Quotient spaces & Ricci curvature

Martin Kell

(joint with F. Galat. García, A. Monelino, G. Sosa)
$\mathbb{G} \rightarrow \mathcal{M}$

$\mathcal{M}^* = \mathcal{M}/\mathbb{G}$
\[ G \rightarrow M \]
\[ \downarrow \]
\[ M^* = M / G \]
\[ d_m(x, y) = d_M(gx, gy) \]
\[ d^*(x^*, y^*) = \inf_{g \in G} d_M(gx, gy) \]
\[ G \rightarrow M \]
\[ M^* = M/G \]

\[ d_M(gx, gy) = d_M(x, y) \]
\[ d^* (x^*, y^*) = \inf_{g \in G} d_M(gx, y) \]

**Facts:**

- \((M, d)\) geodesic \(\implies\) \((M^*, d^*)\) geodesic
- \(\sec_M \geq k \implies \sec_{M^*} \geq k\)
\[ G \rightarrow M \quad \quad d_M(gx, gy) = d_M(x, y) \]

\[ M^* = M / G \quad \quad d^*(x^*, y^*) = \inf_{g \in G} d(gx, y) \]

**Facts:**

1. \((M, d)\) geodesic \(\Rightarrow (M^*, d^*)\) geodesic
2. \(\text{sec}_M \geq k \Rightarrow \text{sec}_{M^*} \geq k\)
3. \((\text{Pro-Wilhelm})\) \(\exists\) warped product \(M = B \times_{\phi} F\), \(F\) homogeneous
   \(\text{Ric}_M > 0\), \(\text{Ric}_{M^*} = \text{Ric}_B \neq 0\)
\[ G \rightarrow M \quad \text{d}_M(gx, gy) = \text{d}_M(x, y) \]
\[ M^g = M / G \quad \text{d}^g(x^g, y^g) = \text{d}(G(x), G(y)) \]
\[ = \inf_{y \in G} \text{d}(gx, y) \]

**Facts:**

- \((M, d)\) geodesic \(\Rightarrow\) \((M^g, d^g)\) geodesic
- \(\sec_M \geq k \Rightarrow \sec_{M^g} \geq k\)
- (Pro-Wilhelm) \(\exists\) warped product \(M = B \times F, F\) homogeneous
  \(\Ric_M > 0, \Ric_{M^g} = \Ric_B \neq 0\)

**Facts true for metric foliations (eq. submetrics)**

\[ M^g = \{ F_x \}_{x \in M}, \quad \text{d}^g(x^g, y^g) = \text{d}(F_x, F_y) = \inf_{y' \in F_y} \text{d}(x, y') \]
Warped Products

\((\mathbb{R}, g_B), (\mathbb{F}, g_F)\) Riemannian manifolds

\[ M = \mathbb{R} \times \mathbb{F} \quad \text{for} \quad f : \mathbb{R} \to \mathbb{R} \]

\[ g_M = g_B + e^{-2f} g_F \]
Warped Products

\((B, g_B), (F, g_F)\) Riemannian manifolds

\[ M = B \times F \quad \text{for} \quad f : B \to \mathbb{R} \]

\[ g_M = g_B + e^{-2f} g_F \]

**Facts:**

1. \( \{ f \times x \}_{x \in B} \) is a metric foliation
2. If \( F \) is homogenous then \( \text{Isom}(F) \to M \) with \( (v^i, df^i) \mapsto (\{ v^i \}, \{ df^i \}) \)
3. \( v, w \in TB \subset TM \quad N = \partial l/\partial v \quad M, \quad u = \partial l/\partial w \)

\[ \text{Ric}_M(v, w) = \text{Ric}_B(v, w) + \text{Hess} g_B(v, w) - \frac{df(v) df(w)}{N - u} \]
Warped Products

\((B, g_B), (F, g_F)\) Riemannian manifolds

\[ M = B \times F \quad \text{for} \quad f: B \to \mathbb{R} \]

\[ g_M = g_B + e^{-2f} g_F \]

Facts:
1. \( \mathcal{F} = \{ f \times F \}_{x \in B} \) is a metric foliation
2. if \( F \) is homogeneous then \( \text{Isom}(F) \to M \) with \( (v_0, df(\xi)) \cdot (v_1, df(\eta)) = \langle d\xi(v_0) v_1, d\eta(v_1) \rangle \)
3. \( v, w \in TB \subset TM \quad N = \text{d}1/\text{d}u \cdot M, \quad u = \text{d}1/\text{d}w \cdot B \)

\[ \text{Ric}_M(v,w) = \text{Ric}_B(v,w) + \text{Hess}_{g_B}(v,w) - \frac{df(v)df(w)}{N - u} \]

Bakry-Émery Ricci tensor: \( \text{Ric}^B_{B,v} \)

Conclusion:
If \( \text{Ric}_M \geq k \) then \( \text{Ric}^B_{B,v} \geq k \)
Problem: In general $\left( M^g, c^g \right)$ is not a manifold.
Problem: In general, \((M^v, d^v)\) is not a manifold.
Solution: Synthetic lower Ricci bounds on \((M, d, m)\)
**Problem:** In general $(M^\nu, d^\nu)$ is not a manifold.

