{\varphi_n\}$ in $\mathcal{S}$ and a sequence $\{\lambda_n\}$ such that $T\varphi_n = \lambda_n\varphi_n$, $n = 1, 2, \ldots$. Moreover, the sequence $\{\lambda_n\}$ is convergent to zero, and nonnegative if $T$ is in the positive part of $\mathcal{I}$. We call this sequence a characteristic sequence of $T$.

We next observe that if $T$ belongs to the positive part of an ideal $\mathcal{J}$ and $\{\psi_n\}$ is another complete orthonormal set in $\mathcal{S}$, then the member $A$ of $\mathcal{I}$ defined by the equations $A\psi_n = \lambda_n\psi_n$, $n = 1, 2, \ldots$, also belongs to the positive part of $\mathcal{J}$, since $A = UTU^{-1}$, where $U$ is unitary. Hence, if $\mathcal{J}$ is an arbitrary ideal in $\mathcal{B}$, $\mathcal{J} \subseteq \mathcal{I}$ the set of all characteristic sequences $\{\lambda_n\}$ belonging to members of $\mathcal{J}_0$ may without ambiguity be called the spectral set of $\mathcal{J}$.

We now characterize intrinsically those subsets of the class of nonnegative sequences with limit zero which occur as spectral sets. As is indicated in the introduction, this result is due to J. v. Neumann.

**Definition 1.1.** Let $\mathcal{I}$ denote the class of all infinite sequences of nonnegative
numbers which converge to zero. A subset $\mathcal{I}$ of $\mathcal{X}$ is called an ideal set if it has the following properties:

(i) If $\{\lambda_n\}$ is in $\mathcal{I}$ and $\pi$ denotes an arbitrary permutation of the positive integers, $\{\lambda_{\pi(n)}\}$ is in $\mathcal{I}$;

(ii) if $\{\lambda_n\}$ and $\{\mu_n\}$ are in $\mathcal{I}$, so is $|\lambda_n + \mu_n|$;

(iii) if $\{\lambda_n\}$ is in $\mathcal{I}$, $\{\mu_n\}$ in $\mathcal{X}$ and the inequality $\lambda_n \geq \mu_n$ holds for all $n$, then $\{\mu_n\}$ is in $\mathcal{I}$.

Our object now is to prove that every spectral set is an ideal set, and conversely. We require first two lemmas concerning ideal sets.

**Lemma 1.1.** Let $\mathcal{I}$ be an ideal set, $\{\lambda_n\}$ an element of $\mathcal{I}$ with infinitely many terms different from zero. Then the subsequence of positive terms of $\{\lambda_n\}$ belongs to $\mathcal{I}$.

We distinguish two cases according as $\{\lambda_n\}$ contains a finite or an infinite number of zeros. In view of condition (i) of Definition 1, we can assume in the first case $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$ and $0 < \lambda_{N+1} \leq \lambda_n$, $n = N + 1, N + 2, \ldots$. We then have to show that $\{\lambda_{N+n}\}$ belongs to $\mathcal{I}$. Let $\{\mu_n\}$ be the sequence defined by the equations

\[
\mu_n = \lambda_{N+n}, \quad n = 1, 2, \ldots, N, \\
\mu_n = 0, \quad n = N + 1, N + 2, \ldots.
\]

Then $\{\mu_n\}$ is dominated by a permutation of $\{\lambda_n\}$ and hence belongs to $\mathcal{I}$, by conditions (i) and (iii) of Definition 1. Moreover, by condition (ii), $\{\lambda_n + \mu_n\}$ belongs to $\mathcal{I}$. But we have $\lambda_{N+n} \leq \lambda_n + \mu_n$, $n = 1, 2, \ldots$, and hence $\{\lambda_{N+n}\}$ belongs to $\mathcal{I}$.

Now suppose $\{\lambda_n\}$ contains an infinite number of zeros. Again invoking (i), we assume

\[
\lambda_{2k} \leq \lambda_{2(k+1)} > 0, \quad \lambda_{2k-1} = 0, \quad k = 1, 2, \ldots.
\]

We must then show that $\{\lambda_{2n}\}$ belongs to $\mathcal{I}$. We define a sequence $\{\mu_n\}$ by the equations

\[
\mu_{2k} = \lambda_{2k-1}, \quad \mu_{2k-1} = \lambda_{2k}, \quad k = 1, 2, \ldots.
\]

Then $\{\mu_n\}$ is a permutation of $\{\lambda_n\}$ and hence belongs to $\mathcal{I}$. But then $\{\lambda_n + \mu_n\}$ belongs to $\mathcal{I}$, and we have

\[
\lambda_{2n} \leq \lambda_n + \mu_n
\]

Thus, by (iii), $\{\lambda_{2n}\}$ belongs to $\mathcal{I}$ as we wished to prove.

**Lemma 1.2.** Let $\mathcal{I}$ be an ideal set, $\{\lambda_n\}$ an element of $\mathcal{I}$ consisting solely of positive terms. Then any sequence $\{\mu_n\}$ which contains $\{\lambda_n\}$ as a subsequence and which, except for this subsequence, consists solely of zeros, is a member of $\mathcal{I}$.

We define

\[
\lambda^{(1)}_{2k-1} = 0, \quad \lambda^{(1)}_{2k} = \lambda_{2k}, \quad k = 1, 2, \ldots, \\
\lambda^{(2)}_{2k-1} = \lambda_{2k-1}, \quad \lambda^{(2)}_{2k} = 0, \quad k = 1, 2, \ldots.
\]
Then \( \{\lambda_n^{(1)}\} \) and \( \{\lambda_n^{(2)}\} \) are both dominated by \( \{\lambda_n\} \) and therefore belong to \( \mathfrak{I} \). Letting \( j_k \) denote the integer immediately preceding \( k/3 \), we now define two permutations \( \pi_1, \pi_2 \) by the equations,

\[
\begin{align*}
\pi_1(k + j_k) &= 2k - 1, \\
\pi_2(4k) &= 2k, \\
\pi_2(4k - 2) &= 2k - 1, \\
\pi_2(k + j_{k+2}) &= 2k, \\
k &= 1, 2, \ldots
\end{align*}
\]

The effect of the first of these permutations is to take the set of all integers divisible by 4 into the set of all even integers, preserving order; and to take the set of all integers not divisible by 4 into the set of all odd integers, again preserving order. The effect of the second is to take all even integers not divisible by 4 into the set of all odd integers, and the complementary set of integers into the set of all even integers, order being preserved in both cases. Hence the sequence

\[
\{\nu_n\} = \{\lambda_n^{(1)}(n) + \lambda_n^{(2)}(n)\}
\]

which clearly belongs to \( \mathfrak{I} \), is related to the sequence \( \{\lambda_n\} \) by the equations

\[
\nu_{2k} = \lambda_k, \quad \nu_{2k-1} = 0, \quad k = 1, 2, \ldots.
\]

But any sequence \( \{\mu_n\} \) derived from \( \{\lambda_n\} \) in the manner described in the theorem, and containing an infinite number of zero terms, is a permutation of \( \{\nu_n\} \) and therefore belongs to \( \mathfrak{I} \).

Thus, to complete the proof, we have only to dispose of the case that \( \{\mu_n\} \) contains only a finite number of zero terms. In this case it is convenient to assume that \( \{\lambda_n\} \) is monotone. We can then complete the proof by showing that the sequence \( \{\mu_n\} \) defined by the equations

\[
\mu_n = 0, \quad n = 1, 2, \ldots, N - 1, \quad \mu_{N+n-1} = \lambda_n, \quad n = 1, 2, \ldots
\]

belongs to \( \mathfrak{I} \). To do this we set

\[
\nu_n = \lambda_1, \quad n = 1, 2, \ldots, N, \quad \nu_n = \lambda_2, \quad n = N + 1, \quad N + 2, \ldots, 2N, \ldots
\]

Then, from the validity of the lemma in the case of an infinite number of zeros and property (ii) of ideal sets we can conclude that \( \{\nu_n\} \) is in \( \mathfrak{I} \). But, since we clearly have

\[
\mu_n \preceq \nu_n
\]

it follows that \( \{\mu_n\} \) is in \( \mathfrak{I} \).

**Theorem 1.6.** Let \( \mathcal{I} \) be an ideal in \( \mathfrak{B} \), \( \mathcal{I} \subseteq \mathfrak{I} \), and let \( \mathfrak{I} \) be its spectral set. Then \( \mathfrak{I} \) is an ideal set. Conversely, if \( \mathfrak{I} \) is an arbitrary ideal set in \( \mathfrak{B} \), there exists an ideal in \( \mathfrak{B} \) whose spectral set is \( \mathfrak{I} \). This correspondence between ideals in \( \mathfrak{B} \) and ideal sets in \( \mathfrak{I} \) is an isomorphism with respect to the relation \( \subseteq \).

If \( \mathcal{I} \) is a two-sided ideal in \( \mathfrak{B} \), \( \mathcal{I} \subseteq \mathfrak{I} \), we consider an arbitrary member \( A \) of \( \mathcal{I}_0 \) in diagonal form; \( A_{n\varphi_n} = \lambda_n\varphi_n, \ n = 1, 2, \ldots \), where \( \{\varphi_n\} \) is a complete orthonormal set in \( \mathfrak{B} \). Then \( \{\lambda_n\} \) is in the spectral set \( \mathfrak{I} \) of \( \mathcal{I} \), and since \( A \) can also
be put in diagonal form with reference to any permutation of the sequence \( \{ \varphi_n \} \), all permutations of \( \{ \lambda_n \} \) clearly belong to \( \mathfrak{F} \) too.

Now let \( B \) be any other member of \( \mathfrak{F} \), also in diagonal form; \( B \psi_n = \mu_n \psi_n \), \( n = 1, 2, \ldots \), and let \( U \) be the unitary operator defined on \( \{ \varphi_n \} \) by the equations \( U \varphi_n = \psi_n \), \( n = 1, 2, \ldots \). Then \( A + U^{-1} BU \) is in \( \mathfrak{F} \) and
\[
(A + U^{-1} BU) \varphi_n = (\lambda_n + \mu_n) \varphi_n , \quad n = 1, 2, \ldots .
\]
Thus \( \{ \lambda_n + \mu_n \} \) belongs to \( \mathfrak{F} \).

Finally, let \( \{ \mu_n \} \) be a sequence such that \( \mu_n \leq \lambda_n \), \( n = 1, 2, \ldots \). We define an operator \( B_0 \) on \( \{ \varphi_n \} \) by the equations
\[
B_0 \varphi_n = (\mu_n/\lambda_n) \varphi_n , \quad \lambda_n \neq 0 , \\
B_0 \varphi_n = 0 , \quad \lambda_n = 0 .
\]
Then \( B_0 \) has a closed linear extension \( B \) which belongs to \( \mathfrak{F} \). Thus \( AB \) belongs to \( \mathfrak{F} \). But \( AB \varphi_n = \mu_n \varphi_n \), \( n = 1, 2, \ldots \), and therefore \( AB \) belongs to \( \mathfrak{F} \), \( \{ \mu_n \} \) to \( \mathfrak{F} \).

We turn now to the converse part of the theorem, denoting by \( \mathfrak{F} \) an arbitrary ideal set in \( \mathfrak{H} \). We designate by \( \mathfrak{A} \) the set of all totally continuous operators \( A \) such that a characteristic sequence of \( (A^*A)^{\frac{1}{2}} \) belongs to \( \mathfrak{F} \). Since, if \( \mathfrak{A} \) is an ideal, \( \mathfrak{F} \) is obviously its spectral set, we have only to show that \( \mathfrak{A} \) is an ideal.

We begin by considering an arbitrary pair of operators \( A \) and \( B \) of \( \mathfrak{A} \), with the object of showing that \( A + B \) belongs to \( \mathfrak{A} \). To this end we consider the operators
\[
D_1 = (A^*A)^{\frac{1}{2}}, \quad D_2 = (B^*B)^{\frac{1}{2}}, \quad D = (A^* + B^*)(A + B)^{\frac{1}{2}}
\]
and characteristic sequences \( \{ \lambda_n \}, \{ \mu_n \}, \{ \nu_n \} \) of \( D_1, D_2, D \), respectively. We have then to show that \( \{ \nu_n \} \) belongs to \( \mathfrak{F} \).

For convenience, we assume that the positive terms of each sequence are arranged in montone order; in addition, if any one of the sequences contains only a finite number of positive terms, we assume that the sequence itself is monotone. This assumption is clearly not restrictive, in view of condition (i) of Definition 1. Now let \( \{ \lambda'_n \} \) be identical with the subsequence of positive terms of \( \{ \lambda_n \} \) if that subsequence is infinite, identical with \( \{ \lambda_n \} \) itself in the alternative case, and let \( \{ \mu'_n \} \) and \( \{ \nu'_n \} \) be defined in the same way with reference to \( \{ \mu_n \} \) and \( \{ \nu_n \} \) respectively. Then, by Lemma 1.1, \( \{ \lambda'_n \} \) and \( \{ \mu'_n \} \) belong to \( \mathfrak{F} \); and by Lemma 1.2, \( \{ \nu_n \} \) belongs to \( \mathfrak{F} \) if \( \{ \nu'_n \} \) does. Thus we can prove that \( A + B \) belongs to \( \mathfrak{A} \) by showing that \( \{ \nu'_n \} \) belongs to \( \mathfrak{F} \).

