Optimal transport and non-branching geodesics

Martin Kell

Pisa, November 15th 2018
\( L^p \)-Monge problem

- Solve for good \( \mu \) and arbitrary \( \nu \) the following

\[
\inf_{\nu = T_* \mu} \int d^p(x, T(x)) d\mu(x)
\]

- When is the solution unique?
\(L^p\)-Monge-Kantorovich problem

- Show that the minimum of
  \[\inf_{\pi \in \Pi(\mu, \nu)} \int d^p(x, y) \, d\pi(x, y)\]
  is supported on a graph of measurable map.

- For \(p = 1\) almost never true.
- For \(p \in (1, \infty)\) depends on the geometry and on \(\mu\).
- If true then
  - the optimal coupling is unique
  - Monge = Kantorovich.
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- the optimal coupling is unique
- Monge = Kantorovich.
Dependence on $\mu$ and the geometry of geodesics

BLACKBOARD
Non-branching geodesics

- Assumption: \((M, d, m)\) a complete geodesic measure space

**Definition (non-branching)**

A geodesic space \((M, d)\) is non-branching if for all geodesics \(\gamma, \eta : [0, 1] \rightarrow M\) with \(\gamma_0 = \eta_0\) and \(\gamma_t = \eta_t\) for some \(t \in (0, 1)\) it holds \(\gamma_t = \eta_t\) for all \(t \in [0, 1]\).

Equivalently:
If \(m\) is a midpoint of \((x, y)\) and \((x, z)\) then \(y = z\).
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Equivalently:

If \(m\) is a midpoint of \((x, y)\) and \((x, z)\) then \(y = z\).
Lemma (no intermediate overlap)

A geodesic space is non-branching if the following holds: Whenever for two geodesics $\gamma$ and $\eta$ satisfy

$$d^p(\gamma_0, \gamma_1) + d^p(\eta_0, \eta_1) \leq d^p(\gamma_0, \eta_1) + d^p(\eta_0, \gamma_1)$$

and $\gamma_t = \eta_t$ for some $t \in (0, 1)$ then $\gamma \equiv \eta$. 
Examples of non-branching space

- Riemannian/Finsler manifolds (geodesic = “ODE solution”)
- Alexandrov spaces (comparison condition)
- Busemann $G$-spaces (unique continuation property)
- $\text{CAT}(\kappa) \oplus \text{RCD}(K,N)$-space [Kapovich-Ketterer '17]
  $\implies$ works also for $\text{MCP}_{loc}(K,N)$-spaces that are (locally) Busemann convex
- subRiemannian Heisenberg(-type) groups [Ambrosio-Rigot '04]
- subRiemannian Engel group [Ardentov-Sachkov '11,'15]
- Open: Ricci limits, RCD-spaces, Carnot groups
Some history of $L^p$-Monge-Kantorovich, $p > 1$

- Theorems using Rademacher Theorem
  - in $\mathbb{R}^n$ [Brenier '91, Gangbo-McCann '96]
  - Riemannian manifolds [McCann '01, Gigli '11]
  - Finsler manifolds [Villani '09, Ohta '09]
  - Heisenberg groups [Ambrosio-Rigot '04]
  - nice subRiemannian manifolds [Figalli-Riffort '10]
  - Alexandrov spaces [Bertrand '08/'15, Schultz-Rajala '18]

- Anyone missing?
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- Theorems using optimal transport and non-branching
  - non-branching CD($K,N$)-spaces [Gigli '12]
  - strongly non-branching doubling spaces with interpolation property [Ambrosio-Rajala '14]
  - non-branching spaces with very weak MCP [Cavalletti-Huesmann '15]

- using weaker essentially non-branching (e.n.b.) condition
  - strong CD($K,\infty$)-spaces [Rajala-Sturm '14]
  - RCD($K,N$)-spaces [Gigli-Rajala-Sturm '16]
  - e.n.b. MCP($K,N$)-spaces [Cavalletti-Mondino '17]
  - e.n.b. spaces with very weak MCP [K. '17]
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Notation

• $\Gamma \subset M \times M$

$$\Gamma_t := \{\gamma_t | \gamma \in \text{Geo}_{[0,1]}(M, d), (\gamma_0, \gamma_1) \in \Gamma\}$$