**Solution:** Synthetic lower Ricci bounds on $(M, d, m)$

**Definition:** (curvature dimension $CD^*(k, n)$)
**Problem:** In general \((M^k, d^k)\) is not a manifold.

**Solution:** Synthetic lower Ricci bounds on \((M, d, m)\)

**Definition** \((\text{curvature dimension } \mathcal{CD}(k, n))\)

\[
\forall \mu_0, \mu_1 \in P_2(M) \exists \sigma \in \text{Opt}(\text{Geo}_2(\mu_0, \mu_1), \mathcal{CD}(k, n))
\]

\[
-\int f^+ \frac{d^k}{m} \leq -\int T^+_{k,n}(d(x,y)) \rho^\frac{k-1}{k} (x) dx + T^+_{k,n}(d(x,y)) \rho^\frac{k-1}{k} (y) d\pi(x,y)
\]

where \( \mu^+ = \rho^+ \mu + \mu^* = (e_k, \sigma), \ \pi = (e, e_k) \sigma \)
Problem: In general, \((M^0, d^x)\) is not a manifold.
Solution: Synthetic lower Ricci bounds on \((M, d, m)\)

Definition (curvature dimension \(CD(k, n)\))

\[
\forall \mu_0, \mu_1 \in \mathcal{P}_2(M) \exists \sigma \in \text{OptGeo}_2(\mu_0, \mu_1)
\]

\[
-\int f^+ \, dm \leq \int \left( T^+_{k, n}(d(x,y)) \rho^{\frac{1}{4}}(x) + T^+_{k, n}(d(x,y)) \rho^{\frac{1}{4}}(y) \right) d\pi(x, y)
\]

where \(\mu_1 = \rho^+ \nu + \mu^s := (e_1, \sigma), \quad \pi := (e, e_1) \ast \sigma\)
Assumption: $G$ compact group with Haar measure $\mu_G$
$G$ acts isometrically, i.e. $d(gx, gy) = d(x, y)$
measure preserving, i.e. $g_*\mu = \mu$

$$\mathcal{P}^G(M) := \{ \mu \in \mathcal{P}(M) : g_*\mu = \mu \forall g \in G \}$$
Assumption: \( G \) compact group with Haar measure \( \mu_G \), \( G \) acts isometrically, i.e. \( d(gx,gy) = d(x,y) \), measure-preserving, i.e. \( g \cdot \mu = \mu \)

\[ \mathcal{P}_2^G(M) := \{ \mu \in \mathcal{P}_2(M) : g \cdot \mu = \mu \text{ for all } g \in G \} \]

Lifting measures & functions
Assumption: \( G \) compact group with Haar measure \( \mu \).
\( G \) acts isometrically, i.e. \( d(gx, gy) = d(x, y) \)
measure-preserving, i.e. \( g \cdot \mu = \mu \)

\[
P^G_2(M) := \{ \mu \in P_2(M) : g \cdot \mu = \mu \ \forall g \in G \}
\]

Lifting measures & functions

Lemma: \( \hat{\ell}(x) := \ell(x^*) \); \( \hat{f} \in L^2(\hat{\mu}) \Rightarrow \hat{f} \in L^2(\mu) \)
\[ \forall \mu \in P_2(M^*) \ \exists! \ \hat{\mu} \in P^G_2(M) : p_* \hat{\mu} = \mu \]
\[ \mu = \ell \mu^* \Rightarrow \hat{\mu} = \hat{\ell} \mu \]
Theorem: \( \Lambda: \mu \mapsto \hat{\mu} \) is an isometric embedding of \( \mathcal{P}_2(M^*) \) into \( \mathcal{P}_2(M) \).

\( P^* : \mathcal{P}_2(M) \to \mathcal{P}_2(M^*) \) is a submetry.
Theorem: \( \Lambda : \mu \mapsto \hat{\mu} \) is an isometric embedding of \( \mathcal{P}_2(M^*) \) into \( \mathcal{P}_2(M) \).

\[ \mathcal{P}_*: \mathcal{P}_2(M) \to \mathcal{P}_2(M^*) \text{ is a submetry} \]

Proposition: \( \forall \mu, \nu \in \mathcal{P}_2(M), \pi \in \text{Opt}_2(\hat{\mu}, \hat{\nu}) \)

\[ (x, y) \in \text{supp} \pi = d(x, y) = d(x^*, y^*) \]

In particular, \( (\rho, \rho^*) \pi \in \text{Opt}(\mu, \nu) \).

and \( \mathcal{P}_* \left( \text{Opt}_{2} \left( \hat{\mu}, \hat{\nu} \right) \right) = \text{Opt}_{2} \left( \mu, \nu \right) \)
Lemma: If $\varphi: M^* \to \mathbb{R} [\alpha - \omega]$ is $C^2$-concave
then $\hat{\varphi}: \tilde{M} \to \mathbb{R} [\alpha - \omega]$ is $C^2$-concave.