To establish the latter result, we note first the relation
\[
\]
which implies that \( 2(D_1^2 + D_2^2) - D^2 \) is nonnegative definite. Holding this fact in reserve, we then recall the theorem of Courant [4],\(^\text{14}\) characterizing the

---

\(^\text{14}\) The paper cited deals only with integral operators with continuous kernels. However, the more general result required here is readily obtained.
sequence \( \{v_n'^2\} \) associated with the operator \( D^2 \) in terms of maxima and minima. For our purposes, we may state this result as follows:

\[
\nu_n'^2 = \min_{\dim (\mathfrak{M}) \leq n-1} \max_{f \in \mathcal{S} \otimes \mathfrak{M}} (D^2 f, f)
\]

where \( \dim (\mathfrak{M}) \) is the dimension number of \( \mathfrak{M} \). Similarly,

\[
\lambda_n'^2 = \min_{\dim (\mathfrak{N}) \leq n-1} \max_{f \in \mathcal{S} \otimes \mathfrak{N}} (D^2 f, f),
\]

\[
\mu_n'^2 = \min_{\dim (\mathfrak{P}) \leq n-1} \max_{f \in \mathcal{S} \otimes \mathfrak{P}} (D^2 f, f).
\]

On the basis of these relations we are able to conclude that the inequality

\[
\nu_n'^2 \leq 2(\lambda_n'^2 + \mu_n'^2)
\]

is valid provided we have \( j + k \leq n + 1 \). For if \( \nu_n'^2 > 2(\lambda_n'^2 + \mu_n'^2) \) holds, we have, in view of (2) and (3),

\[
\nu_n'^2/2 > \min_{\dim (\mathfrak{M}) \leq j-1} \max_{f \in \mathcal{S} \otimes \mathfrak{M}} (D^2 f, f) + \min_{\dim (\mathfrak{N}) \leq k-1} \max_{f \in \mathcal{S} \otimes \mathfrak{N}} (D^2 f, f)
\]

and this implies

\[
\nu_n'^2/2 > \min_{\dim (\mathfrak{M} + \mathfrak{N}) \leq j+k-2} \left[ \max_{f \in \mathcal{S} \otimes (\mathfrak{M} \oplus \mathfrak{N})} (D^2 f, f) + \max_{f \in \mathcal{S} \otimes \mathfrak{N}} (D^2 f, f) \right]
\]

which in turn implies

\[
\nu_n'^2/2 > \min_{\dim (\mathfrak{M} + \mathfrak{N}) \leq j+k-2} \left[ \max_{f \in \mathcal{S} \otimes (\mathfrak{M} \oplus \mathfrak{N})} (D^2 f + D^2 f, f) \right],
\]

\( f \) being restricted in every case to satisfy \( |f| = 1 \). But then, since \( D_1^2 + D_2^2 - D^2/2 \) is nonnegative definite, we have

\[
\nu_n'^2 > \min_{\dim (\mathfrak{M} + \mathfrak{N}) \leq j+k-2} \max_{f \in \mathcal{S} \otimes (\mathfrak{M} \oplus \mathfrak{N})} (D^2 f, f)
\]

which contradicts (1) unless \( j + k - 2 \) is greater than \( n - 1 \), or unless \( j + k \) exceeds \( n + 1 \). Thus (4) holds for \( j + k \leq n + 1 \), as stated, and we have

\[
\nu_n' \leq 2(\lambda_n' + \mu_k')
\]

for \( j + k \leq n + 1 \).

Now let \( r_n \) be \( n/2 \) if \( n \) is even, \( (n+1)/2 \) if \( n \) is odd. Then, from (5) we have

\[
\nu_n' \leq 2(\lambda_{r_n}' + \mu_{r_n}')
\]

Moreover, the sequences \( \{\lambda_n''\} \), \( \{\mu_n''\} \) defined by the equations

\[
\lambda_{2k-1}'' = 0, \quad \lambda_{2k}'' = \lambda_k', \quad k = 1, 2, \ldots,
\]

\[
\mu_{2k-1}'' = 0, \quad \mu_{2k}'' = \mu_k', \quad k = 1, 2, \ldots,
\]

belong to \( \mathfrak{S} \) by Lemma 1.2; and since

\[
\lambda_{r_n}' = \lambda_n'' + \lambda_{r(n)}', \quad \mu_{r_n}'' = \mu_n'' + \pi_{r(n)}', \quad n = 1, 2, \ldots
\]
where \( \pi(2k - 1) = 2k, \pi(2k) = 2k - 1 \), it follows from conditions (i) and (iii) of Definition 1.1 that the sequences \( \{\lambda_n\} \) and \( \{\mu'_n\} \) both belong to \( \mathcal{I} \). But then, by (6) and conditions (ii) and (iii) of our definition, \( \{\nu_n\} \) belongs to \( \mathcal{I} \); and this completes the proof of our assertion that \( A + B \) belongs to \( \mathcal{J} \).

Now let \( A \) be an arbitrary element of \( \mathcal{J} \), \( D_1 = (A^*A)^{1/2} \), \( X \) an arbitrary element of \( \mathcal{B} \), \( D_x^2 = (A^*X^*X A)^{1/2} \). Since \( A \) is in \( \mathcal{J} \), \( XA \) and \( D_2 \) are also. Let \( \{\lambda_n\} \), \( \{\mu_n\} \), \( \{\lambda'_n\} \), \( \{\mu'_n\} \) have the same meanings as above with reference to \( D_1 \) and \( D_2 \). Then \( \{\lambda'_n\} \) is in \( \mathcal{J} \) and we can show that \( XA \) is in \( \mathcal{J} \) by showing that \( \{\mu'_n\} \) is in \( \mathcal{J} \); the arguments here are the same as above.

To establish the latter, we again apply the theorem of Courant;

\[
\lambda_{n}^{2} = \min_{\dim(\mathcal{R}) \leq n-1} \max_{f \in \mathcal{R}} (D_{f}^{2}f,f),
\]

\[
\mu_{n}^{2} = \min_{\dim(\mathcal{R}) \leq n-1} \max_{f \in \mathcal{R}} (D_{x}^{2}f,f).
\]

But \( (D_{f}^{2}f,f) = (Af,Af) \), \( (D_{x}^{2}f,f) = (XAf,XAf) \). Since \( XA \) is in \( \mathcal{J} \) by showing that \( \{\mu'_n\} \) is in \( \mathcal{J} \); the arguments here are the same as above.

Thus \( \mathcal{J} \) is a left ideal and in view of Theorem 1.2, we can show that \( \mathcal{J} \) is two-sided by showing that \( \mathcal{J} \) is closed with respect to the operation \( * \). But this is an immediate consequence of Lemmas 1.1 and 1.2 and the well-known fact that \( (A^*A)^{1} \) and \( (AA^*)^{1} \) have the same positive characteristic values, each with the same multiplicity.\(^{15}\)

The concluding assertion of the theorem is obvious.

**Theorem 1.7.** Let \( \mathcal{F} \) denote the class of all operators \( A \) in \( \mathcal{B} \) such that \( \mathcal{R}(A) \) has a finite dimension number. Then \( \mathcal{F} \) is a two-sided ideal in \( \mathcal{B} \). If \( \mathcal{J} \) is an arbitrary two-sided ideal, in \( \mathcal{B} \), then \( \mathcal{J} = (0) \) or \( \mathcal{J} \subseteq \mathcal{F} \).

To establish the first assertion, we have only to note that the class \( \mathcal{F} \) of all sequences in \( \mathcal{F} \) with only a finite number of terms different from zero constitutes an ideal set. To establish the second we first observe that any ideal set different from the one containing only the sequence all of whose terms are zero, contains a sequence \( \{\lambda_n\} \) with \( \lambda_1 = 0, \lambda_n = 0 \) for \( n \neq 1 \), by virtue of (i) and (iii). Hence, by (ii) and (iii) it contains all sequences of this sort and hence, by (i) and (ii), contains \( \mathcal{F} \).

It is worthwhile to observe here that the effect of Theorem 1.6 is to establish an isomorphism between the lattice \( L_1 \) of two-sided ideals in \( \mathcal{B} \) and a certain sub-lattice \( L_2 \) of the lattice of ideals in the ring \( \mathcal{B} \) of all bounded sequences of complex numbers. To make this clear we require two preliminary results concerning \( \mathcal{B} \); the first theorem is the analogue for \( \mathcal{B} \) of Theorem 1.5 for \( \mathcal{B} \) the second is analogous to Theorems 1.3 and 1.4.

---

\(^{15}\) By [12], Satz 7, for example.
Theorem 1.8. Let $\mathfrak{B}$ be the ring of all bounded sequences of complex numbers, $\mathfrak{I}$ an ideal in $\mathfrak{B}$. Let $\mathfrak{I}_0$ be the set of all sequences $\{\| \lambda_n \| \}$ such that $\{\lambda_n \}$ is in $\mathfrak{I}$.

Then $\mathfrak{I}$ contains $\mathfrak{I}_0$, and if $\mathfrak{I}_1$ is an ideal in $\mathfrak{B}$ containing $\mathfrak{I}_0$, then $\mathfrak{I}_1 \supseteq \mathfrak{I}$.

The proof is straightforward and is left to the reader.

Theorem 1.9. Let $L_2$ be the lattice of all ideals $\mathfrak{I}$ in $\mathfrak{B}$ which satisfy the following condition; if $\{a_n\}$ is in $\mathfrak{I}$, and $\pi$ is a permutation of the positive integers $\{a_{\pi(n)}\}$ is in $\mathfrak{I}$.

Then the set $\mathfrak{I}$ of sequences convergent to zero belongs to $L_2$, and if $\mathfrak{I}$ is an arbitrary element of $L_2$, either $\mathfrak{I} = \mathfrak{B}$ or $\mathfrak{I} \subseteq \mathfrak{I}_0$.

The first assertion is obvious. Now if $\mathfrak{I}$ is a member of $L_2$ which is not contained in $\mathfrak{I}$, $\mathfrak{I}$ contains a sequence $\{a_n\}$ which has a subsequence $\{a_{n_k}\}$ such that $\{a_{n_k}^{-1}\}$ is bounded, and which converges to a number different from zero. Furthermore, the sequence $\{b_n\}$ with $b_{n_k} = a_{n_k}$, $b_n = 0$, $n \neq n_k$ can be written $\{b_n\} = \{c_n\} \{a_n\}$, where $c_n = 1$, $c_n = 0$, $n \neq n_k$, and hence $\{b_n\}$ belongs to $\mathfrak{I}$. We now write the sequence $\{b_n\}$ in the form $\{b_n\} = \{\rho_n e^{i\theta_n}\}$, $\rho_n > 0$, and observe that $\{\rho_n\}$ also belongs to $\mathfrak{I}$ by Theorem 1.8. But then by the same sort of argument that was used to prove Lemma 1.1, we can show that $\{\rho_n\}$ belongs to $\mathfrak{I}$, and hence $\{a_{n_k}\}$ does also. Hence $\{e_k\} = \{a_{n_k}\} \{a_{n_k}^{-1}\}$ belongs to $\mathfrak{I}$ and $e_k = 1$, $k = 1, 2, \ldots$. Thus $\mathfrak{I} = \mathfrak{B}$.

Theorem 1.10. Let $\mathfrak{I}$ be a member of $L_2$, $\mathfrak{I}_0$ the subset of $\mathfrak{I}$ defined in Theorem 1.8. Then either $\mathfrak{I}_0 = \mathfrak{B}_0$, where $\mathfrak{B}_0$ the set of all sequences of nonnegative numbers in $\mathfrak{B}$, or every sequence in $\mathfrak{I}_0$ is convergent to zero and $\mathfrak{I}_0$ is an ideal set in the sense of Definition 1.1.

Theorem 1.10 follows at once from Theorems 1.8 and 1.9.

Theorem 1.11. Let $L_1$ be the lattice of all two-sided ideals in $\mathfrak{B}$, let the set $\mathfrak{B}_0$ be called the spectral set of $\mathfrak{B}$, and let the class of ideal sets be extended to include $\mathfrak{B}_0$. Then $L_1$ is lattice-isomorphic to the extended class of ideal sets, each member of $L_1$ corresponding under this isomorphism to its spectral set. Similarly, the lattice $L_2$ is lattice-isomorphic to the extended class of all ideal sets, each member $\mathfrak{I}$ of $L_2$ corresponding to its $\mathfrak{I}_0$. Thus $L_1$ and $L_2$ are lattice-isomorphic and under this isomorphism $\mathfrak{B}$ corresponds to $\mathfrak{B}$, $\mathfrak{I}$ to the class of all sequences convergent to zero, $\mathfrak{I}$ to the class of all sequences with only a finite number of terms different from zero.

Theorem 1.11 is obvious on the basis of preceding results and we omit the proof. Before proceeding, however, we wish to make the following observations: The ring $\mathfrak{B}$ can be imbedded in $\mathfrak{B}$ in a very simple way; we have merely to choose a complete orthonormal set $\{\varphi_n\}$ in $\mathfrak{S}$ and identify the element $\{a_n\}$ of $\mathfrak{B}$ with the closed linear operator $A$ in $\mathfrak{S}$ which is defined on $\{\varphi_n\}$ by the equations $A\varphi_n = a_n\varphi_n$, $n = 1, 2, \ldots$. Moreover, if we consider $\mathfrak{B}$ in terms of this identification, each two-sided ideal $\mathfrak{I}$ in $\mathfrak{B}$ corresponds under the isomorphism of Theorem 1.11 merely to its intersection with $\mathfrak{B}$. This suggests an alternative attack on the problem solved by Theorem 1.6; however, in so far as we can determine, it is not possible to devise any essentially different proof of that theorem on this basis.
2. Congruences in $\mathcal{B}$

We pass now to the study of congruences modulo an ideal $\mathcal{I}$ in $\mathcal{B}$. Following the standard procedure of abstract algebra, we consider the class $\mathcal{B}/\mathcal{I}$ whose elements $\alpha$, $\beta$, $\ldots$ are the residue classes of $\mathcal{B}$ with respect to $\mathcal{I}$; by definition, two members $A$ and $B$ of $\mathcal{B}$ belong to the same element $\alpha$ of $\mathcal{B}/\mathcal{I}$ if and only if $A - B$ is in $\mathcal{I}$. If $\alpha$ and $\beta$ are arbitrary elements of $\mathcal{B}/\mathcal{I}$, we define $\alpha + \beta$ as the class of all elements $A + B$ of $\mathcal{B}$ such that $A$ is in $\alpha$, $B$ in $\beta$; similarly, we define $\alpha \beta$ as the class of all $AB$ in $\mathcal{B}$ such that $A$ is in $\alpha$, $B$ in $\beta$. Then, from the general theorem\textsuperscript{16} which is controlling in such situations, we have

**Theorem 2.1.** If $\alpha$ and $\beta$ are elements of $\mathcal{B}/\mathcal{I}$, so also are $\alpha + \beta$ and $\alpha \beta$. With addition and multiplication defined in this way $\mathcal{B}/\mathcal{I}$ is a ring; that is to say, $\mathcal{B}/\mathcal{I}$ is a commutative group with respect to the operation $+$, and further, the following formal laws are satisfied:

$$\alpha(\beta + \gamma) = \alpha \beta + \alpha \gamma, \quad (\beta + \gamma) \alpha = \beta \alpha + \gamma \alpha.$$ 

Moreover, $\mathcal{B}/\mathcal{I}$ possesses a unit.