• $A \subset M$ and $x \in M$

$$A_{t,x} = (A \times \{x\})_t$$

Remark

In the following fix a $p \in (1, \infty)$ so that optimal = $d^p$-optimal, cyclically monotone = $d^p$-cyclically monotone.
Very weak MCP condition

Definition ([Cavalletti-Huesmann ’15])

A metric measure space is qualitatively non-degenerate if for all $R > 0$ there is a function $f_R : (0, 1) \rightarrow (0, \infty)$ with $C_R = \limsup_{t \rightarrow 0} f_R(t) > \frac{1}{2}$ such that whenever $\{x\}, A \subset B_R(x_0)$ then

$$m(A_t, x) \geq f_R(t)m(A).$$

Remark

Note that $2C_R > 1$. 

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10/33
**Optimal transport maps**

**Definition (Good Transport Behavior)**

A metric measure space \((M,d,\mathfrak{m})\) has **good transport behavior** \((GTB)_p\) if for all \(\mu \in \mathcal{P}^{ac}_p(M)\) and all \(\nu \in \mathcal{P}_p(M)\) every optimal coupling \(\pi\) is induced by a transport map \(T\), i.e. \(\pi = (\text{id} \times T)_\ast \mu\).

**Theorem ([Cavalletti-Huesmann ’15])**

Assume \((M,d,\mathfrak{m})\) is qualitatively non-degenerate and non-branching. Then \((M,d,\mathfrak{m})\) has good transport behavior \((GTB)_p\) for all \(p \in (1,\infty)\).
Definition (Good Transport Behavior)

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Assume \((M,d,m)\) is qualitatively non-degenerate and non-branching. Then \((M,d,m)\) has good transport behavior \((GTB)_p\) for all \(p \in (1,\infty)\).
Proof for $\nu = (1 - \lambda)\delta_{x_1} + \lambda\delta_{x_2}, \ x_1 \neq x_2$

- Choose some optimal coupling $\pi$ and note
  $$\text{supp}\, \pi = A_1 \times \{x_1\} \cup A_2 \times \{x_2\} \cup A \times \{x_1, x_2\}.$$ 

- Observation:
  - $\pi$ is induced by a transport map iff $m(A) = 0$.
  - by non-branching for $t \in (0, 1)$
    $$A_{t,x_1} \cap A_{t,x_2} = \emptyset.$$ 
  - by qualitative non-degeneracy (and $A$ is compact)
    $$m(A) \geq \limsup_{t \to 0} m(A_{t,x_1} \cup A_{t,x_2})$$
    $$= \limsup_{t \to 0} m(A_{t,x_1}) + m(A_{t,x_2})$$
    $$= 2 \limsup_{t \to 0} f(t)m(A) = 2C_R m(A).$$ 

- Conclusion: $m(A) = 0$ and $\pi$ is induced by a transport map.
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Proof for \( \nu = \sum_{i=1}^{n} \lambda_i \delta_{x_i} \)

- Choose some optimal coupling \( \pi \) then

\[
\text{supp} \pi = \bigcup_{i=1}^{n} A_i \times \{x_i\} \cup \bigcup_{i<j} A_{ij} \times \{x_i, x_j\}.
\]

- By previous slide \( m(A_{ij}) = 0 \).
- Hence

\[
T(x) = \begin{cases} 
  x_i & x \in A_i \\
  x & \text{otherwise}
\end{cases}
\]

is a transport map.
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Observation

- For distinct \(x, y \in M\) and compact \(A \subset M\)

\[
A_- = \{z \in M \mid d(x, z) = d(y, z)\}
\]

\[
A_\neq = A \setminus A_-.
\]

- If \(A \times \{x, y\}\) is cyclically monotone then even without non-branching

\[
(A_\neq)_{t,x} \cap (A_\neq)_{t,y} = \emptyset.
\]

- Hence if

\[
\forall x \neq y : m(\{z \in M \mid d(x, z) = d(y, z)\}) = 0
\]

then any optimal coupling between \(\mu \ll m\) and \(\nu\) discrete is induced by a transport map.