In particular, $(x, y) \in E^c \hat{\varphi} \Rightarrow d(x, y) = d^c(x^*, y^*)$. 
**Lemma**: If \( \phi : M^* \to \mathbb{R} \cup \{+\infty\} \) is \( C_2 \)-concave then \( \hat{\phi} : M \to \mathbb{R} \cup \{+\infty\} \) is \( C_2 \)-concave.

In particular, \( (x,y) \in \mathcal{S}^\hat{\phi} \Rightarrow d(x,y) = d^*(x^*,y^*) \).

**Pf**: \( \hat{\phi}(x) = \ell(x^*) \)
Lemma: If $\Psi : M^* \to \mathbb{R}_0^\infty$ is $C^2$-concave then $\hat{\Psi} : M \to \mathbb{R}_0^\infty$ is $C^2$-concave.

In particular, $(x, y) \in \mathcal{E} \hat{\Psi} \Rightarrow d(x, y) = d^*(x^*, y^*)$.

Proof: $\hat{\Psi}(x) = \ell(x^*) = \inf_{y^* \in M^*} d^*(x^*, y^*) - \ell(y^*)$.
Lemma: If $\varphi : M^* \rightarrow \mathbb{R}$ is $C_2$-concave, then $\hat{\varphi} : M \rightarrow \mathbb{R}$ is $C_2$-concave.

In particular, $(x,y) \in \mathcal{E}_\varphi \Rightarrow d(x,y) = d'(x,y)$.

Proof:

$\hat{\varphi}(x) = \ell(x) = \inf_{y \in M^*} d(x,y) - \varphi(y)$

$= \inf_{y \in M} d(x,y) - \varphi(y)$
Lemma: If $\Psi : \mathcal{M}^* \to [\mathcal{R} u^* - [\mathcal{M}^* - 0])$ is $c_2$-concave then $\tilde{\Phi} : \mathcal{M} \to [\mathcal{R} u^* - [\mathcal{M}^* - 0])$ is $c_2$-concave.

In particular, $(x, y) \in \mathcal{S}^c \tilde{\Phi} \Rightarrow d(x, y) = d^*(x, y)$.

Proof: $\tilde{\Phi}(x) = (\ell(x^*)) = \inf_{y \in \mathcal{M}^*} d(x^*, y^*) = d(x^*, y) - \Phi(y)$

= $\inf_{y \in \mathcal{M}} d(x, y) - \Phi(y)$

= $\inf_{y \in \mathcal{M}} d(x, y) - \Phi(y)$
Lemma: If $\Psi: M^* \to \mathbb{R}^{\mathbb{S}-\infty}$ is $C_2$-concave then $\hat{\Psi}: M \to \mathbb{R}^{\mathbb{S}-\infty}$ is $C_2$-concave.

In particular, $(x, \gamma) \in \mathbb{S}^\circ \hat{\Psi} \Rightarrow d(x, \gamma) = d^*(x^*, \gamma^*)$.

Proof:
\[
\hat{\Psi}(x) = \ell(x^*) = \inf_{\gamma \in M^*} d(x^*, \gamma) - \Phi(\gamma^*)
\]
\[
= \inf_{\gamma \in M} d(x, \gamma) - \Phi(\gamma)
\]
\[
= \inf_{\gamma \in M} d(x, \gamma) - \hat{\Phi}(\gamma)
\]

- $\hat{\Psi}(x) + \hat{\Phi}(\gamma) = \Psi(x^*) + \Phi(\gamma^*)$
- $d(x, \gamma) = d^*(x^*, \gamma^*)$
**Lemma:** If $\Phi: M^* \rightarrow \mathbb{R}^{n \times m}$ is $C^2$-concave, then $\hat{\Phi}: M \rightarrow \mathbb{R}^{n \times m}$ is $C^2$-concave.

In particular, $(x, y) \in \partial \Phi \Rightarrow d(x, y) = d^*(x^*, y^*)$.

**Proof:**

\[ \hat{\Phi}(x) \equiv \ell(x^*) = \inf_{x^* \in M^*} d(x^*, y) - \Phi(y^*) \]
\[ = \inf_{y \in M} d(x, y) - \Phi(y) \]
\[ = \inf_{y \in M} \left( d(x, y) - \Phi(y) \right) \]

- $\hat{\Phi}(x) + \hat{\Phi}(y) = \Phi(x^*) + \Phi(y^*)$
- $d(x, y) \geq d^*(x^*, y^*)$

\(\square\)
Pf of Prop:
Pf of Prop:

Choose $\pi \in \text{Opt}_1(\mu, \nu)$ and $(\xi, \psi)$ dual solution.
Pf of Prop:

- Choose a \( \pi \in \text{Opt}_1(\mu, \nu) \) and \((\xi, \eta)\) dual solution.
- It suffices to show \((\xi, \eta)\) is a dual solution.
Pf of Prop:

1. Choose \( \pi \in \text{Opt}_1(\mu, \nu) \) and \((\ell, \psi)\) dual solution.
2. It suffices to show \((\ell, \psi)\) is a dual solution.
3. \( \exists (x, y) \mapsto \pi_{x,y} \in \text{Opt}_2(\hat{S}_x, \hat{S}_y) \) measurable.
Pf of Prop:

1. Choose $\pi \in \text{Opt}_1(\mu, \nu)$ and $(\lambda, \Psi)$ dual solution.
2. It suffices to show $(\lambda, \Psi)$ is a dual solution
3. $\exists (x, y) \mapsto \pi_{x,y} \in \text{Opt}_2(\tilde{s}_x, \tilde{s}_y)$ measurable
4. $\tilde{\pi} := \int \pi_{x,y} \, d\tilde{n}(x,y) \in \Pi(\tilde{\mu}, \tilde{\nu})$
Proof of Prop:

1. Choose \( \pi \in \text{Opt}_1(\mu, v) \) and \((\xi, \eta)\) dual solution.
2. It suffices to show \((\xi, \eta)\) is a dual solution.
3. \( \exists (x_i, y_i) \mapsto \pi_{x_i, y_i} \in \text{Opt}_2(\hat{S}_{x_i}, \hat{S}_{y_i}) \) measurable
4. \( \tilde{\pi} : = \int \pi_{x_i, y_i} \, d\hat{\nu}(x_i, y_i) \in \Pi(\hat{\mu}, \hat{\nu}) \)
5. \( \int d^2(x_i, y_i) \, d\hat{\pi}(x_i, y_i) = \int d^2(x_i, y_i) \, d\hat{\nu}(x_i, y_i) \)
Proof:

. Choose \( \tilde{\pi} \in \text{Opt}_1(\mu, \nu) \) and \((Q, \Psi)\) dual solution.
. It suffices to show \((Q, \Psi)\) is a dual solution.
. \( \exists (x_i, y_i) \mapsto \pi_{x_i, y_i} \in \text{Opt}_2(\hat{S}_{x_i}, \hat{S}_{y_i}) \) measurable.
. \( \hat{\pi} := \int \pi_{x_i, y_i} \, d\hat{\nu}(x_i, y_i) \in \Pi(\hat{\mu}, \hat{\nu}) \)

\[
\int d^2(x, y) \, d\hat{\pi}(x, y) = \int d^2(x', y') \, d\hat{\pi}(x', y') \\
= \int \phi \, d\hat{\mu} + \int \psi \, d\hat{\nu} = \int \Psi \, d\hat{\mu} + \int \hat{\Psi} \, d\hat{\nu}
\]
Corollary: The quotient of a $CD(k,N)$-space is a $CD(k,N)$-space.
Corollary: The quotient of a $CD^*(k,N)$-space is a $CD^*(k,N)$-space.

Pf.: Let $\mu_0, \mu_1 \in \mathcal{P}_2(M)$

... Choose $\sigma \in \text{OptGeo}_2(\hat{\mu}_0, \hat{\mu}_1)$ s.t. $CD^*(k,N)$-ineq. holds
Corollary: The quotient of a $CD^*(k,N)$-space is a $CD^*(k,N)$-space.

Pf.: Let $\mu_0, \mu_1 \in \mathcal{P}_2^c(M)$

- Choose $\sigma \in \text{OptGeo}_2(\tilde{\mu}_0, \tilde{\mu}_1)$ s.t. $CD^*(k,N)$-ineq. holds
- $\tilde{\sigma} := \int g_x \sigma \, d\mu_\infty(g)$, $\tilde{\tau} := (e_0 e_1)_* \tilde{\sigma}$
- $\text{Ent}_\nu((e_1)_* \tilde{\sigma}) \leq \text{Ent}_\nu((e_1)_* \sigma)$
Corollary: The quotient of a $\mathcal{CD}(k,N)$-space is a $\mathcal{CD}(k,N)$-space.

Pf.: Let $\mu_0, \mu_\ast \in \mathcal{P}^\ast\ell_2(M^k)$
- Choose $\sigma \in \text{OptGeo}_2(\hat{\mu}_0, \hat{\mu}_\ast)$ s.t. $\mathcal{CD}(k,N)$-ineq. holds
- $\tilde{\sigma} := \int x \sigma \, d\mu_\ast(g)$, $\tilde{\pi} := (e_0, e_\ast)_{\ast} \tilde{\sigma}$
- $\text{Ent}_\nu((e_0)_{\ast} \tilde{\sigma}) \leq \text{Ent}_\nu((e_\ast)_{\ast} \sigma)$
- $\int \tau_{k,\nu}(d(x_0,x)) \hat{\sigma}_{\ast}^{-1} \pi(x) \, d\pi(x_0,x) = \int \tau_{k,\nu}(d(x_0,x)) \hat{\sigma}_{\ast}^{-1} \pi(x) \, d\pi(x_0,x)$
**Corollary:** The quotient of a $CD^*(k,N)$-space is a $CD(k,N)$-space.