It may be noted that except in the case $\mathcal{I} = \mathcal{B}$, the subclass of $\mathcal{B}/\mathcal{I}$ each of whose elements contains a scalar multiple of the identity in $\mathcal{B}$, is isomorphic to the class of scalar multiples of the identity in $\mathcal{B}$, since no two of these elements of $\mathcal{B}$ can have difference in $\mathcal{I}$ unless they are identical. It is convenient therefore to use italic letters for these elements as well as for the corresponding elements of $\mathcal{B}$. In addition we shall use the symbol 1 for the unit in $\mathcal{B}/\mathcal{I}$; that is, for the element of $\mathcal{B}/\mathcal{I}$ whose members have the form $I + T$, where $I$ is the identity in $\mathcal{B}$ and $T$ belongs to $\mathcal{I}$.

It is worth pointing out here that the ring $\mathcal{B}/\mathcal{I}$, $\mathcal{I} \neq \mathcal{B}$, is certainly non-commutative; to verify this one needs only to consider two orthogonal projections $E$ and $F$ whose ranges are Hilbert spaces and whose sum is the identity, a partially isometric operator $W$ which maps $E$ on $F$, and the operator $WW^* - WW^{**} = E - F$ which belongs to no ideal except $\mathcal{B}$ itself. Later we shall show that the center of $\mathcal{B}/\mathcal{I}$, $\mathcal{I} \neq \mathcal{B}$, is the set of all scalar multiples of unity (Theorem 2.9).

**Theorem 2.2.** If $\alpha$ is an arbitrary element of $\mathcal{B}/\mathcal{I}$, the class $\alpha^*$ of all members $A^*$ of $\mathcal{B}$ such that $A$ is in $\alpha$ is in $\mathcal{B}/\mathcal{I}$ also. The operation $^*$ so defined in $\mathcal{B}/\mathcal{I}$ obeys the following laws:

$$\alpha^{**} = \alpha, \quad (\alpha + \beta)^* = \alpha^* + \beta^*, \quad (\alpha \beta)^* = \beta^* \alpha^*.$$ 

That $\alpha^*$ is in $\mathcal{B}/\mathcal{I}$ follows at once from the fact that $\mathcal{I}^* = \mathcal{I}$; and the three laws stated in the theorem are readily verified on the basis of their validity in $\mathcal{B}$.

Thus we see that the rings $\mathcal{B}/\mathcal{I}$ have all of the formal properties of matrix algebras and are homomorphs of $\mathcal{B}$ with respect to the operations $+$, $\cdot$, $^*$, $\cdot$.

\textsuperscript{16}[1], pp. 252-253. The missing details necessary for our purposes are readily supplied. Cf. the discussion of the commutative case in [19].
moreover, they are of course the only homomorphs of $\mathcal{B}$ with respect to these operations.

It is now desirable to consider these homomorphisms with respect to the following important notions in operator theory; for an operator to be self-adjoint, to be idempotent, to be partially isometric, to be unitary. Hence we are led to define these concepts in $\mathcal{B}/\mathcal{I}$ without explicit reference to their meanings in $\mathcal{B}$.

**Definition 2.1.** An element $\alpha$ of $\mathcal{B}/\mathcal{I}$ is called self-adjoint if $\alpha = \alpha^*$; a self-adjoint element $\epsilon$ of $\mathcal{B}/\mathcal{I}$ is called idempotent if $\epsilon^2 = \epsilon$; an element $\omega$ of $\mathcal{B}/\mathcal{I}$ is called partially isometric if $\omega^*\omega = \epsilon$ is idempotent, unitary if $\omega^*\omega = \omega\omega^* = 1$.

It is easy to see that under the homomorphisms $\mathcal{B} \to \mathcal{B}/\mathcal{I}$ the image of every-self adjoint operator is self-adjoint and that analogous assertions hold for projections, partially isometric operators, and unitary operators. We shall now show that with reference to the first two of these concepts the converse statements are also true.

**Theorem 2.3.** If $\alpha$ is a self-adjoint element of $\mathcal{B}/\mathcal{I}$, $\alpha$ contains a self-adjoint member of $\mathcal{B}$, and conversely.

Let $\alpha$ be self-adjoint, $A$ an element of $\alpha$. Then $A^* - A$ is in $\mathcal{I}$ and hence $A + (A^* - A)/2 = (A + A^*)/2$ is in $\alpha$. The converse, as we have already noted, is obvious.

**Theorem 2.4.** If $\epsilon$ is an idempotent element of $\mathcal{B}/\mathcal{I}$, there exists a projection $E$ in $\mathcal{B}$ which belongs to $\epsilon$, and conversely.

The theorem is obvious for $\mathcal{I} = \mathcal{B}$; we assume therefore $\mathcal{I} \subset \mathcal{T}$. By Theorem 2.3, $\epsilon$ contains a self-adjoint transformation $A$, and since $\epsilon$ is idempotent, $A^2 - A = A(A - I)$ is in $\mathcal{I}$ and thus in $\mathcal{T}$. Hence $A^2 - A$ can be reduced to diagonal form, and therefore $A$ can also; $A\varphi_n = \lambda_n\varphi_n$, $n = 1, 2, \ldots$, where $\{\varphi_n\}$ is a complete orthonormal set in $\mathcal{S}$. But then it follows that $\{\lambda_n\}$ contains a subsequence $\{\lambda_n^{(0)}\}$ convergent to zero, and such that the remaining terms of $\{\lambda_n\}$ form a subsequence, say $\{\lambda_n^{(1)}\}$, convergent to 1, since under any other circumstances $A(A - I)$ would fail to be totally continuous. Moreover, we can clearly assume that $\{\lambda_n^{(1)}\}$ contains no terms with the value zero and that $\{\lambda_n^{(0)}\}$ contains no terms with the value unity.

Now let $\mathcal{M}_0$ be the subspace of $\mathcal{S}$ determined by the characteristic elements of $A$ corresponding to terms of $\{\lambda_n^{(0)}\}$, $\mathcal{M}_1$ the subspace determined by the other characteristic elements of $A$, $E_0$ and $E_1$ the projections with ranges $\mathcal{M}_0$ and $\mathcal{M}_1$, respectively. We shall show that $E_1$ belongs to $\epsilon$. To do this we note first that in $\mathcal{M}_0$, $A - I$ induces a transformation with bounded inverse. Hence if $B$ is equal to this inverse in $\mathcal{M}_0$ and to zero in $\mathcal{M}_1$,

$$E_0(A^2 - A)BE_0 = E_0AE_0$$

is in $\mathcal{I}$, since $A^2 - A$ is. Similarly, it follows that

$$E_1(A - I)E_1 = E_1AE_1 - E_1$$

is in $\mathcal{I}$. But then, adding the right members of the two preceding equations we find that
is in $\mathcal{J}$, and from this it follows that $E_1$ belongs to $\mathcal{J}$.

Again the converse part of the theorem is obvious, so the proof is complete.

**Theorem 2.5.** Let $\mathcal{J}$ be an ideal in $\mathcal{B}$, $\mathcal{J}_0$ the set of all nonnegative definite self-adjoint elements of $\mathcal{J}$, $\mathcal{J}_0^2$ the class of all squares of elements of $\mathcal{J}_0$. Then a necessary and sufficient condition that every partially isometric element $\omega$ of $\mathcal{B}/\mathcal{J}$ contain a partially isometric transformation $W$ is that $\mathcal{J}_0 = \mathcal{J}_0^2$.

Again the case $\mathcal{J} = \mathcal{B}$ is trivial, so we assume $\mathcal{J} \subseteq \mathcal{T}$. Let $\omega$ be partially isometric, $V$ an element of $\omega$, $V = UB$ its canonical decomposition. Then $V^*V = B^2$ belongs to an idempotent element $\epsilon$ of $\mathcal{B}/\mathcal{J}$. Hence, if we identify $B^2$ with the self-adjoint transformation $A$ which appears in the proof of Theorem 2.4, we can invoke that theorem to establish the existence of a projection $E$ such that $B^2 - E$ is in $\mathcal{J}$. Moreover, an inspection of the proof reveals also that $B^2$ commutes with $E$ and induces in the range of $E$ a transformation with bounded inverse. In particular, this implies that $E$ has for its range a subspace of the initial set of $U$ and thus that $UE$ is partially isometric, since $EU^*UE = E$.\(^{17}\)

We shall now show that under the condition of the theorem $B - E$ is in $\mathcal{J}$. We note first that since $B^2$ and $E$ commute, we have $B^2 - E = (B - E)(B + E)$, and since $B$ is nonnegative, $B + E$ induces in $\mathcal{H}(E)$ a transformation with bounded inverse. Hence, if $C$ is equal to this inverse in $\mathcal{H}(E)$, and to zero in $\mathcal{J} \oplus \mathcal{H}(E)$, we have $(B^2 - E)C = EB - E$, and $EB - E$ is in $\mathcal{J}$. Moreover $(I - E)(B^2 - E) = (I - E)B^2$ is in $\mathcal{J}$. But if $\mathcal{J}$ has the property described in the theorem, $[(I - E)B^2] = (I - E)B$ is in $\mathcal{J}$, and thus

$$(I - E)B + EB - E = B - E$$

is in $\mathcal{J}$ as we wished to show. But then $V - UE = U(B - E)$ is in $\mathcal{J}$ and $W = UE$, which we have already shown to be partially isometric, belongs to $\omega$.

It remains therefore for the converse part of the theorem to be proved. To this end, we suppose that $\mathcal{J}$ contains a nonnegative definite self-adjoint transformation $B^2$ such that $B$ does not belong to $\mathcal{J}$, and denote by $\omega$ the congruence class in $\mathcal{B}/\mathcal{J}$ to which $B$ belongs. Then, since $B^*B = B^2$ belongs to $\mathcal{J}$, we have $\omega$ partially isometric by definition. But if $V$ is a partially isometric transformation in $\omega$, $V^*V = E$ must be congruent to $B^2$ modulo $\mathcal{J}$, which is to say that $E$ is congruent to zero modulo $\mathcal{J}$. But this implies that $E$ has range with finite dimension number and the range of $E$ is the initial set of $V$. Thus $V$ is in $\mathcal{J}$ and hence in $\mathcal{J}$. However, $V - B$ is in $\mathcal{J}$ since $V$ is in $\omega$, and as $B$ is by assumption not in $\mathcal{J}$, we have a contradiction. Hence, the condition of the theorem is necessary as well as sufficient.

We may note in passing that the ideals $(0), \mathcal{J}, \mathcal{J}_0, \mathcal{B}$ all satisfy the condition of Theorem 2.5, but that these are not the only ideals which do so. Consider, for example, the class of all sequences $\{\lambda_n\}$ of nonnegative numbers such that

\(^{17}\) By [9], Lemma 4.3.2.
\[ \sum_{n=1}^{\infty} \lambda_n \text{ converges for some } p. \] It is easily seen that this class is an ideal set in the sense of Definition 1.1. and that the corresponding ideal in \( B \) has the property of Theorem 2.5.

For the sake of completeness, we state the following obvious theorem:

**Theorem 2.6.** If \( W \) is a partially isometric member of \( \mathfrak{B} \), the congruence class of \( W \) in \( \mathfrak{B}/\mathcal{I} \) is partially isometric in \( \mathfrak{B}/\mathcal{I} \).

**Theorem 2.7.** Let \( \mathcal{I} \) be an ideal in \( \mathfrak{B} \). Then, if \( \omega \) is a unitary element of \( \mathfrak{B}/\mathcal{I} \), \( \omega \) contains a maximal partially isometric transformation\(^\text{18} \) with deficiency-index \((0, n)\) or \((n, 0)\), \( n < \mathcal{N}_0 \).

Again the case \( \mathcal{I} = \mathfrak{B} \) is trivial, so we assume \( \mathcal{I} \neq \mathfrak{B} \). Let \( \omega \) be unitary, \( V \) a member of \( \omega \), \( V = UB \) its canonical decomposition. Since \( \omega \) is also partially isometric, the first part of the proof of Theorem 2.5 applies to yield the following results: there exists a projection \( E \) with range in the initial set of \( U \), and which commutes with \( B \), such that \( V^*V - E = B^2 - E \) and \( EB - E \) are in \( \mathcal{I} \). But since \( V^*V \) is congruent to \( I \) modulo \( \mathcal{I} \), \( I - E \) is in \( \mathcal{I} \) and thus \((I - E)B \) is in \( \mathcal{I} \).

Therefore

\[ B - E = (I - E)B + EB - E \]

is in \( \mathcal{I} \). But then \( V - UE = UB - UE \) is in \( \mathcal{I} \). Hence \( W = UE \) is a partially isometric operator which belongs to \( \omega \). Moreover, since \( \omega \) is unitary, \( I - WW^* \) is in \( \mathcal{I} \) and hence, since this operator is a projection and belongs to \( \mathcal{I} \), its range must have a finite dimension number. Similarly, \( I - W^*W \) has range with a finite dimension number. Thus both the initial and final sets of \( \mathcal{I} \) have orthogonal complements with finite dimension numbers. Therefore, if \( W_1 \) is the contraction of \( W \) with domain \( \mathfrak{R}(E) \) and \( X \) a maximal partially isometric extension of \( W_1 \), \( X \) has the property required in the theorem and \( X - W \) is in \( \mathcal{I} \). Thus \( X \) belongs to \( \omega \), and the theorem is proved.

It is important to observe that every unitary element of \( \mathfrak{B}/\mathcal{I} \) does not contain a unitary member of \( \mathfrak{B} \), except in the trivial cases \( \mathcal{I} = \mathfrak{B} \), \( \mathcal{I} = (0) \).

To prove this, we consider an isometric transformation \( X \) with deficiency-index \((0, n)\), \( n < \mathcal{N}_0 \), and the congruence class \( \omega \) modulo \( \mathcal{I} \), to which \( X \) belongs. Then \( \omega \) is clearly unitary in \( \mathfrak{B}/\mathcal{I} \) provided \( \mathcal{I} \neq (0) \). Now suppose \( U \) is a unitary transformation in \( \omega \). Then \( U - X \) is in \( \mathcal{I} \), and thus in \( \mathcal{I} \), if \( \mathcal{I} \neq \mathfrak{B} \). Hence \( I - U^{-1}X \) is in \( \mathcal{I} \) and \( U^{-1}X \) also has deficiency-index \((0, n)\). But by a lemma which the author has proved elsewhere, this is possible if and only if \( n = 0 \).\(^\text{19} \)

**Theorem 2.8.** Let \( \mathcal{I} \) be an ideal in \( \mathfrak{B} \) different from \((0)\), \( W \) a partially isometric operator in \( \mathfrak{B} \) with deficiency-index \((m, n)\), \( m, n < \mathcal{N}_0 \). Then the congruence class \( \omega \) in \( \mathfrak{B}/\mathcal{I} \) to which \( W \) belongs is unitary.