- This holds for any normed space and measure assigning zero mass to hyperplanes.
Observation

- For distinct \( x, y \in M \) and compact \( A \subset M \)

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A_\equiv = \{ z \in M \mid d(x, z) = d(y, z) \}
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- This holds for any normed space and measure assigning zero mass to hyperplanes.
Proof for general $\nu$

**Theorem (Selection dichotomy, see e.g. [K. '17])**

If some optimal coupling $\pi$ is not induced by a transport map then there is a compact set $K \subset \text{supp}(p_1)_*\pi$ with $\pi(K \times M) > 0$ and two continuous maps $T_1, T_2 : M \to M$ with $T_1(K) \cap T_2(K) = \emptyset$ such that

$$\Gamma^{(1)} \cup \Gamma^{(2)}$$

is cyclically monotone where $\Gamma^{(i)} = \text{graph}_K T_i$.

**Lemma**

If $(M, d)$ is non-branching then

$$\Gamma^{(1)}_t \cap \Gamma^{(2)}_t = \emptyset$$

and

$$m(K) \geq \limsup_{t \to 0} \left[ m(\Gamma^{(1)}_t) + m(\Gamma^{(2)}_t) \right]$$
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Lemma

If \((M, d, m)\) is non-branching and qualitatively non-degenerate then

\[
m(\Gamma_t^{(i)}) \geq f(t)m(K).
\]

Idea of proof.

Let \(\nu_n \rightarrow (T_i)_*\mu|_K\) with \(\nu_n\) discrete then eventually

\[
\Gamma_t^{(i),n} \subseteq (\Gamma_t^{(i)})_\varepsilon,
\]

so that

\[
m(\Gamma_t^{(i)}) \geq \limsup_{n \rightarrow 0} m(\Gamma_t^{(i),n}) \geq f(t)m(K).
\]
Lemma

If \((M, d, m)\) is non-branching and qualitatively non-degenerate then

\[
m(\Gamma_t^{(i)}) \geq f(t)m(K).
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Idea of proof.

Let \(\nu_n \to (T_i)_\# \mu|_K\) with \(\nu_n\) discrete then eventually

\[
\Gamma_t^{(i),n} \subseteq (\Gamma_t^{(i)})_\epsilon
\]

so that

\[
m(\Gamma_t^{(i)}) \geq \limsup_{n \to 0} m(\Gamma_t^{(i),n}) \geq f(t)m(K).
\]
• If the claim was wrong then using we arrive at the following contradiction

\[
m(K) \geq \limsup_{t \to 0} m(\Gamma_{t}^{(1)}) + m(\Gamma_{t}^{(2)}) \\
\geq 2 \limsup_{t \to 0} f(t)m(K) = 2C_{R}m(K).
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• Thus, any optimal transport between \( \mu \ll m \) and arbitrary \( \nu \) is induced by a transport map.
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Thus, any optimal transport between \( \mu \ll m \) and arbitrary \( \nu \) is induced by a transport map.
Ingredients of the proof

- (weak) non-branching property

\[ m(\Gamma_t^{(1)} \cap \Gamma_t^{(2)}) = 0 \]
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• qualitative non-degeneracy
Ingredients of the proof

- (weak) non-branching property

$$m(\Gamma_t^{(1)} \cap \Gamma_t^{(2)}) = 0$$

- qualitative non-degeneracy $\implies$ not an optimal transport property
Partial extension to $p = 1$, see e.g. [K.-Suhr 18]

- For simplicity let $M = \mathbb{R}^n$

**Lemma**

If $\Gamma \subset \{x_n < 0\} \times \{x_n = 0\}$ is $d$-cyclically monotone then for all distinct geodesics $\gamma, \eta$ with

$$(\gamma_0, \gamma_1), (\eta_0, \eta_1) \in \Gamma$$

it holds $\gamma_t \neq \eta_t$.

**Corollary**

Assume $\text{supp} \mu \times \text{supp} \nu \subset \{x_n < 0\} \times \{x_n = 0\}$ then any $d$-optimal coupling is induced by a transport map.