**Pf.:** Let $\mu_0, \mu_\ast \in P^*_2(M^*)$

1. Choose $\sigma \in \text{OptGeo}_2(\mu_0, \mu_\ast)$ s.t. $CD^*(k,N)$-ineq. holds
2. $\tilde{\sigma} := \int g_x \sigma \, d\mu_{\ast}(g)$, $\tilde{\pi} := (e_0, e_\ast)_\ast \tilde{\sigma}$
3. $\text{Ent}_\ast (e_\ast)_\ast \tilde{\sigma} \leq \text{Ent}_\ast (e_\ast) \sigma$

$$\int \left( \begin{array}{c} \tau_{k,N}^* (d(x_0, x_\ast)) \\ \frac{1}{2} \hat{p}_{e_\ast} (x_\ast) \end{array} \right) d \tilde{\pi} (x_0, x_\ast) = \int \tau_{k,N}^* (d(x_0, x_\ast)) \hat{p}_{e_\ast} (x_\ast) d \tilde{\pi} (x_0, x_\ast)$$

where $\tilde{\pi} := (p, p)_\ast \tilde{\pi}$.  \( \Box \)
Remark: "Lifting is isometric embedding" only needs

\[ d(F_x, F_y) = d(x, y) \]

or

\[ \exists \{\nu_x\}_{x \in M^*} \text{ with } \text{ supp } \nu_x \subset F_x \]

\[ w_2(\nu_x, \nu_y) = d^*_{\gamma}(x, y) \]

"Preservation of (distorted) Entropy" needs

\[ M_x = \nu_x \text{ where } \nu_x = \int m_x \, dm_x(x) \text{ (disintegration over } p: M \to M^*) \]

\[ \rightarrow \text{ metric measure foliation} \]
Lots of new examples
Lots of new examples

\[(M_i, d_i, \mu_i)_{i=1,2} \quad RCD^*(\mathcal{N}, 1, \mu_i)\]

Join \(M_n \times M_2 := \) "Space of directions of \( C_0(M_n) \times C_0(M_2) \)"

\(- RCD^* (\mathcal{N}, n+2, \mathcal{N}, n+2) \quad \text{[Gigli, ketterer]} \)
Lots of new examples

\((M_i,d_i,n_i)_{i=1,2} \quad \text{RCD}^*(\mu_i,1,\mu_i)\)

Join

\(M_1 * M_2 := \text{Space of directions of } C^0(M_1) \times C^0(M_2)\)\)

- \(\text{RCD}^*(N_1+1, N_1 + 1)\) [Gigli, ketterer]
- not Alexandrov if \(\text{sec}_{M_i} \neq 1\)
- if \(\text{Isom}(M_i)\) large then \(\text{Isom}(M_1 * M_2)\) 'larger'

\(\rightarrow M_1 * M_2 \overset{G}{\longrightarrow} \text{is } \text{CD}^* \forall G \leq \text{Isom}(M_1 * M_2)\)
(see later for \(\text{RCD}^*\))
Sturm's super Ricci flow

$G$ acts on $(M, d_t, m_t)_{t \in [0, T]}$
Sturm's super Ricci flow

$G$ acts on $(M, d_t, m_t)_{t \in [0, T)}$

**Definition (Super Ricci Flow)**

$(M, d_t, m_t)$ is a super $N$-Ricci flow if for a.e. $t \in (0, T)$ and all $t \mapsto m_t \ll m_t$ geodesic in $\mathcal{P}_2(M)$

$$\frac{d}{dt} \operatorname{Ent}_t(m_t) - \frac{4}{N} \operatorname{W}^2_t(m_0, m_t) + \frac{4}{N} \left| \operatorname{Ent}_t(m_t) - \operatorname{Ent}_t(m_0) \right|^2$$
Sturm's super Ricci flow

\( G \) acts on \((M, d_t, \mu_t)_{t \in [0, \Gamma]}\)

**Definition (Super Ricci flow)**

\((M, d_t, \mu_t)\) is a super \(N\)-Ricci flow if for a.e. \( t \in (0, \Gamma) \) and all \( \tau \mapsto \mu_\tau \ll \mu_t \) geodesic in \( P_2^N(M) \)

\[
\frac{d}{dz} \text{Ent}_t(\mu_\tau) - \frac{d}{dz} \text{Ent}_t(\mu_0) \\
\geq -\frac{4}{z} \mathcal{W}^2_t(\mu_0, \mu_\tau) + \frac{1}{z} \left| \text{Ent}_t(\mu_\tau) - \text{Ent}_t(\mu_0) \right|^2
\]

**Theorem:** The quotient of an equivariant super \(N\)-Ricci flow is a super \(N\)-Ricci flow.
What else is preserved?
What else is preserved?

- (essentially) non-branching
- Good Transport Behavior (Existence of transport maps)
- MCP under - essentially non-branching \cite{Cavalletti-Monaco}:
  \[ \forall x, y \in M : m(\{ z \in M | d(x, z) = d(y, z) \}) = 0 \ [k] \]
What else is preserved?

- (essentially) non-branching
- Good Transport Behavior (Existence of transport maps)

\[ \text{MCP under - essentially non-branching [Cavalletti-Mondino]} \]
\[ \forall x,y \in M : m(\{ z \in M | d(x,z) = d(y,z) \}) = 0 \quad [k.] \]

Open: doubling, Poincaré?
What else is preserved?