If \( W \) has the properties stated then \( I - W^*W \) and \( I - WW^* \) are projections

---

\(^{18}\) We call a partially isometric operator maximal if the isometric transformation which determines it is maximal. Similarly, we shall have occasion to refer to the deficiency-index of a partially isometric operator.

\(^{19}\) [3], Lemma 4.1.
which belong to $\mathcal{F}$. Thus, since $\mathcal{F} \subseteq \mathcal{J}$ by Theorem 1.7, both of these operators belong to $\mathcal{J}$ and $\omega$ is unitary by definition.

We conclude this section with

**Theorem 2.9.** Let $\mathcal{J}$ be an ideal in $\mathcal{B}$, $\mathcal{J} \neq \mathcal{B}$. Then the center of $\mathcal{B}/\mathcal{J}$, that is, the set of all elements of $\mathcal{B}/\mathcal{J}$ which commute with every element of $\mathcal{B}/\mathcal{J}$, is the set of all elements $\lambda \cdot 1$, where $\lambda$ is a complex number.

It is clear that the center contains the set of all scalar multiples of the identity; hence we need only show that it contains no other members.

We begin by showing that it is sufficient to consider merely the self-adjoint members of the center. For suppose $\alpha$ belongs to the center. Then $\alpha \beta^* - \beta^* \alpha = 0$ for all $\beta$ in $\mathcal{B}/\mathcal{J}$ and hence $(\alpha \beta^* - \beta^* \alpha)* = \beta \alpha^* - \alpha^* \beta = 0$ for all $\beta$ in $\mathcal{B}/\mathcal{J}$. Thus $\alpha^*$ belongs to the center, and consequently the self-adjoint elements $\alpha + \alpha^*$ and $i(\alpha - \alpha^*)$ do also. Now suppose $\alpha + \alpha^* = \lambda \cdot 1$, $i(\alpha - \alpha^*) = \mu \cdot 1$. Then, eliminating $\alpha^*$, we have $\alpha = (\mu + i\lambda)/2i$. Hence we have only to prove that every self-adjoint member of the center is a scalar multiple of the identity.

In terms of operators in $\mathcal{B}$, this problem reduces to the following: to show that every self-adjoint operator $A$ in $\mathcal{B}$ such that $AB - BA$ is in $\mathcal{J}$ for all $B$ in $\mathcal{B}$ is of the form $T + \lambda I$, where $T$ is in $\mathcal{J}$.

We consider first the case $\mathcal{J} = \mathcal{T}$. So we consider a self-adjoint operator $A$ so that $AB - BA$ is totally continuous for all $B$ in $\mathcal{B}$. If $A$ is not of the form $T + \lambda I$, $T$ in $\mathcal{T}$, the spectrum of $A$ must contain two distinct points, each of which is either a limit point of the spectrum of $A$ or a characteristic value of infinite multiplicity; for, otherwise, the spectrum of $A$ consists solely of isolated characteristic values of finite multiplicity together with one point $\mu$ which is either a limit point or a characteristic value of infinite multiplicity, and in this case $A - \mu I$ clearly belongs to $\mathcal{T}$. Hence if $E(\lambda)$ is the resolution of the identity of $A$, there exist numbers $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ in the spectrum of $A$, $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3$ such that $E(\lambda_1) - E(\lambda_0)$ and $E(\lambda_3) - E(\lambda_2)$ have ranges $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, which are Hilbert spaces.

Next let us consider the partially isometric operator $W$ with initial set $\mathcal{M}_1$ and final set $\mathcal{M}_2$. We then have, for $f$ in $\mathcal{M}_1$

$$|WAf - AWf| \geq |AWf| - |WAf|,$$

and thus, since we also have

$$|AWf| \geq \lambda_2 |Wf| = \lambda_2 |f|, |WAf| = |Af| \leq \lambda_1 |f|$$

we obtain

$$|WAf - AWf| \geq (\lambda_2 - \lambda_1)f.$$

Hence $(WA - AW)$ induces on $\mathcal{M}_1$ a transformation with bounded inverse and therefore $(WA - AW)\mathcal{M}_1$ is a Hilbert space. Consequently, by a lemma previously referred to, $WA - AW$ is not in $\mathcal{T}$. Hence the assumption that $A$ is not of the form $T + \lambda I$, $T$ in $\mathcal{T}$, is untenable, and the theorem is established for $\mathcal{J} = \mathcal{T}$. 

**bounded operators in hilbert space** 853
Consider now an arbitrary ideal $\mathcal{I}$, $\mathcal{F} \subset \mathcal{J} \subset \mathcal{I}$, $\mathcal{I} \neq \mathcal{F}$, $\mathcal{J} \neq \mathcal{F}$. As before, we consider a self-adjoint operator $A$ such that $AB - BA$ is in $\mathcal{I}$ for all $B$ in $\mathcal{B}$. Since $\mathcal{I} \subset \mathcal{J}$, it follows from the preceding result that $A$ is of the form $T + \lambda I$, $T$ in $\mathcal{J}$. Thus $TB - BT$ is in $\mathcal{I}$ for all $B$ in $\mathcal{B}$. Now let $\{\varphi_n\}$ be a complete orthonormal set of characteristic elements of $T$; $T\varphi_n = \lambda_n \varphi_n$, $n = 1, 2, \ldots$, and let us suppose that $T$ is not in $\mathcal{I}$. Moreover, let us assume that the sequence $\{\varphi_n\}$ is so arranged that $\lambda_1$ is different from zero, while the positive terms of $|\lambda_n|$ are in monotone order. Let $\{\mu_n\}$ be a sequence of positive numbers in the spectral set of $\mathcal{I}$ with $\mu_1 < |\lambda_1|$; further let $\{\lambda_{nk}\}$ be an infinite subsequence of $|\lambda_n|$ such that $0 < |\lambda_{nk}| \leq \mu_k$, $k = 1, 2, \ldots$. Finally, let $E$ be the projection with range $\mathcal{M}$ determined by the orthonormal set $\{\varphi_{nk}\}$ and let $W$ be a partially isometric transformation with initial set $\mathcal{S}$ and final set $\mathcal{M}$.

Then $WT - TW$ is in $\mathcal{I}$. Hence $W*WT* - TWW* = WTW* - TE$ is in $\mathcal{I}$. But then, since $|\lambda_{nk}|$ is in the spectral set of $\mathcal{I}$ by choice of that subsequence, $TE$ is in $\mathcal{I}$. Hence $W*WT*W = T$ is in $\mathcal{I}$ too, which is a contradiction.

It remains to prove the theorem for the case $\mathcal{I} = \mathcal{F}$. Let us suppose that $BA - AB$ is in $\mathcal{F}$ for all $B$ in $\mathcal{B}$ and that $A$ is not in $\mathcal{F}$. Then $A$ is of the form $T + \lambda I$, where $T$ is in $\mathcal{F}$ and not in $\mathcal{I}$. Hence there exists an infinite orthonormal set $\{\varphi_n\}$ in $\mathcal{S}$ such that $T\varphi_n = \lambda_n \varphi_n$, $\lambda_n \neq 0$, $n = 1, 2, \ldots$, $\lambda_n \neq \lambda_m$ if $m \neq n$. Hence, if $U$ is defined by the equations

\[ U\varphi_n = \varphi_{n-1}, \quad U\varphi_{n-1} = \varphi_n, \quad n = 1, 2, \ldots, \]

in the closed linear manifold determined by $\{\varphi_n\}$, $U = I$ in $\mathcal{S} \cap \mathcal{M}$, $UTU^{-1} - T$ is not in $\mathcal{F}$ and hence $UT - TU$ is not either. Therefore the assumption that $A$ is not in $\mathcal{F}$ leads to a contradiction and the theorem is proved for $\mathcal{I} = \mathcal{F}$.

3. A metric in $\mathcal{B}/\mathcal{F}$

We now confine our attention to the case $\mathcal{I} = \mathcal{F}$, beginning with the definition of a norm in $\mathcal{B}/\mathcal{F}$.

Throughout the remainder of the paper we employ the notation $|A|$ for the bound of the operator $A$ of $\mathcal{B}$.

**Definition 3.1.** Let $\alpha$ be an arbitrary element of $\mathcal{B}/\mathcal{F}$. We define $|\alpha|$, called the norm of $\alpha$, by the equation

\[ |\alpha| = \text{g. l. b. } |A|. \]

**Theorem 3.1.** The norm $|\alpha|$ in $\mathcal{B}/\mathcal{F}$ has the following properties:

1. $|\alpha| \geq 0$, the equality sign holding if and only if $\alpha = 0$;
2. $|\alpha + \beta| \leq |\alpha| + |\beta|$;
3. $|\alpha\beta| \leq |\alpha||\beta|$;
4. $|a\alpha| = |a| |\alpha|$;
5. $|\alpha^*| = |\alpha|$;
6. $|1| = 1$.

---

*If no such subsequence exists, $T$ is in $\mathcal{F}$ and hence in $\mathcal{I}$.\"
The validity of the laws (2) — (5) is an immediate consequence of the definition of $|\alpha|$ in terms of the norm $|A|$ in $B$, and the fact that the latter function has those properties. The same is true of the assertion $|\alpha| \geq 0$. Moreover, in view of (2) and (3) we can conclude that the set of elements $\alpha$ of $B/T$ for which $|\alpha| = 0$ is a two-sided ideal in $B/T$. Hence, since $T$ is a prime ideal in $B$, we have either $\alpha = 0$ when and only when $|\alpha| = 0$, or $\alpha = 0$ for all $\alpha$ in $B/T$.

Thus to complete the proof it is necessary only to establish (6). We have then to show that

$$\text{g. l. b. } |I + T| = 1.$$ 

We note first

$$|I + (T + T^*)/2| \leq |(I + T)/2| + |(I + T^*)/2| = |I + T|;$$

hence we need only show g. l. b. $|I + T| = 1$ for $T \in T, T = T^*$. But, if $T$ is so restricted, we can find, for any $\varepsilon > 0$, a real number $\lambda$ with $|\lambda| < \varepsilon$ such that $T\varphi = \lambda\varphi$ for some $\varphi \neq 0$ in $\mathcal{F}$, by virtue of the spectral properties of $T$. Thus $|(I + T)\varphi|/|\varphi| > (1 - \varepsilon)$ and hence $|I + T|$ exceeds $1 - \varepsilon$. But then, since $\varepsilon$ is an arbitrary positive number, we have $|I + T| \geq 1$. Hence, since $|I| = 1$, (6) follows.

**Theorem 3.2.** With $|\alpha - \beta|$ interpreted as the distance between $\alpha$ and $\beta$, $B/T$ is a complete linear metric space.

That $B/T$ is a metric space follows from Theorem 3.1 while its linear properties are evident. To show that it is complete we must prove that, for every sequence $\{\alpha_n\}$ in $B/T$ such that

$$\lim_{n,m \to \infty} |\alpha_n - \alpha_m| = 0,$$

there exists an element $\alpha$ such that

$$\lim_{n \to \infty} |\alpha_n - \alpha| = 0.$$

Let $\{\alpha_n\}$ be a sequence satisfying the first of these conditions. We choose a subsequence $\{\alpha_{n_k}\}$ such that

$$|\alpha_{n_k} - \alpha_n| \leq \frac{1}{2^{k+1}},$$

for $n \geq n_k$.

We then choose an arbitrary element $A_{n_1}$ of $\alpha_{n_1}$ and an element $C_1$ of $\alpha_{n_2} - \alpha_{n_1}$ such that $|C_1| \leq \frac{1}{2}$. Setting $A_{n_2} = C_1 + A_{n_1}$, we have $A_{n_2}$ in $\alpha_{n_2}$

$$|A_{n_2} - A_{n_1}| \leq \frac{1}{2}.$$

---

21 For the fact that $T$ is divisorless in $B$ implies that $B/T$ contains no two-sided ideals except $(0)$ and $B/T$ itself. Cf. [19], pp. 56-57 for a discussion of ideals in commutative rings which is readily generalized to cover the case in hand.
Continuing this process we determine a sequence \( \{A_n\} \), with \( A_n \in \alpha_n \), such that
\[
|A_{n+1} - A_n| \leq \frac{1}{2^n}.
\]
Thus
\[
|A_{n+j} - A_n| \leq \sum_{k=n}^{j-1} |A_{n+k+1} - A_{n+k}| \leq \frac{1}{2^{j-1}}.
\]
Hence there exist an element \( A \) of \( \mathcal{B} \) such that
\[
\lim_{k \to \infty} |A_n - A| = 0,
\]
and if \( \alpha \) is the residue class to which \( A \) belongs we have, in consequence,
\[
\lim_{n \to \infty} |\alpha_n - \alpha| = \lim_{k \to \infty} |\alpha_n - \alpha| = 0.
\]

We note in passing that the space \( \mathcal{B}/\mathcal{I} \) is non-separable. For every idempotent except 0 in \( \mathcal{B}/\mathcal{I} \) can be shown to have the norm 1, and if \( E(\lambda) \) is the resolution of the identity in \( \mathcal{B} \) of a transformation with spectrum the entire interval \( 0 \leq \lambda \leq 1 \), the set \( \epsilon(\lambda) \) of idempotents in \( \mathcal{B}/\mathcal{I} \) such that \( \epsilon(\lambda) \) contains \( E(\lambda) \) has the property that \( \epsilon(\lambda_2) - \epsilon(\lambda_1) \) is an idempotent different from zero if \( 0 \leq \lambda_2 < \lambda_1 \leq 1 \).

4. The space \( \mathcal{B} \)

We now propose to realize \( \mathcal{B}/\mathcal{I} \) as an algebraic ring of operators in a certain complex Euclidean space. To define a space \( \mathcal{B} \) suitable for this purpose, we make use of a concept of generalized limit introduced by Banach and Mazur.\(^\text{22}\) In the interests of greater generality, however, we shall employ a less restrictive concept of generalized limit than that of these writers, and we begin with a discussion of this concept.\(^\text{23}\)

We consider a linear functional defined for all bounded sequences \( \{x_n\} \) of real numbers, denoted by \( \text{Lim} \ x_n \), which has the following properties:

- \( \text{Lim} \ x_n + y_n = \text{Lim} \ x_n + \text{Lim} \ y_n; \)
- \( \text{Lim} \ x_n \geq 0, \) for \( x_n \geq 0, \) \( n = 1, 2, \ldots \);
- \( \text{Lim} \ x_n \) is independent of \( x_p \) for each integer \( p; \)
- \( \text{Lim} \ 1 = 1. \)

\(^{22}\) [2], p. 34.