**Remark**

Works for non-bran., qual. non-deg. spaces if $\nu$ is supported in a level set of a dual solution.
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Lorentzian setting [K.-Suhr ’18]

- Lagrangian for $p \in (0, 1]$
  \[
  \mathcal{L}_p(v) = \begin{cases} 
    -(-g(v,v))^\frac{p}{2} & v \in \bar{C} \\
    \infty & \text{otherwise}
  \end{cases}
  \]
  induces cost function $c_p : M \times M \to (-\infty, 0] \cup \{\infty\}$

- For $p \in (0, 1)$, geodesics connecting $(x, y) \in c_p^{-1}((-\infty, 0))$ are non-branching.

- For $p = 1$ and hyperbolic spacetimes, introduce smooth time function $\tau : M \to \mathbb{R}$ with
  \[
  \forall v \in \bar{C} : d\tau(v) > 0
  \]
  and then all causal geodesics can be parametrized \textit{time-affinely} and geodesics with endpoints in
  \[
  \{\tau < a\} \times \{\tau = a\}
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Spaces with good transport behavior (I)

• Assume \((M, d, \mathfrak{m})\) has good transport behavior \((GTB)_p\)

  • for every \(\mu_0 \ll \mathfrak{m}\) and \(\mu_1\) the optimal coupling is unique and induced by a transport map
  • for every \(\mu_0 \ll \mathfrak{m}\) and \(\mu_1\) the geodesic \(t \mapsto \mu_t\) is unique
  • if, in addition, \(\mathfrak{m}\) is qual. non-deg. then \(\mu_t \ll \mathfrak{m}\)
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Obstructions to good transport behavior (I)

- Let \( T = [0, 1]_1 \cup [0, 1]_3 \cup [0, 1]_3 \) be the tripod glued at 0.
- \((T,d,m)\) is a \( CAT(0) \)-space, but will never have good transport behavior.
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Essentially non-branching (I)

- A subset of geodesics $\mathcal{G} \subset \text{Geo}_{[0,1]}(M,d)$ is called non-branching if for all $\gamma, \eta \in \mathcal{G}$ with $\gamma_0 = \eta_0$ or $\gamma_1 = \eta_1$ it holds whenever $\gamma_t = \eta_t$ for some $t \in (0,1)$ it holds $\gamma_t = \eta_t$ for all $t \in [0,1]$.

Definition ([Rajala-Sturm ’14])

The space $(M, d, m)$ is essentially non-branching $(\text{ENB})_p$ if for every optimal dynamical coupling $\sigma$ with $(e_0) \ast \sigma, (e_1) \ast \sigma \ll m$ is concentrated on a non-branching set.

- May alter condition: For each $t_1, \ldots, t_n \in (0,1)$, $\sigma$ is concentrated on a cyclically monotone set $\Gamma$ such that for all distinct geodesics $\gamma$ and $\eta$ with endpoints in $\Gamma$ it holds $\gamma_{t_i} \neq \eta_{t_i}$. 
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The space $(M,d,\mathfrak{m})$ is essentially non-branching $(\text{ENB})_p$ if for every optimal dynamical coupling $\sigma$ with $(e_0)_\ast \sigma, (e_1)_\ast \sigma \ll \mathfrak{m}$ is concentrated on a non-branching set.

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Definition ([Rajala-Sturm '14])

The space $(M,d,\mu)$ is essentially non-branching ($\text{ENB}_p$) if for every optimal dynamical coupling $\sigma$ with $(e_0)_*\sigma, (e_1)_*\sigma \ll \mu$ is concentrated on a non-branching set.

- May alter condition: For each $t_1, \ldots, t_n \in (0,1)$, $\sigma$ is concentrated on a cyclically monotone set $\Gamma$ such that for all distinct geodesics $\gamma$ and $\eta$ with endpoints in $\Gamma$ it holds $\gamma_{t_i} \neq \eta_{t_i}$.
A subset of geodesics $G \subset \text{Geo}_{[0,1]}(M,d)$ is called non-branching to the left if for all $\gamma, \eta \in G$ with $\gamma_0 = \eta_0$ it holds whenever $\gamma_t = \eta_t$ for some $t \in (0,1)$ it holds $\gamma_t = \eta_t$ for all $t \in [0,1]$. 