- (essentially) non-branching
- Good Transport Behavior (Existence of transport maps)
- MCP under - essentially non-branching \([\text{Cavalletti-Monaco}]\)
  \[ \forall x \neq y \in M : m(\{z \in M | d(x,z)=d(y,z)\}) = 0 \ [k] \]

Open: doubling, Poincaré?

Question: How about the RCD?
Lipschitz function on quotient spaces
Lipschitz function on quotient spaces

\[ \text{Lip}_f(x,r) := \sup_{y \in B_r(x)} \frac{|f(y)-f(x)|}{r} \]

\[ \text{Lip}_f(x) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

\[ \text{Lip}_f(W) := \lim_{r \to 0} \text{Lip}_f(x,r) \]
**Lipschitz function on quotient spaces**

\[
\text{Lip}\ f(x,r) := \sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{r}
\]

\[
\text{Lip}\ f(x) := \lim_{r \to 0} \text{Lip}\ f(x,r)
\]

\[
\text{Lip}\ f\ (\omega) := \lim_{r \to 0} \text{Lip}\ f(x,r)
\]

**Def:** \((M,d,\mu)\) is a diff. space if for all Lipschitz functions \(f\)

\[
\text{Lip}\ f = \text{Lip}\ f \quad \text{m.-a.e.}
\]

"Lip-Lip-condition"
Lipschitz function on quotient spaces

\[ \text{Lip}_f(x,r) := \sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{r} \]

\[ \text{Lip}_f(x) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

\[ \text{Lip}_f(w) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

**Def:** \((M,d,\mu)\) is a diff. space if for all Lipschitz functions \(f\)

\[ \text{Lip} f = \text{Lip} f \quad \mu\text{-a.e.} \]

"Lip-lip-condition"

**Lemma:** \(f \in \text{Lip}(\mathbb{R}^n), x \in \Omega, r > 0 : \text{Lip}^\mathbb{R} f(x,r) = \text{Lip}^\mathbb{R} f(x,r). \)
Lipschitz function on quotient spaces

\[ \text{Lip}_f(x,r) := \sup_{y \in B_r(x)} \frac{|f(y)-f(x)|}{r} \]

\[ \text{Lip}_f(x) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

\[ \text{Lip}_f(\omega) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

**Def:** \((M,d,\mu)\) is a diff. space if for all Lipschitz functions \(f\)

\[ \text{Lip}_f = \text{Lip}_f \quad \text{m-a.e.} \]

"Lip-lip-condition"

**Lemma:** \(f \in \text{Lip}(M^\omega), x \in M, r > 0 : \text{Lip}_f(x,r) = \text{Lip}_f(x,r) \exp(x) \)

**Pf:**

\[ \sup_{y \in B_r(x)} \frac{|f(y)-f(x)|}{r} = \sup_{y \in B_r(x)} \frac{|f(y)-f(x)|}{r} = \sup_{y \in B_r(x)} \frac{|f(y)-f(x)|}{r} \]

\[ \square \]
Lipschitz function on quotient spaces

\[ \text{Lip}_f(x,r) := \sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{r} \]

\[ \text{Lip}_f(x) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

\[ \text{Lip}_f(\omega) := \lim_{r \to 0} \text{Lip}_f(x,r) \]

**Def:** \((M,d,\mu)\) is a diff. space if for all Lipschitz functions \(f\)

\[ \text{Lip}_f = \text{Lip}_f \quad \text{\(\mu\)-a.e.} \]

"Lip-lip-condition"

**Lemma:** \(f \in \text{Lip}(\mathcal{M}), x \in \mathcal{M}, r > 0 : \text{Lip}^\mathcal{M}_f(x,r) = \text{Lip}^\mathcal{H}_f(x,r) \).

**Pf:**
\[
\sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{r} = \sup_{y \in B_r(x)} \frac{|f(y^*) - f(x^*)|}{r} = \sup_{y \in B_r(x)} \frac{|f(y^*) - f(x^*)|}{r}
\]

Corollary The Lip-lip-condition is preserved under taking quotients.
**Conclusion:**

- If \((M,d,m)\) is inf. Hilbertian and both \((M,d,m)\) & \((M',d',m')\) are \(\mathbb{P}^1\)-spaces, then \((M',d',m')\) is inf. Hilbertian.
- \(\text{RCD}^\nu(k,\nu), \nu\in[1,\infty)\), is preserved.
Conclusion:

- If $(M, d, m)$ is inf. Hilbertian and both $(M, d, m)$ & $(M', d', m')$ are $\mathbb{P}$-spaces then $(M', d', m')$ is inf. Hilbertian.
- $\text{RCD}^*(k, \nu), \nu \in \mathcal{P}_1(m)$, is preserved.