\(^{23}\) The possibility of generalizing the notion of Banach in this way was pointed out to us by J. v. Neumann; originally we had employed the Banach limit.
For subsequent use, we note that (a) and (b) imply
\[(e) \quad \lim_{n \to \infty} x_n \geq \lim_{n \to \infty} y_n, \quad \text{if} \quad x_n \geq y_n \quad \text{for} \quad n = 1, 2, \ldots.
\]

The reader will observe that the four preceding conditions differ from the four basic properties of the Banach limit, as given in the reference cited, in the following respects: first, we do not require homogeneity; second, and more important, the Banach limit has the property
\[(1) \quad \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n
\]
in place of our (c). Since from (1) one has
\[\lim_{n \to \infty} x_{n+r+1} = \lim_{n \to \infty} x_n,
\]
it is clear that (1) implies (c).

We now wish to show that homogeneity is a consequence of conditions (a) - (d) and that the use of (c) instead of (1) does not affect the other essential properties of $\lim_{n \to \infty} x_n$.

To begin we observe that (a) implies
\[(2) \quad \lim_{n \to \infty} rx_n = r \lim_{n \to \infty} x_n \quad \text{for all rational} \quad r.
\]

We now consider an arbitrary bounded sequence $\{x_n\}$, and rational upper and lower bounds, $R$ and $r$, respectively, of $\{x_n\}$. Invoking (2), (d), and (e), we then obtain
\[R = \lim_{n \to \infty} R \geq \lim_{n \to \infty} x_n \geq \lim_{n \to \infty} r = r.
\]
Hence, since $R$ is any rational upper bound of $\{x_n\}$, and $r$ any rational lower bound, we have
\[(3) \quad \text{l. u. b.} \quad x_n \geq \lim_{n \to \infty} x_n \geq \text{g. l. b.} \quad x_n
\]
Moreover, if we now invoke (c) in conjunction with (3), it becomes clear at once that we must have
\[(f) \quad \limsup_{n \to \infty} x_n \geq \lim_{n \to \infty} x_n \geq \liminf_{n \to \infty} x_n
\]
Thus, we have the important property
\[(g) \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n \quad \text{whenever} \quad \lim_{n \to \infty} x_n \text{exists.}
\]

\[\text{\textsuperscript{24}} \text{The argument of Banach, loc. cit., thus serves to establish the existence of a functional with the properties (a)-(d). An elegant and simple direct proof of the existence of such functionals has been obtained by Ulam and Kakutani independently, but has not been published. Moreover, it is not difficult to show that there exist functionals satisfying (a)-(d) but not (1). This latter fact, however, is not of essential importance in the present paper, but rather in certain related investigations. Added in proof: Since the completion of this paper J. v. Neumann has developed a general theory of limits of the sort used here. His results will appear in a forthcoming number of the Annals of Mathematics Studies dealing with the theory of measure.}\]
Furthermore, since it is now clear that \( \lim \) is a bounded additive functional on the space of all bounded sequences, we can conclude that it is homogeneous:

\[
\lim a x_n = a \lim x_n, \quad \text{for all numbers } a.
\]

We now extend the notion of generalized limit to bounded sequences of complex numbers in the obvious way; if \( \{x_n\} \) is such a sequence we set

\[
\lim x_n = \lim \Re x_n + i \lim \Im x_n.
\]

It is then readily proved that properties (a), (c), (d), (g), (h), persist in the complex case.

Hereafter, as occasion requires, we shall refer to the properties of \( \lim x_n \) as given above by letter without other comment.

We turn now to the construction of our space \( \mathcal{Q} \). First, we consider the class \( \mathcal{Q}'' \) of all sequences \( \{f_n\} \) in Hilbert space \( \mathcal{S} \) which are weakly convergent to zero; this class is evidently a module when we define addition and scalar multiplication by the equations

\[
\{f_n\} + \{g_n\} = \{f_n + g_n\},
\]

\[
a\{f_n\} = \{af_n\}.
\]

In \( \mathcal{Q}'' \), we define \( (\{f_n\}, \{g_n\}) \) by the equation

\[
(\{f_n\}, \{g_n\}) = \lim_{n \to \infty} (f_n, g_n),
\]

invoking the boundedness of the sequences \( \{f_n\} \) and \( \{g_n\} \) to assure the boundedness of the sequence of numbers \( \{(f_n, g_n)\} \). Then from the properties of \( \lim \) given above and the properties of the inner product in \( \mathcal{S} \), we have

\[
(a\{f_n\} + b\{g_n\}, \{h_n\}) = a(\{f_n\}, \{h_n\}) + b(\{g_n\}, \{h_n\}),
\]

\[
(\{f_n\}, \{g_n\}) = (\{g_n\}, \{f_n\}),
\]

\[
(\{f_n\}, \{f_n\}) = 0.
\]

Thus \( \{f_n\}, \{g_n\} \) has all the properties requisite for an inner product in \( \mathcal{Q}'' \) except that requiring that \( (\{f_n\}, \{f_n\}) = 0 \) if and only if \( \{f_n\} = 0 \), and it is easy to see that this requirement is not fulfilled, since \( \mathcal{Q}'' \) contains sequences strongly convergent to zero. However, as A. E. Taylor [18] has pointed out, this requirement is not an essential one, since we can regard it not as a postulate but as a definition of zero and thus of equality. In our case, this means that we must identify the sequences \( \{f_n\} \) and \( \{g_n\} \) provided

\[
\lim_{n \to \infty} |f_n - g_n|^2 = 0.
\]
In this way, we obtain a class $\mathcal{Q}'$ of elements $f, g, \ldots$, the quotient group of the additive group $\mathcal{Q}''$ by the subgroup of elements $\{f_n\}$ such that

$$\lim_{n \to \infty} |f_n|^2 = 0.$$ 

If under this homomorphism $\{f_n\} \to f, \{g_n\} \to g$, we define

$$(f, g) = (\{f_n\}, \{g_n\}).$$

Thus we achieve in $\mathcal{Q}'$ a (possibly incomplete) complex Euclidean space; that is to say, $\mathcal{Q}'$ is a module and the function $(f, g)$ has the properties

$$(af + bg, h) = a(f, h) + b(g, h),$$

$$(f, g) = (g, f),$$

$$(f, f) \geq 0,$$

$$(f, f) = 0 \text{ implies } f = 0.$$ 

Since the pertinent facts in this connection are discussed in the paper of Taylor cited above, it is unnecessary for us to dwell on them here.

It now remains for us to consider the space $\mathcal{Q}'$ with reference to the matters of completeness and separability. The answers to both questions are provided through the two simple lemmas which follow.

**Lemma 4.1.** The cardinal number of $\mathcal{Q}'$ is not greater than the cardinal number $c$ of the continuum.

Let $\{\varphi_m\}$ be an arbitrary complete orthonormal set in $\mathcal{Q}$, $\{f_n\}$ an arbitrary element of $\mathcal{Q}''$. Then, if

$$f_n = \sum_{m=1}^{\infty} a_{n,m}\varphi_m$$

$\{a_{n,m}\}$ is a bounded matrix. Thus the cardinal number of $\mathcal{Q}''$ does not exceed the cardinal number of the class of all bounded infinite matrices, and the cardinal number of the latter class is $c$. Hence the cardinal number of $\mathcal{Q}'$ is certainly less than $c$.

**Lemma 4.2.** The space $\mathcal{Q}'$ contains an orthonormal set with cardinal number $c$.

We consider an enumeration $\{r_\alpha\}$ of the positive rational numbers, a complete orthonormal set $\{\varphi_n\}$ in $\mathcal{Q}$, and the correspondence $r_\alpha \leftrightarrow \varphi_n$ between them. We denote by $\{\varphi_\alpha\}$ the class of all infinite subsequences of $\{\varphi_n\}$, $\alpha$ running over some set which we leave undesignated. Now let $a_1$ and $a_2$ be any two distinct positive numbers, $\{r'_\alpha\}$ and $\{r''_\alpha\}$ infinite subsequences of $\{r_\alpha\}$ convergent to $a_1$ and $a_2$ respectively. Corresponding to $\{r'_\alpha\}$ and $\{r''_\alpha\}$ we have two subsequences of $\{\varphi_n\}$ which belong to $\{\varphi_\alpha\}$; we denote these by $\{\varphi_{\alpha_1}^{a_1}\}$ and $\{\varphi_{\alpha_2}^{a_2}\}$, respectively. Then, for $n$ larger than some integer $N$, we have $\varphi_{\alpha_1}^{a_1} \neq \varphi_{\alpha_2}^{a_2}$ and hence

$$\lim_{n \to \infty} (\varphi_{\alpha_1}^{a_1}, \varphi_{\alpha_2}^{a_2}) = 0.$$
Moreover,
\[ \lim_{n \to \infty} |\varphi_n|^2 = 1, \]
for all \( \alpha \). Hence, if \( \varphi_\alpha \) denotes the element of \( \mathcal{S}' \) containing \( \{\varphi_\alpha^*\} \), it follows that the set of all \( \{\varphi_\alpha\} \) contains an orthonormal set with cardinal number \( c \).

**Theorem 4.1.** The space \( \mathcal{S}' \) is incomplete.

Consider an orthonormal set \( \{\varphi_\alpha\} \) in \( \mathcal{S}' \) with cardinal number at least \( c \). Then the set of all \( \sum a_\alpha \varphi_\alpha \) with \( \sum |a_\alpha|^2 < \alpha \) has cardinal number at least \( 2^c \), and hence, in view of Lemma 4.1, this set cannot belong to \( \mathcal{S}' \). Therefore \( \mathcal{S}' \) is incomplete.\(^{25}\)

We now denote by \( \mathcal{S} \) the space obtained by completing \( \mathcal{S}' \); the details of this construction are described in [5] and in [13], so we need not consider them here.

**Theorem 4.2.** The dimension number of \( \mathcal{S} \) is \( c \).

Since \( \mathcal{S}' \) is dense in \( \mathcal{S} \), the dimension number of \( \mathcal{S} \) cannot exceed \( c \), by Lemma 4.1. But by Lemma 4.2, it cannot be less than \( c \).

5. The algebraic ring \( \mathfrak{M} \) and congruence modulo \( \mathcal{I} \) in \( \mathfrak{B} \)

We now consider transformations induced in the space \( \mathcal{S} \) by means of members of the ring \( \mathfrak{B} \).

**Lemma 5.1.** Let \( A \) be an arbitrary bounded everywhere-defined transformation in \( \mathcal{S} \), \( \{f_n\} \) an arbitrary sequence of the class \( \mathcal{S}'' \). Then \( \{Af_n\} \) is in \( \mathcal{S}'' \) and \( \lim_{n \to \infty} |Af_n|^2 = 0 \) if \( \lim_{n \to \infty} |f_n|^2 = 0 \).

That \( \{Af_n\} \) is in \( \mathcal{S}'' \) follows at once from the fact that a bounded transformation is weakly continuous. And since we have \( |Af_n|^2 \leq |A|^2 |f_n|^2 \), \( n = 1, 2, \ldots \), it follows from property (e) of \( \lim \) that \( \lim_{n \to \infty} |f_n|^2 = 0 \) implies
\[ \lim_{n \to \infty} |Af_n|^2 = 0. \]

**Theorem 5.1.** Let \( A \) be an arbitrary member of \( \mathfrak{B} \). Then, if \( f \) is an arbitrary element of \( \mathcal{S} \), and \( \{f_n\} \) belongs to \( f \), we set
\[ g = T_1(A)f, \]
where \( g \) is the element of \( \mathcal{S} \) containing \( \{Af_n\} \). The transformation \( T_1(A) \) so defined in \( \mathcal{S} \) is a single valued linear bounded transformation with bound not exceeding the bound of \( A \) in \( \mathcal{S} \). Thus \( T_1(A) \) has a unique closed bounded extension \( T(A) \) with domain \( \mathcal{S} \), and the bound of \( T(A) \) does not exceed the bound of \( A \) in \( \mathcal{S} \).

That \( T_1(A) \) is single-valued follows at once from Lemma 5.1, while its linear character is a consequence of the linearity of \( A \). Since, in addition,
\[ |T_1(A)f|^2 = \lim_{n \to \infty} |Af_n|^2 \leq \lim_{n \to \infty} |A|^2 |f_n|^2 = |A|^2 |f|^2 \]

---

\(^{25}\) This simple proof of Theorem 4.1 was suggested by J. v. Neumann. The theorem can also be proved directly.
it is evident that $T_1(A)$ is bounded with bound less than or equal to $|A|$. Thus the transformation $T_1(A) = T(A)$ exists and has domain $\mathcal{L}$, while its bound is clearly the same as that of $T_1(A)$.

**Theorem 5.2.** The class $\mathcal{M}$ of operators $T(A)$ in $\mathcal{L}$, defined for all $A$ in $\mathcal{B}$, is an algebraic ring of operators in the class of all bounded everywhere defined operators in $\mathcal{L}$, and is a homomorphism of $\mathcal{B}$ with respect to the operations $+, \cdot, ^*$, and scalar multiplication; that is,

$$T(A + B) = T(A) + T(B), \quad T(AB) = T(A)T(B),$$

$$T(A^*) = T^*(A), \quad T(aA) = aT(A).$$

All of these relations are quite obvious, except possibly $T(A^*) = T^*(A)$. To prove this we consider two arbitrary elements $f$ and $g$ of $\mathcal{L}$' and sequences $\{f_n\}$ and $\{g_n\}$ belonging to $f$ and $g$, respectively. Then

$$(T(A)f, g) = \lim_{n \to \infty} (f_n, g_n) = \lim_{n \to \infty} (f_n, A^*g_n) = (f, T(A^*)g).$$

Thus $T(A^*)$ and $T^*(A)$ coincide on $\mathcal{L}$, and therefore throughout $\mathcal{L}$.

**Lemma 5.2.** A necessary and sufficient condition that $T(A)$ be the transformation in $\mathcal{L}$ which takes every element of $\mathcal{L}$ into zero is that $A$ belong to the ideal $\mathcal{I}$ of totally continuous operators in $\mathcal{B}$. Thus $T(A) = T(B)$ if and only if $A$ is congruent to $B$ modulo $\mathcal{I}$.