**Theorem ([K. ’17])**

If $(M,d,m)$ has good transport behavior then any dynamical coupling $\sigma$ with $(e_0)_* \sigma \ll m$ is concentrated on a set that is non-branching to the left. In particular, $(M,d,m)$ is essentially non-branching $(\text{ENB})_p$. 
• A subset of geodesics $\mathcal{G} \subset \text{Geo}_{0,1}(M,d)$ is called non-branching to the left if for all $\gamma, \eta \in \mathcal{G}$ with $\gamma_0 = \eta_0$ it holds whenever $\gamma_t = \eta_t$ for some $t \in (0,1)$ it holds $\gamma_t = \eta_t$ for all $t \in [0,1]$.

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Choose three mutually singular measures $m_1, m_2$ and $m_3$ on $[0,1]$.

For the tripod $(T,d)$, regard them as measures on $[0,1_i]$ and set $m = m_1 + m_2 + m_3$

Observations:
- $(T,d,m)$ is essentially non-branching
- all optimal couplings $\pi$ with $(p_i)_\ast \pi \ll m$ are induced by transport map
Choose three mutually singular measures $m_1, m_2$, and $m_3$ on $[0, 1]$.

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Observations:

- $(T, d, m)$ is essentially non-branching
- all optimal couplings $\pi$ with $(p_i)_* \pi \ll m$ are induced by transport map.
Theorem ([K. ’17])

If $(M, d, m)$ is essentially non-branching $(ENB)_p$ and qualitatively non-degenerate then it has good transport behavior $(GTB)_p$. 
Idea of the proof

- As $\nu \not\ll m$ essentially non-branching cannot be directly used
- Construct dynamical optimal coupling $\sigma$ with $\left(e_\epsilon\right)_*\sigma, \left(e_{1-\epsilon}\right)_*\sigma \ll m$ with
  \[
  m|_{\Gamma_{\epsilon, 1-\epsilon}} \ll \left(e_\epsilon\right)_*\sigma \ll m|_{\Gamma_{\epsilon, 1-\epsilon}}
  \]
  for a cyclically monotone set $\Gamma$ with $(e_0, e_{1-\epsilon})_*\sigma(\Gamma) = 1$.
- Essentially non-branching implies
  \[
  m(\Gamma^{(1)}_\epsilon \cap \Gamma^{(2)}_\epsilon) = 0
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  when $\Gamma^{(1)}$ is given via the Selection Dichotomy.
- A proof a la Cavalletti–Huesmann gives the result.
Idea of the proof

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\left. m \right|_{\Gamma_{\frac{\epsilon}{1-\epsilon}}} \ll (e_\epsilon) \ast \sigma \ll \left. m \right|_{\Gamma_{\frac{\epsilon}{1-\epsilon}}}
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for a cyclically monotone set \( \Gamma \) with \( (e_0, e_1 - \epsilon) \ast \sigma(\Gamma) = 1 \).
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- A proof a la Cavalletti–Huesmann gives the result.
Intermediate summary

- Assume \((M, d, m)\) essentially non-branching and qualitatively non-degenerate
- Let \(\mu_0 = f_0 m\) and \(\mu_1\) arbitrary
- Conclusion:
  1. (uniqueness) unique optimal dynamical coupling \(\sigma\)
  2. (good transport behavior)

\[(e_0, e_1)_* \sigma = (\text{id} \times T_1)_* \mu_0\]

3. (strong interpolation property)

\[(e_t)_* \sigma = f_t m\]

4. (strong bounded density property)

\[f_t(\gamma_t) \leq \frac{1}{f_R(t)} f_0(\gamma_0).\]
Theorem ([K. ’17])

Assume \((M, d, m_1)\) and \((M, d, m_2)\) are both essentially non-branching \((ENB)_p\) and qualitatively non-degenerate then \(m_1\) and \(m_2\) are mutually absolutely continuous.