More generally

Theorem: If $(M, d, m)$ is geodesic then

$$i : W^{1, q}(M', m') \rightarrow W^{1, q}(M, m), \quad f \mapsto \hat{f}$$

is an isometric embedding onto the space of $G$-invariant Sobolev functions.
More on RCD

If $G$ acts Lipschitz & co-Lipschitz on orbits then
More on RCD

If $G$ acts Lipschitz & co-Lipschitz on orbits then

$$\dim u(x) = \dim G(x) + \dim u(x^e)$$

local dimension
More on RCD

If $G$ acts Lipschitz & co-Lipschitz on orbits then

$$\dim u(x) = \dim G_o(x) + \dim u(x)$$

local dimension

$\Rightarrow$ Rigidity if $u(x) \in \{0, 1, 2\}$, i.e. $\dim G_o(x)$ large

Note: $\dim G = \dim G_x + \dim G_o(x)$
Principle Orbit Theorem

\[ G_x = \{ g \in G \mid g \cdot x = x \} \]
Principle Orbit Theorem

$$G_x = \{ g \in G \mid gx = x \}$$

**Theorem:** If $$(M, d, m)$$ has $$(G, T, B)$$ then there is an (open) set $$U \subset M$$ of full measure such that

$$\forall x, y \in U: G(x) \& G(y)$$ are homeomorphic

$$G_x \& G_y$$ are conjugate
Principle Orbit Theorem

\[ G_x = \{ g \in G \mid gx = x \} \]

Theorem: If \((M, d, \mu)\) has \((GTB)\) then there is an (open) set \(U \subset M\) of full measure such that

\[ \forall x, y \in U : G(x) \& G(y) \text{ are homeomorphic} \]

\[ G_x \& G_y \text{ are conjugate} \]

Proof: \(Q_x(y) := d^2(G(x), y) = (d^2(x, \cdot))^{-1}(y)\) is \(c_1\)-concave
Principle Orbit Theorem

\[ G_x = \{ g \in G \mid gx = x \} \]

Theorem: If \((M,d,m)\) has \((GTB)\) then there is an (open) set \(U \subseteq M\) of full measure such that

\[ \forall x,y \in U: G(x) \& G(y) \text{ are homeomorphic} \]

\[ G_x \& G_y \text{ are conjugate} \]

Proof: \(C_x(y) = d^2(G(x),y) = (d^2(x,\cdot))(y)\) is \(c_\cdot\)-concave

\((GTB) \Rightarrow \) for a.e. \(y \in M\) \(\mathcal{G}^c G(y) = \Sigma_x \exists y_x \subseteq G_x\)

\[ G_y \subseteq G_{xy} \]
Principle Orbit Theorem

\[ G_x = \{ g \in G \mid gx = x \} \]

**Theorem:** If \((M,d,\mu)\) has \((G\text{TB})\) then there is an (open) set \(U \subset M\) of full measure such that

\[ \forall x,y \in U: G(x) \cong G(y) \text{ are homeomorphic} \]

\[ G_x \cong G_y \text{ are conjugate} \]

**Proof:**
\[ q_x(y) := d^2(G(x),y) = (d(x',\cdot))^2(y) \text{ is } C^2 \text{-concave} \]

\( (G\text{TB}) \Rightarrow \) for a.e. \( y \in M: \mathcal{G}^c(y) = \{ x | x \cong y \} \subset G_x \)

\[ G_y \leq G_{xy} \]

From topological group theory:
\[ x \mapsto \text{type}(G_x) \text{ is l.s.c.} \]
Principle Orbit Theorem

\[ G_x = \{ g \in G \mid g x = x \} \]

**Theorem:** If \((M,d,m)\) has \((GTB)\) then there is an (open) set \(U \subset M\) of full measure such that

\[ \forall x,y \in U: G(x) \& G(y) \text{ are homeomorphic} \]

\[ G_x \& G_y \text{ are conjugate} \]

**Pf.:**
1. \(c_x(y) := d^2(g(x),y) = (d^2(x,\cdot))(y)\) is \(c_t\)-concave
2. \((GTB) \Rightarrow \text{ for a.e. } y \in M: \quad \mathcal{G}(y) = \{ x \} \subset G_x \]
3. From topological group theory: \( x \mapsto \text{type}(G_x) \) is l.s.c.

Choose \( x^* \in M \) with \( \text{type}(G_{x^*}) \) minimal gives result.
Cohomogeneity One Actions

Theorem: If $(M,\omega,\mu)$ has (G7B) and $M^\bullet \cong S^1$
then $M$ is homeomorphic to an $S^1$-fiber bundle.
Cohomogeneity One Actions

**Theorem:** If \((M,\mathfrak{g},\mu)\) has (GTB) and \(M^* \cong S^1\), then \(M\) is homomorphic to an \(S^1\)-fiber bundle.

**Pr:** - let \(U_x\) be as before.
Cohomogeneity One Actions

**Theorem:** If $(M,d,m)$ has (GTB) and $M^s \cong S^1$, then $M$ is homomorphic to an $S^1$-fiber bundle.

**Pr:**
- Let $U_x$ be as before.
- For $m$-a.e. $y \in U_x$, $x_x(t) \in U$
Cohomogeneity One Actions

**Theorem:** If \((M,d,m)\) has (GTB) and \(M^s \cong S^1\), then \(M\) is homomorphic to an \(S^1\)-fiber bundle.

**Pr:**
- Let \(U_x\) be as before.
- For \(m\text{-a.e. } y \in U_x\), \(\xi_x^y(t) \in U\)
- Choose \(x_1, x_2 \in M\) s.t. \(y \in p^{-1}(\xi_{x_1}^{x_2}(t))\)
Cohomogeneity One Actions

Theorem: If $(M,\omega,m)$ has (GTB) and $M^s \cong S^1$
then $M$ is isomorphic to an $S^1$-fiber bundle.