From the homomorphism $\mathcal{B} \to \mathcal{M}$, it follows that the set of all $A$ in $\mathcal{B}$ such that $T(A) = 0$ is a two-sided ideal $\mathcal{J}$ in $\mathcal{B}$. Moreover, since a totally continuous transformation $A$ in $\mathcal{B}$ takes weakly convergent sequences into strongly convergent ones, it follows from property (e) of $\lim$ that $\mathcal{J} \supseteq \mathcal{I}$. Hence, by Theorem 1.4, we have either $\mathcal{I} = \mathcal{J}$ or $\mathcal{J} = \mathcal{B}$. But since $\mathcal{J}$ clearly fails to contain the identity in $\mathcal{B}$, we must conclude that $\mathcal{I} = \mathcal{J}$, which establishes the lemma.

Since it also follows immediately from the homomorphism $\mathcal{B} \to \mathcal{M}$ that $\mathcal{M}$ is isomorphic to the ring $\mathcal{B}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of all $A$ in $\mathcal{B}$ such that $T(A) = 0$, we can now conclude that $\mathcal{M}$ is isomorphic to $\mathcal{B}/\mathcal{J}$. More precisely, we have

**Theorem 5.3.** The algebraic ring $\mathcal{M}$ is isomorphic to $\mathcal{B}/\mathcal{J}$ with respect to the operations $+, \cdot, ^*, a \cdot$, an element $T(\alpha)$ of $\mathcal{M}$ corresponding to the element $\alpha$ of $\mathcal{B}/\mathcal{J}$ if and only if $\alpha$ belongs to $\alpha$.

**Definition 5.1.** If $\alpha$ is an arbitrary element of $\mathcal{B}/\mathcal{J}$, we define $\alpha_T(\alpha)$ as the element of $\mathcal{M}$ corresponding to $\alpha$ under the isomorphism of Theorem 5.3.

Evidently $T(\alpha)$ is identical with $T(A)$, for all $A$ in $\alpha$, and we shall continue to use both notations for elements of $\mathcal{M}$ as occasion requires.

**Theorem 5.4.** An element $T(\alpha)$ of $\mathcal{M}$ is self-adjoint, partially isometric, or unitary, respectively if and only if $\alpha$ has that property in the sense of Definition 2.1. An element $T(\alpha)$ of $\mathcal{M}$ is a projection if and only if $\alpha$ is an idempotent according to that definition.

The assertion of the theorem concerning self-adjointness is obvious. To prove the other parts of the theorem we note first that since the properties which form
the various criteria of Definition 2.1 are all defined in terms of the operations and $*$, it follows from Theorem 5.3 that an element $\alpha$ of $\mathcal{B}/\mathcal{T}$ possesses one of them if and only if $T(\alpha)$ does. But for transformations, each of these properties is characteristic of the class of transformations in question: more precisely, an everywhere defined bounded linear operator $T$ in $\mathcal{L}$ is a projection if and only if $T^2 = T^* = T$,\footnote{[17], Theorems 2.35, 2.36.} is partially isometric if and only if $T^*T$ is a projection,\footnote{[9], Lemma 4.3.2.} and thus is obviously unitary if and only if $T^*T$ and $TT^*$ are equal to the identity in $\mathcal{L}$. Hence the theorem follows.

The reader will note that Theorems 2.3–2.7 can be interpreted now to yield assertions concerning the homomorphism $\mathcal{B} \to \mathfrak{M}$; the details here are obvious and we omit them.

**Theorem 5.5.** Let $\alpha$ be an arbitrary element of $\mathcal{B}/\mathcal{T}$. Then the bound of the operator $T(\alpha)$ in $\mathcal{L}$ is $|\alpha|$; in other words, the isomorphism $\mathcal{B}/\mathcal{T} \leftrightarrow \mathfrak{M}$ is an isometry. Thus $\mathfrak{M}$ is closed in the uniform topology for operators.

Evidently the concluding assertion is a consequence of the first one and Theorem 3.2. Hence we need only show $|T(\alpha)| = |\alpha|$.

From the final statement of Theorem 5.1 and the definition of the norm in $\mathcal{B}/\mathcal{T}$, we have at once

\[ |T(\alpha)| \leq |\alpha|. \tag{1} \]

Hence we need only establish

\[ |T(\alpha)| \geq |\alpha|. \tag{2} \]

To prove (2), we first select an arbitrary element $A$ of $\alpha$, with canonical decomposition $A = WB$. We then denote by $\lambda$ the lowest upper bound of those points of the spectrum of $B$ which are either limit points of the spectrum or characteristic values of infinite multiplicity, and by $S$ the set of points $\mu$ in the spectrum of $B$ such that $\mu$ exceeds $\lambda$. Then $S$ clearly consists entirely of isolated points, each a characteristic value of finite multiplicity. Furthermore, either $S$ is a finite sequence $\{\mu_n\}$ or an infinite sequence with $\lambda$ as limit. Hence if $\mathfrak{M}_n$ is the characteristic manifold of $B$ corresponding to $\mu_n$, $n = 1, 2, \ldots$, and we set $C = B - \lambda I$ on $\mathfrak{M} = \sum_n \mathfrak{M}_n$, $C = 0$ on $\mathcal{S} \ominus \mathfrak{M}$, $C$ belongs to $\mathcal{T}$. Thus, if $B_1 = B - C$, then $A_1 = WB_1$ belongs to $\alpha$. Moreover, this is evidently the canonical decomposition of $A_1$, so $|A_1| = |B_1|$. Hence we have

\[ |B_1| \geq |\alpha|. \tag{3} \]

Now let us consider the transformation $T(B_1)$ in $\mathcal{L}$. We distinguish two cases, according as the sequence $\{\mu_n\}$ is infinite or finite. If $\{\mu_n\}$ is infinite, $\mathfrak{M} = \sum_n \mathfrak{M}_n$ is a Hilbert space and contains an infinite orthonormal set $\{\phi_n\}$. Furthermore, $B_1 \phi_n = \lambda \phi_n$, $n = 1, 2, \ldots$. Thus, if $\varphi$ is the element of $\mathcal{L}'$ to which
\{\varphi_n\} belongs, we have
\[(T(B_1)\varphi, \varphi) = \lim_{n \to \infty} (B_1\varphi_n, \varphi_n) = \lambda.\]
(4)
\[(\varphi, \varphi) = \lim_{n \to \infty} (\varphi_n, \varphi_n) = 1.\]

But clearly \(\lambda = |B_1|\), and hence we have
\[(5) \quad |T(B_1)| \geq |B_1|.\]

Now suppose \(\{\mu_n\}\) is finite, and let \(E(\lambda)\) be the resolution of the identity for \(B_1\) in \(\mathcal{S}\). Then, since \(\lambda\) is a limit point of the spectrum of \(B_1\), there exists a monotone increasing sequence \(\{\lambda_n\}\) with limit \(\lambda\), such that \(E(\lambda_{n+1}) - E(\lambda_n)\) is different from zero, \(n = 1, 2, \ldots\). Hence we can select an orthonormal set \(\{\varphi_n\}\) with \(\varphi_n\) in the range of \(E(\lambda_{n+1}) - E(\lambda_n)\), \(n = 1, 2, \ldots\). Moreover, for every \(n\), we have \(\lambda_n \leq (B_1\varphi_n, \varphi_n) \leq \lambda_{n+1}\). Hence if \(\varphi\) is the element of \(\mathcal{S}'\) containing \(\{\varphi_n\}\), we have (4) in this case also.

Finally, since \(B_1 = W^*A_1\), we have \(T(B_1) = T(W^*)T(A_1)\) and since \(T(W^*)\) is partially isometric by Theorems 5.4 and 2.6, it follows that we have \(T(W^*)| = 1\), and hence that
\[(6) \quad |T(A_1)| \geq |T(B_1)|\]
holds. But \(T(A_1) = T(\alpha)\), and hence from (3), (5) and (6) we have (2) which completes the proof of the theorem.

We now wish to prove that \(\mathcal{M}\) is not closed in the weak or strong topologies for operators. The proof reposes on the following lemma concerning monotone sequences of projections in \(\mathcal{M}\).

**Lemma 5.3.** Let \(\{T(\varepsilon_n)\}\) be a sequence of projections in \(\mathcal{M}\) such that \(T(\varepsilon_{n+1}) \leq T(\varepsilon_n)\), \(n = 1, 2, \ldots\). Then, if \(\lim_{n \to \infty} T(\varepsilon_n) = 0\), \(T(\varepsilon_n) = 0\) for all \(n\) greater than some integer \(M\).

By Theorem 5.4, each of the terms of \(\{\varepsilon_n\}\) is an idempotent, and consequently by Theorem 2.4, contains a projection \(E_\alpha\) of \(\mathcal{B}\). The sequence \(\{E_n\}\), however, is evidently not necessarily monotone, and our next step is to show that there exists a monotone non-increasing sequence \(\{F_n\}\) of projections in \(\mathcal{B}\) such that \(E_n - F_n\) is in \(\mathcal{F}\), \(n = 1, 2, \ldots\). We begin by setting \(F_1 = E_1\) and then, assuming that \(F_n\) is determined for all \(n \leq N\), we show that \(F_{N+1}\) can be defined.

We note first that
\[T(\varepsilon_n)T(\varepsilon_{n+1})T(\varepsilon_n) = T(\varepsilon_{n+1})\]
and hence that \(F_nE_{N+1}F_N - E_{N+1}\) is in \(\mathcal{F}\). Consequently, it follows that
\[(F_nE_{N+1}F_N)^2 - F_nE_{N+1}F_N\]
is in \(\mathcal{F}\). Thus, if \(\mathcal{M}_N\) is the range of \(F_N\), \(F_NE_{N+1}F_N\) induces in \(\mathcal{M}_N\) a self-adjoint transformation \(E_{N+1}^*\) congruent to its square modulo the class of all totally continuous operators in \(\mathcal{M}_N\). Hence, by Theorem 2.4, there exists in \(\mathcal{M}_N\) a projection \(F_{N+1}'\) congruent to \(E_{N+1}^*\) modulo that class. Therefore if \(F_{N+1}\) is the projection in \(\mathcal{S}\) which is equal to \(F_{N+1}'\) in \(\mathcal{M}_N\), equal to
zero in $\mathcal{S} \ominus \mathcal{M}_N$, then $F_{N+1} - F_N E_{N+1} F_N$ is in $\mathcal{S}$. But then $F_{N+1} - E_{N+1}$ is in $\mathcal{S}$ and since $F_{N+1}$ is equal to zero in $\mathcal{S} \ominus \mathcal{M}_N$, we have $F_{N+1} \leq F_N$. Thus the sequence $\{F_n\}$ with the stated properties exists.

Now let us suppose that $T(\epsilon_n)$ is never zero. Then, if $\lim_{n \to \infty} T(\epsilon_n) = 0$, $T(\epsilon_n)$ must be different from $T(\epsilon_{n+1})$ for an infinite number of values of $n$. Hence we can select a subsequence $\{T(\epsilon_{n_k})\}$ such that $T(\epsilon_{n_{k+1}}) < T(\epsilon_{n_k})$ holds, $k = 1, 2, \ldots$. Consequently, we clearly have $F_{n_{k+1}} < F_{n_k}$, $k = 1, 2, \ldots$, since $T(\epsilon_n) = T(F_n)$. Therefore, if $\mathcal{N}_k$ is the range of $F_{n_k}$, none of the spaces $\mathcal{N}_k \oplus \mathcal{N}_{k+1}$ is empty, and we can select an orthonormal set $\{\varphi_k\}$ with $\varphi_k$ in $\mathcal{N}_k \ominus \mathcal{N}_{k+1}$, $k = 1, 2, \ldots$. Let $\mathcal{N}_0$ be the closed linear manifold determined by $\{\varphi_1\}$, $F_0$ the projection with range $\mathcal{N}_0$. Then $F_0 - F_0 F_{n_k} F_0$ is the projection with range determined by $\{\varphi_j\}$, $j = 1, 2, \ldots k$, and hence belongs to $\mathcal{S}$. Consequently $T(F_0) T(F_{n_k}) T(F_0) = T(F_0)$ and $T(F_0)$ is clearly not zero. Hence, since we obviously have

$$\lim_{k \to \infty} T(F_0) T(F_{n_k}) T(F_0) = 0$$

if $\lim_{n \to \infty} T(\epsilon_n) = 0$, it follows that the latter is impossible. Therefore $T(\epsilon_n) = 0$ for all $n$ sufficiently large, as we wished to show.

It is of some interest to note the following alternative statement of the preceding lemma; if $\{T(\epsilon_n)\}$ is an infinite sequence of orthogonal projections in $\mathcal{M}$, and if $\sum_{n=1}^{\infty} T(\epsilon_n)$ is the identity in $\mathcal{S}$, then all but a finite number of terms of the sequence $T(\epsilon_n)$ are zero.

**Theorem 5.6.** The algebraic ring $\mathcal{M}$ is not closed in either the weak or strong topology for operators.

We consider a monotone sequence $\{T(\epsilon_n)\}$ of projections in $\mathcal{M}$, with $T(\epsilon_{n+1}) > T(\epsilon_n)$, $n = 1, 2, \ldots$. Such a sequence is readily generated by means of a sequence $\{E_n\}$ of projections in $\mathcal{B}$ such that $E_{n+1} - E_n$ is a projection with range a Hilbert space, $n = 1, 2, \ldots$; we have only to choose $\epsilon_n$ as the residue class of $E_n$. The sequence $T(\epsilon_n)$, being monotone, is convergent in both the weak and the strong topologies and has a limit which is a projection. Let us suppose that this limit is in $\mathcal{M}$ and hence that it has the form $T(\epsilon)$ where $\epsilon$ is an idempotent in $\mathcal{B}/\mathcal{S}$. Then $|T(\epsilon_n) - T(\epsilon)|$ is a monotone non-increasing sequence of projections in $\mathcal{M}$ with limit zero and thus $T(\epsilon_n) = T(\epsilon)$ for all $n$ greater than some integer $N$, by Lemma 5.3. This, however, is impossible in view of the fact that $\{T(\epsilon_n)\}$ was chosen with $T(\epsilon_{n+1}) > T(\epsilon_n)$, $n = 1, 2, \ldots$; we must conclude therefore that the sequence $\{T(\epsilon_n)\}$ has no limit in $\mathcal{M}$.