Corollary

For \(i = 1, 2\) let \((M, d, m_i)\) be RCD\((K_i, N_i)\)-spaces with \(N_i < \infty\). Then \(m_1\) and \(m_2\) are mutually absolutely continuous.
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For $i = 1, 2$ let $(M,d,m_i)$ be $\text{RCD}(K_i,N_i)$-spaces with $N_i < \infty$. Then $m_1$ and $m_2$ are mutually absolutely continuous.
1st Proof of the Measure Rigidity (I)

- Decompose $m_2 = f m_1 + m_2^s$
- Assume, by contradiction, $m_2^s \neq 0$.
- Observation: We must have $f \neq 0$ by strong interpolation property.
- The following claim implies gives a contradiction.

**Claim**

$m_2^s$ is both essentially non-branching and qualitatively non-degenerate.
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**Claim**

\( m_2^s \) is both essentially non-branching and qualitatively non-degenerate
1st Proof (II) - proof of the claim

• Note if $m_1(A_t,x) = m_1(A) = 0$ and then

$$m^s_2(A_t,x) = m_2(A_t,x) \geq f_R(t)m_2(A) = f_R(t)m^s_2(A).$$

• Observation: Since $m_2(A_t,x) > 0$ for all $t \in (0,1]$, it is possible to show that for $m_2$-a.e. $x \in A$ there is a unique geodesic $\gamma(x)$ such that

$$x \in \gamma(x)((0,1))$$

Lemma

There is $K \subset \subset A$ with $m_2(K) > 0$ and $t \mapsto \mu_t$ geodesic with $\mu_1 \ll m_2$, $\mu_{t_0} = \frac{1}{m_2(K)}m_2|_K$ and $\mu_1 = \delta_x$.

• Lemma implies if $\mu_{t_0} \perp m_1$ then $\mu_t \perp m_1$ for all $t \in [0,1)$ which yields the claim.
Note if $m_1(A_{t,x}) = m_1(A) = 0$ and then

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\[
x \in \gamma(x)((0,1))
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**Lemma**

*There is $K \subset A$ with $m_2(K) > 0$ and $t \mapsto \mu_t$ geodesic with $\mu_1 \ll m_2$, $\mu_{t_0} = \frac{1}{m_2(K)}m_2|_K$ and $\mu_1 = \delta_x$.***

- Lemma implies if $\mu_{t_0} \perp m_1$ then $\mu_t \perp m_1$ for all $t \in [0,1)$ which yields the claim.
Assume $m^s_s \neq 0$ and choose $\mu_0 = \frac{1}{m^s_2(K)}m^s_2|_K$ and $\mu_1 \ll m_1$

By strong intersection property

$$\mu_t \ll m_1, \mu_t \ll m_2$$

hence $\mu_t \perp m^s_2$

By bounded density property for $\mu_t = f_t m_2$

$$\|f_t\|_\infty \leq \frac{1}{f_R(t)m^s_2(K)}$$

Arrive at contradiction using the following lemma.

**Lemma (Self-intersection [CH '14, K. '17])**

If $\mu = \frac{1}{m(K)}m|_K$ and $\mu_n = f_n m$ with $W_p(\mu_n, \mu) \to 0$ and $\|f_n\|_\infty \leq C$ then $\mu \not\perp \mu_n$ for all sufficiently large $n$. 

Optimal transport and non-branching geodesics

Martin Kell

32/33
• Assume $m^S_s \neq 0$ and choose $\mu_0 = \frac{1}{m^S_2(K)} m^S_2 \mid_K$ and $\mu_1 \ll m_1$

• By strong intersection property

\[ \mu_t \ll m_1, \mu_t \ll m_2 \]

hence $\mu_t \perp m^S_2$

• By bounded density property for $\mu_t = f_t m_2$

\[ \|f_t\|_\infty \leq \frac{1}{f_R(t) m^S_2(K)} \]

• Arrive at contradiction using the following lemma.

**Lemma (Self-intersection \([CH \ '14, K. \ '17]\))**

If $\mu = \frac{1}{m(K)} m \mid_K$ and $\mu_n = f_n m$ with $W_p(\mu_n, \mu) \to 0$ and $\|f_n\|_\infty \leq C$ then $\mu \nabla \mu_n$ for all sufficiently large $n$. 

Optimal transport and non-branching geodesics
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2nd Proof of the Measure Rigidity

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Thank you for your attention