We proceed now to examine the relationship between the spectrum of a self-adjoint member of $\mathcal{M}$ and the corresponding self-adjoint transformations in $\mathcal{B}$. We begin with a formal definition.

**Definition 5.2.** Let $A$ be an arbitrary self-adjoint transformation in $\mathcal{S}$. Then a point $\lambda$ of the spectrum of $A$ which is a limit point of the spectrum or a character-
istic value of infinite multiplicity is called a point of condensation of the spectrum. The set of all such points is called the condensed spectrum of $A$. The complementary set in the $\lambda$-plane is called the augmented resolvent set of $A$.

**Theorem 5.7.** Let $A$ be a self-adjoint transformation in $\mathcal{H}$. Then a necessary and sufficient condition that $\lambda$ belong to the augmented resolvent set of $A$ is that the manifold $\mathcal{M}$ of solutions of the equation $Af - \lambda f = 0$ have a finite dimension number and that in $\mathcal{H} \ominus \mathcal{M}$, $A - \lambda I$ induce a transformation with bounded inverse.

Let $\mathcal{M}$ be the manifold of zeros of $A$, $A_1$ the transformation induced in $\mathcal{H} \ominus \mathcal{M}$ by $A$. Then the condition of the theorem may be stated in this way: $\mathcal{M}$ is finite-dimensional and $\lambda$ is in the resolvent set of $A_1$. But the latter is possible if and only if $\lambda$ is a finite distance from the spectrum of $A_1$, which is to say that $\lambda$ is an isolated point of the spectrum of $A$ or belongs to the resolvent set of $A$. Thus the theorem follows.

**Theorem 5.8.** Let $A$ be a self-adjoint transformation in $\mathcal{H}$. Then the resolvent set of the transformation $T(A)$ in $\mathfrak{Q}$ is the augmented resolvent set of $A$ and the spectrum of $T(A)$ is the condensed spectrum of $A$. Every point in the spectrum of $T(A)$ is a characteristic value with multiplicity $c$.

Let $\lambda$ belong to the augmented resolvent set of $A$, and let $\mathcal{M}$ be the manifold of zeros of $A - \lambda I$, $E$ the projection operator of $\mathcal{H} \ominus \mathcal{M}$. Let $B$ be equal in $\mathcal{M}$ to zero, and in $\mathcal{H} \ominus \mathcal{M}$ to the inverse of the transformation induced in that space by $A - \lambda I$. Then $B$ is in $\mathcal{B}$ and $B(A - \lambda I) = E$. Thus $T(B)T(A - \lambda I) = T(E)$. But $T(E)$ is the identity in $\mathfrak{Q}$, since $I - E$ is in $\mathfrak{F}$, and hence $T(B) = [T(A) - \lambda T(I)]^{-1}$. Therefore $\lambda$ is in the resolvent set of $T(A)$.

Now suppose $\lambda$ is in the condensed spectrum of $A$. Then $\lambda$ is either a characteristic value of infinite multiplicity or a limit point of the spectrum of $A$, and in either case we can select an orthonormal set $\{\varphi_n\}$ in $\mathcal{H}$ such that

$$\lim_{n \to \infty} (A \varphi_n, \varphi_n) = \lambda$$

$$\lim_{n \to \infty} (A \varphi_n, A \varphi_n) = \lambda^2.$$  

Thus

$$\lim_{n \to \infty} |A \varphi_n - \lambda \varphi_n|^2 = \lim_{n \to \infty} (|A \varphi_n|^2 + \lambda^2 |\varphi_n|^2 - 2\lambda (A \varphi_n, \varphi_n)) = 0.$$  

But then, if $\{\psi_n\}$ is any subsequence of $\{\varphi_n\}$ and $\psi$ the element of $\mathfrak{Q}'$ containing $\{\varphi_n\}$, we have

$$T(A)\psi = \lambda \psi.$$  

Moreover, there exist $c$ such subsequences such that any two have only a finite number of terms in common and consequently $c$ orthogonal elements of $\mathfrak{Q}'$ satisfying the preceding equation. Therefore $\lambda$ is a characteristic value of $T(A)$ with multiplicity $c$.

\textsuperscript{28} These are the Häufungspunkte of the spectrum in the sense of Weyl, [20].
We have now shown that every point of the augmented resolvent set of $A$ is in the resolvent set of $T(A)$ and that every point of the condensed spectrum of $A$ is in the spectrum of $T(A)$. But since the augmented resolvent set and the condensed spectrum of $A$ together constitute the entire $\lambda$-plane, they must be respectively the entire resolvent set and the entire spectrum of $A$.

It should be observed that we cannot infer from theorem 5.8 that the sum of the characteristic manifolds of $T(A)$ is $\mathfrak{P}$. Whether or not this is so we are unable to say at present.

We now have from Theorem 5.8 the classical theorem of Weyl [20].

**Theorem 5.9.** Let $A$ and $B$ be two self-adjoint transformations in $\mathfrak{H}$, such that $A - B$ is totally continuous. Then $A$ and $B$ have the same condensed spectrum and the same augmented resolvent set.\(^{29}\)

The theorem is obvious since $T(A) = T(B)$.

We proceed now to determine the resolution of the identity corresponding to a self-adjoint transformation in $\mathfrak{H}$ by means of the resolution of the identity of a corresponding member of $\mathfrak{B}$. If $A$ is self-adjoint in $\mathfrak{D}$, $E(\lambda)$ its resolution of the identity, it is clear that $T(E(\lambda))$ has many of the properties of a resolution of the identity in $\mathfrak{P}$. Specifically, it is readily shown that the following assertions hold:

1. $T(E(\lambda))$ permutes with $T(A)$;
2. $T(E(\lambda))T(E(\mu)) = T(E(\mu))T(E(\lambda)) = T(E(\mu))$ for $\mu \leq \lambda$;
3. $T(E(\lambda)) = 0$ if $\lambda$ is less than the lower bound of $T(A)$, $T(E(\lambda)) = 1$ if $\lambda$ exceeds the upper bound of $T(A)$;
4. in the range of $T(E(\lambda))$, the upper bound of $T(A)$ does not exceed $\lambda$, and in the range of $1 - T(E(\lambda))$ the lower bound of $T(A)$ is not less than $\lambda$.

In spite of these facts $T(E(\lambda))$ fails usually to be the resolution of the identity for $T(A)$. This is readily seen in view of Lemma 5.3, which assures us that in general we do not have

$$\lim_{\varepsilon \to 0} T(E(\lambda + \varepsilon)) = T(E(\lambda)).$$

Moreover, if $A$ and $B$ are self-adjoint and congruent modulo $\mathfrak{J}$, $E(\lambda)$ and $F(\lambda)$ their respective resolutions of the identity, we do not in general have $E(\lambda) - F(\lambda)$ in $\mathfrak{J}$. Thus for a self-adjoint transformation $T(A)$ in $\mathfrak{H}$ we can exhibit many monotone families of projections with the properties (1) — (4) above, none of which is the resolution of the identity of $A$.

We can however, derive from any one of these families of projections the resolution of the identity belonging to the member of $\mathfrak{M}$ in question. The procedure is described in the following theorem.

**Theorem 5.10.** Let $A$ be a self-adjoint transformation in $\mathfrak{D}$, $E(\lambda)$ its resolution of the identity. Let $E(\lambda)$ be the family of projections in $\mathfrak{P}$ defined by the equation

$$E(\lambda) = \lim_{\varepsilon \to 0} T(E(\lambda + \varepsilon)), \quad \varepsilon > 0.$$

\(^{29}\) Weyl proves also that if $A$ is an arbitrary bounded self-adjoint transformation, there exists a totally continuous self-adjoint transformation $T$ such that $A + T$ has a pure point spectrum. At present we see no direct way to derive this result from ours. Cf. also [14].
Then $E(\lambda)$ is the resolution of the identity in $\mathcal{B}$ of $T(A)$.

The existence of the limit $E(\lambda)$ follows of course from the monotone character of the family $T(E(\lambda))$ which follows in turn from the corresponding property of $E(\lambda)$. We shall now show that $E(\lambda)$ has the following six properties:

1. $E(\lambda)$ permutes with $T(A)$;
2. $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\mu)$ for $\mu \leq \lambda$;
3. $E(\lambda) = 0$ for $\lambda < a$ (a the lower bound of $T(A)$), and $E(\lambda) = 1$ for $\lambda \geq b$ (b the upper bound of $T(A)$);
4. $\lim_{\epsilon \to 0} E(\lambda + \epsilon) = E(\lambda), \epsilon > 0$;
5. in the range of $E(\lambda)$, the upper bound of $T(A)$ does not exceed $\lambda$;
6. in the range of $1 - E(\lambda)$, the lower bound of $T(A)$ is not less than $\lambda$ and if it is equal to $\lambda$ it is not attained.

The validity of (1) follows at once from the permutability of $T(A)$ and $T(E(\lambda))$. To prove (2) we note first that $T(E(\lambda + \epsilon))T(E(\mu)) = T(E(\mu))T(E(\lambda + \epsilon))$ for all $\epsilon > 0$.

Thus, allowing $\epsilon$ to tend to zero, we have

$$E(\lambda)T(E(\mu)) = T(E(\mu))E(\lambda).$$

Hence

$$E(\lambda)T(E(\mu + \epsilon)) = T(E(\mu + \epsilon))E(\lambda)$$

for all $\mu$ and all $\epsilon > 0$. Consequently, again allowing $\epsilon$ to approach zero, we have

$$E(\lambda)E(\mu) = E(\mu)E(\lambda).$$

Moreover, if $\mu \leq \lambda$, we have clearly $E(\mu) \leq E(\lambda)$, and so $E(\lambda)E(\mu) = E(\mu)$. Therefore (2) holds.

Now let $a$ be the lower bound of $T(A)$. Then by Theorem 5.8, $a$ is the lower bound of the condensed spectrum of $A$ and it follows therefore that if $\lambda$ is less than $a$, then the range of $E(\lambda)$ must be finite-dimensional. For otherwise the spectrum of $A$ would have points of condensation less than or equal to $\lambda$. Thus $T(E(\lambda)) = 0$ for $\lambda < a$ and therefore $E(\lambda) = 0$ for $\lambda < a$.

On the other hand, let $b$ be the upper bound of the spectrum of $T(A)$. Then $b$ is the upper bound of the condensed spectrum of $A$ and by a similar argument $I - E(\lambda)$ has a finite-dimensional range, $\lambda \geq b$. Thus $T(E(\lambda + \epsilon)) = T(I)$ for $\lambda \geq b$ and $\epsilon > 0$. But then $E(\lambda)$ is the identity in $\mathcal{B}$ for $\lambda \geq b$. Thus we have (3).

Next we note that for $\epsilon_1 > \epsilon > 0$, we have

$$E(\lambda) \leq E(\lambda + \epsilon) \leq T(E(\lambda + \epsilon)),$$

Hence, allowing $\epsilon$ and $\epsilon_1$ both to approach zero, preserving the relations $\epsilon_1 > \epsilon$, $\epsilon_1 > 0$, we obtain (4).

Finally, we consider the behavior of $T(A)$ in the range of $E(\lambda)$ and its orthog-
onal complement. Since $E(\lambda)AE(\lambda)$ has upper bound not exceeding $\lambda$, it follows that the upper bound of $T(E(\lambda + \epsilon))T(A)T(E(\lambda + \epsilon))$ does not exceed $\lambda + \epsilon$. But for every $f$ in $\mathcal{F}$, we have

$$(E(\lambda)T(A)E(\lambda)f, f) = \lim_{\epsilon \to 0} (T(E(\lambda + \epsilon))T(A)T(E(\lambda + \epsilon))f, f), \quad \epsilon > 0.$$  

Hence the upper bound of $E(\lambda)T(A)E(\lambda)$ does not exceed $\lambda$ and (5) is established.

In entirely similar fashion, it can be shown that in the range of $1 - E(\lambda)$ the lower bound of $T(A)$ is not less than $\lambda$. Moreover, if $\lambda$ is the lower bound and this lower bound is attained, we have for some $f$ in $\mathcal{F}$,

$$(1 - E(\lambda))T(A)(1 - E(\lambda))f = \lambda f.$$  

Hence

$$\lim_{\epsilon \to 0} (T(A)(1 - E(\lambda + \epsilon))f, (1 - E(\lambda + \epsilon))f) = \lambda |f|^2.$$  

But $(T(A)(1 - E(\lambda + \epsilon))f, (1 - E(\lambda + \epsilon))f)$ is monotone non-decreasing as $\epsilon$ approaches zero, and consequently we have

$$(T(A)(1 - E(\lambda + \epsilon))f, (1 - E(\lambda + \epsilon))f) \leq \lambda |f|^2$$  

for some $\epsilon > 0$. This, however is impossible, since in the range of $1 - E(\lambda + \epsilon)$, the lower bound of $T(A)$ is not less than $\lambda$. Hence, we have (6).

These six properties are sufficient to characterize $E(\lambda)$ as the resolution of the identity of $T(A)$. One can for example, argue as follows: On the basis of these six properties the approximation theorem (Theorem 6) of the paper [6] of Lengyel and Stone can be proved and from this result it follows that $E(\lambda)$ permutes with every bounded linear operator defined over $\mathcal{F}$ which permutes with $T(A)$ ([6], Theorem 7). But this fact together with properties (1), (5), (6) above uniquely determines $E(\lambda)$ as the resolution of the identity for $T(A)$ ([6], Theorem 5).

We are now in position to establish certain relationships between the resolutions of the identity belonging to two self-adjoint transformations in $\mathcal{D}$ whose difference is totally continuous.

**Theorem 5.11.** Let $A$ and $B$ be self-adjoint transformations in $\mathcal{D}$ such that $A - B$ is totally continuous. Let $E(\lambda)$ and $F(\lambda)$ be the resolutions of the identity corresponding to $A$ and $B$ respectively. Then, if $\mu$ is in the augmented resolvent set of $A$, $E(\mu) - F(\mu)$ is totally continuous.

Let $E(\lambda)$ be the resolution of the identity of $T(A)$ in $\mathcal{F}$. Then, since $\mu$ is in the resolvent set of $T(A)$, we have for some $\delta > 0$, $E(\mu - \delta) = E(\mu + \delta)$. But, from Theorem 5.10 it is clear that we have

$$E(\mu - \delta) \leq T(E(\mu)) \leq E(\mu + \delta),$$

$$E(\mu - \delta) \leq T(F(\mu)) \leq E(\mu + \delta).$$

\[39\] [6], Theorem 3, for example.
Thus \( T(E(\mu)) = T(F(\mu)) \) and hence \( E(\mu) - F(\mu) \) belongs to \( \mathcal{I} \), as we wished to show.

It is important to note that the requirement that \( \mu \) be in the augmented resolvent set cannot be dropped from Theorem 5.11. Consider, for example, a nonnegative definite self-adjoint transformation \( A \) in \( \mathcal{S} \) which is totally continuous and has an inverse, and its resolution of the identity \( E(\lambda) \). Consider also the transformation 0 in \( \mathcal{S} \), and its resolution of the identity \( F(\lambda) \). Then \( A - 0 \) is in \( \mathcal{I} \), but \( E(0) = 0 \) and \( F(0) = I \).

**Theorem 5.12.** Let \( A \) and \( B \) be self-adjoint transformations in \( \mathcal{S} \) such that \( A - B \) is totally continuous, and let \( E(\lambda) \) and \( F(\lambda) \) be their respective resolutions of the identity. Then, if \( \mu \) is less than \( \lambda \), there exists a totally continuous transformation \( T_{\mu,\lambda} \) in \( B \) such that \( E(\mu) + T_{\mu,\lambda} \) is a projection satisfying the inequality

\[
E(\mu) + T_{\mu,\lambda} \leq F(\lambda).
\]

If \( T_{\mu,\lambda} \) can be chosen so that the equality sign holds, then the only points \( \lambda_0 \) on the interval \( \mu < \lambda_0 < \lambda \) which belong to the spectrum of either \( A \) or \( B \) are characteristic values of finite multiplicity.

Conversely, let \( A \) and \( B \) be self-adjoint transformations in \( \mathcal{S} \) with resolutions of the identity \( E(\lambda) \) and \( F(\lambda) \), respectively. Then, if the inequalities \( T(E(\mu)) \leq T(F(\lambda)) \) and \( T(F(\mu)) \leq T(E(\lambda)) \) hold for all \( \lambda \) and \( \mu \) such that \( \mu < \lambda \), \( A - B \) is totally continuous.

We prove the converse part first. From the inequalities in question, it follows at once that

\[
\lim_{\epsilon \to 0} T(E(\lambda + \epsilon)) = \lim_{\epsilon \to 0} T(F(\lambda + \epsilon)), \quad \epsilon > 0,
\]

for all \( \lambda \) and thus that \( T(A) \) and \( T(B) \) have the same resolution of the identity. Hence \( T(A) = T(B) \), or \( A - B \) is in \( \mathcal{I} \).

Now let \( A - B \) belong to \( \mathcal{I} \), and let \( E(\lambda) \) be the resolution of the identity of \( T(A) \) in \( \mathcal{S} \). Then, for \( \mu < \lambda \), we have

\[
T(E(\mu)) \leq E(\mu) \leq T(F(\lambda)).
\]

Hence \( T(F(\lambda))T(E(\mu))T(F(\lambda)) = T(E(\mu)) \) and therefore \( F(\lambda)E(\mu)F(\lambda) - E(\mu) \) is in \( \mathcal{I} \). Consequently, invoking Theorem 2.4 with reference to the ring of bounded everywhere defined operators in the range of \( F(\lambda) \), we see that there exists a totally continuous operator \( T_0 \) in \( \mathcal{S} \), with the value zero in the range of \( I - F(\lambda) \), such that

\[
F(\lambda)E(\mu)F(\lambda) + T_0
\]

is a projection satisfying

\[
F(\lambda)E(\mu)F(\lambda) + T_0 \leq F(\lambda).
\]

---

\[ \text{Compare §4 of [20] to which this theorem is closely related.} \]
Thus, if we set
\[ T_{\mu,\lambda} = T_0 + F(\mu)E(\mu)F(\lambda) - E(\mu), \]
then \( T_{\mu,\lambda} \) is in \( \mathcal{T} \) and \( E(\mu) + T_{\mu,\lambda} \) is a projection satisfying the inequality of the theorem.

Now suppose we have
\[ E(\mu) + T_{\mu,\lambda} = F(\lambda). \]
Then \( T(E(\mu)) = T(F(\lambda)) \), and since for all \( \lambda_0 \) on the interval \( \mu < \lambda_0 < \lambda \), we have
\[ T(E(\mu)) \leq E(\lambda_0) \leq T(F(\lambda)), \]
it follows that \( E(\lambda) \) is constant on that interval. Hence every point on the interval is in the resolvent set of \( T(A) = T(B) \), and this is equivalent to the concluding statement of the first paragraph of the theorem.

**Corollary.** Let \( A, B, E(\lambda), \) and \( F(\lambda) \) be as in Theorem 5.12. Then, if \( \lambda \neq \mu \),
\[ E(\mu)F(\lambda) - F(\lambda)E(\mu) \]
is totally continuous.

From the inequality of Theorem 5.12, we have, for \( \mu < \lambda \),
\[ F(\lambda)E(\mu) + F(\lambda)T_{\mu,\lambda} = E(\mu)F(\lambda) + T_{\mu,\lambda}F(\lambda), \]
or
\[ F(\lambda)E(\mu) - E(\mu)F(\lambda) = T_{\mu,\lambda}F(\lambda) - F(\lambda)T_{\mu,\lambda}. \]
Thus \( F(\lambda)E(\mu) - E(\mu)F(\lambda) \) is totally continuous for \( \mu < \lambda \), and by symmetry for \( \mu > \lambda \).

This is not necessarily so, however, for \( \lambda = \mu \). For consider any two projections \( E \) and \( F \) in \( \mathcal{S} \) such that \( \mathcal{R}(E) \) and \( \mathcal{R}(F) \) and their orthogonal complements are Hilbert spaces. Let \( A \) be a transformation which is equal to zero in \( \mathcal{R}(E) \) and which induces in \( \mathcal{S} \ominus \mathcal{R}(E) \) a nonnegative definite self-adjoint totally continuous transformation whose inverse exists. Then \( A \) is in \( \mathcal{S} \) and if \( E(\lambda) \) is the resolution of the identity of \( A, E(0) = E \). Similarly, we can define a self-adjoint totally continuous operator \( B \) with resolution of the identity \( F(\lambda) \) such that \( F(0) = F \). But then \( A - B \) is in \( \mathcal{T} \), while in general \( E(0)F(0) - F(0)E(0) \) is not.

It follows also from the preceding example that the hypothesis \( \mu < \lambda \) of Theorem 5.12 cannot be replaced by the hypothesis \( \mu \leq \lambda \).

We conclude this paper with a few observations concerning congruence modulo \( \mathcal{T} \) in \( \mathcal{B} \) which are not restricted to self-adjoint transformations.

**Theorem 5.13.** Let \( A_1 \) and \( A_2 \) be members of \( \mathcal{B} \) whose difference belongs to \( \mathcal{T} \), \( A_1 = W_1B_1, A_2 = W_2B_2 \) their canonical decompositions. Then \( B_1 - B_2 \) is in \( \mathcal{T} \), and if zero is in the augmented resolvent set of \( B_1 \), \( W_1 - W_2 \) is in \( \mathcal{T} \).

Since \( A_1^*A_1 = B_1^2, A_2^*A_2 = B_2^2 \), we have \( T(A_1^*A_1) = T(B_1^2) = T(B_2^2) \). More-
over, \([T(B_1)]^2 = [T(B_2)]^2 = T(B_1^2)\). But \(B_1\) and \(B_2\) are nonnegative definite and hence

\[
\lim_{n \to \infty} (B_1 f_n, f_n) \geq 0, \quad \lim_{n \to \infty} (B_2 f_n, f_n) \geq 0
\]

for all \(\{f_n\}\) in \(\mathcal{L}'\), by property (f) of \(\text{Lim}\). Hence \(T(B_1)\) and \(T(B_2)\) are both nonnegative definite, and since \(T(B_2^2)\) can have only one nonnegative definite square root, we have \(T(B_1) = T(B_2)\) which implies that \(B_1 - B_2\) is in \(\mathcal{J}\).

Now let

\[ T(A_1) = WB \]

be the canonical decomposition of \(T(A_1)\). Then \(B = T(B_1) = T(B_2)\). Moreover,

\[ T(A_1) = T(W_1)T(B_1) = T(W_2)T(B_2). \]

Hence we can infer that the initial set of the partially isometric transformation \(T(W_1)\) contains the initial set of \(W\) (the closure of the range of \(B\)) and that the two transformations are equal there. Hence \(T(W_1)\) is identical with \(W\) provided the initial set of \(T(W_1)\) is the closure of the range of \(B\). But, if the origin is in the augmented resolvent set of \(B_1\), it is in the resolvent set of \(T(B_1) = B\) and hence \(W\) is unitary. Thus under the hypothesis of the theorem we have \(T(W_1) = W\). Furthermore, under that hypothesis, the origin is also in the augmented resolvent set of \(B_2\) and hence a similar argument yields the equation \(T(W_2) = W\). Thus \(T(W_1) = T(W_2)\) and \(W_1 - W_2\) is in \(\mathcal{J}\) as we wished to show.

We wish to emphasize that the hypothesis that zero be in the augmented resolvent set of \(B_1\) cannot be omitted from Theorem 5.13. For example, consider a totally continuous nonnegative definite self-adjoint transformation \(A\) with an inverse. Then if \(A = W_1B_1\) is its canonical decomposition and \(0 = W_2B_2\) is the canonical decomposition of \(0\), we have \(W_1 = I, W_2 = 0, A - 0\) in \(\mathcal{J}\).

We now extend Theorem 5.9 to cover transformations which are not necessarily self-adjoint.

**Definition 5.3.** Let \(A\) be an arbitrary bounded everywhere defined transformation in \(\mathcal{D}\). We define the augmented resolvent set of \(A\) as the set of points \(\lambda\) in the complex plane such that the following conditions are satisfied:

1. the manifold of zeros of \(A - \lambda I\) has a finite dimension number;
2. the range of \(A - \lambda I\) is closed;
3. the orthogonal complement of the range of \(A - \lambda I\) has a finite dimension number.

The complement of the augmented resolvent set is called the condensed spectrum of \(A\). Evidently the augmented resolvent set of \(A\) contains the resolvent set, so that our terminology is justified.

**Theorem 5.14.** Let \(A\) and \(C\) be two bounded everywhere-defined transforma-
tions in $\mathcal{S}$ such that $A - C$ is totally continuous. Then $A$ and $C$ have the same augmented resolvent sets.

It is sufficient for the proof to show that every point in the augmented resolvent set of $A$ is also in the augmented resolvent set of $C$.

Let $\lambda$ be a point of the augmented resolvent set of $A$, $\mathcal{M}$ the manifold of zeros of $A - \lambda I$. Then on $\mathcal{S} \oplus \mathcal{M}$, $A - \lambda I$ induces a transformation $T$ with an inverse whose range is $\mathcal{R}(A - \lambda I)$. Moreover, since $\mathcal{R}(A - \lambda I)$ is closed, $T^{-1}$ is bounded.

Now let $\mathcal{R}$ be the manifold of zeros of $C - \lambda I$. Then if $\mathcal{R}$ is a Hilbert space, $(\mathcal{S} \oplus \mathcal{M}) \cdot \mathcal{R}$ is a Hilbert space, and $T$ has a contraction $T_1$ with domain $(\mathcal{S} \oplus \mathcal{M}) \cdot \mathcal{R}$ which also has a bounded inverse. But in $(\mathcal{S} \oplus \mathcal{M}) \cdot \mathcal{R}$, we have $A = C = T_1$ and this contradicts our hypothesis that $A - C$ is in $\mathcal{T}$. Hence $\mathcal{R}$ must have a finite dimension number.

We now observe that when $\lambda$ is a point of the augmented resolvent set of $A$, $\bar{\lambda}$ is a point of the augmented resolvent set of $A^*$. Hence the preceding argument serves to show that the manifold of zeros of $C^* - \lambda I$ has a finite dimension number. But this manifold is precisely the orthogonal complement of the closure of $\mathcal{R}(C - \lambda I)$, and hence to show that $\lambda$ is in the augmented resolvent set of $C$, it remains only for us to prove that $\mathcal{R}(C - \lambda I)$ is closed.

To establish the latter we consider the canonical decompositions

$$A - \lambda I = W_1 B_1, \quad C - \lambda I = W_2 B_2.$$  

From Theorem 5.13, it follows that $B_1 - B_2$ is totally continuous. Moreover, since the range of $A - \lambda I$ is closed, so also is the range of $B_1$, while the manifold of zeros of $B_1$ is the manifold of zeros of $A - \lambda I$. But then the origin is in the augmented resolvent set of $B_1$ and hence in the augmented resolvent set of $B_2$. Hence $\mathcal{R}(B_2)$ is closed which implies that $\mathcal{R}(C - \lambda I)$ is closed also as we wished to show.

If $\lambda$ is in the augmented resolvent set of $A$, we denote by $R_\lambda$ the transformation which is equal to zero in $\mathcal{S} \oplus \mathcal{R}(A - \lambda I)$ and which takes each element $f$ of $\mathcal{R}(A - \lambda I)$ into that element $g$ in the orthogonal complement of the manifold of zeros of $\mathcal{R}(A - \lambda I)$ which satisfies $(A - \lambda I)g = f$. Thus $R_\lambda(A - \lambda I)$ is the projection with range the orthogonal complement of the manifold of zeros of $A - \lambda I$. We call the family of transformations $R_\lambda$ so defined the augmented resolvent of $A$.

**Theorem 5.15.** Let $A$ and $B$ be two transformations in $\mathcal{S}$ such that $A - B$ is in $\mathcal{T}$. Let $R^{(1)}_\lambda$ and $R^{(2)}_\lambda$ be respectively their augmented resolvents. Then $R^{(1)}_\lambda - R^{(2)}_\lambda$ is in $\mathcal{T}$ for all $\lambda$ for which these transformations are defined.

Consider the transformation $T(A) = T(B)$ in $\mathcal{S}$. If $\lambda$ is in the augmented resolvent set of $A$ (or $B$) it is in the resolvent set of $T(A)$, and $T(R^{(1)}_\lambda)$ and $T(R^{(2)}_\lambda)$ are both inverses of $T(A) - \lambda \cdot 1 = T(B) - \lambda \cdot 1$. But only one such inverse can exist; hence $T(R^{(1)}_\lambda) = T(R^{(2)}_\lambda)$ and $R^{(1)}_\lambda - R^{(2)}_\lambda$ is in $\mathcal{T}$.

**Illinois Institute of Technology**
REFERENCES