Proof: let $U_x$ be as before.
- for $m$-a.e. $y \in U_x$, $\xi_x(1) \in U$
- Choose $x_1, x_2 \in M$ s.t. $y \in \rho^{-1}(\xi_{x_1}(1), (2))$
  $\implies y \in U_{x_1} \cap U_{x_2}$
Cohomogeneity One Actions

Theorem: If \((M,\mathfrak{g},\mathfrak{m})\) has (GTB) and \(\mathfrak{m} \cong S^1\), then \(M\) is homeomorphic to an \(S^1\)-fiber bundle.

Pr: 
- Let \(U_x\) be as before.
- For m-a.e. \(\gamma \in U_x\), \(\xi_x(\gamma(t)) \in U\). 
- Choose \(x_1, x_2 \in M\) s.t. \(\gamma \in \mathcal{P}(\xi_{x_1}(\chi_{x_2}(\gamma)))\)
  \(\implies \gamma \in U_{x_1} \cap U_{x_2}\)
- \(\exists x_1, x_2, x_3, x_4 \in M\). \(\forall y \in M\): \(\gamma \in U_{x_1} \cup U_{x_2}\) for \(i, j \in \{1, 2, 3, 4\}\).
Cohomogeneity One Actions

Theorem: If \((M,\alpha,\omega)\) has (GTB) and \(M^* \cong S^1\) then \(M\) is homeomorphic to an \(S^1\)-fiber bundle.

Proof:
- Let \(U_x\) be as before.
- For m.a.e. \(\gamma \in U_x\), \(e_x(\gamma) \in U\).
- Choose \(x, x_2 \in M\) s.t. \(\gamma \in \hat{p}^{-1}(e_x(e_{x_2}^{-1}(\gamma)))\)
  \(\implies \gamma \in U_{x_1} \cap U_{x_2}\)
- \(\exists x_1, x_2, x_3, x_4 \in M\) s.t. \(\gamma \in U_{x_i} \cap U_{x_j}\) for \(i, j \in \{1, 2, 3, 4\}\)
  \(\implies U_{x_i} = M\), \(G(\gamma) \cong G(\gamma')\) \(\blacksquare\)
Orbi-folds
Orbifolds - spaces locally isometric to $\mathbb{R}^n/\Gamma_u$
**Orbi Folds** - spaces locally isometric to $\mathbb{R}^n / \Gamma_u$

Note: If $U$ is locally $(\mathcal{R})^{\mathbb{C}^0}$ then $V$ is locally $(\mathcal{R})^{\mathbb{C}^0}$

$V = \frac{u}{\Gamma_u}$
Orbi-folds - spaces locally isometric to $\mathbb{R}^n/\Gamma_u$

Note: If $U$ is locally (RCD)*
then $V$ is locally (RCD)*

Can choose $V$ s.t. for (unique) $x, v$
$p^{-1}(x, v)$ is unique fixed point of $\Gamma_u = \{\Gamma_x\}$
and $\Gamma_u$ acts effectively (possibly $\Gamma_u = \text{Id}$)!
Orbifolds - spaces locally isometric to $\mathbb{R}^n / \Gamma_u$

Note: If $U$ is locally (RCD)* then $V$ is locally (RCD)*

Can choose $V$ s.t. for (unique $x, v$
$p^{-1}(x)$ is unique fixed point of $\Gamma_u$ ($= P_u$)
and $\Gamma_u$ acts effectively (possibly $\Gamma_u = \{id\}$)

Corollary (Bishop for orbifolds)

If $\text{Ric}^0 \geq k$ then $\text{vol}_C(\text{Br}(x)) \leq \frac{1}{1 + \text{vol}_C(\text{Br}(x))}
\text{vol}_C(U_k(r))$

define either via $\text{RCD}(k, \infty)$ or $\text{Ric} \geq k$ locally in $U$
Application - Orbispaces & discrete infinite groups
Application - Orbispaces & discrete infinite groups

Theorem: If $(M,d,w)$ is locally isometric to a quotient of a (locally) RCD$^*$-space then it is an RCD$^*$-space.
Application - Orbispaces & discrete infinite groups

Theorem: If \((M,d,w)\) is locally isometric to a quotient of a (locally) QCD*-space then it is an QCD*-space.

Definition: A discrete group acts almost effectively if
\[
\forall x \in M : G_x = \{g \in G : gx = x\} \text{ is finite}
\]

Theorem: If \((M,d,w)\) has (GTB) and \(G\) acts effectively and almost effectively then \(G_x = \{1\} \cup \{m\} \) \(m\)-almost everywhere.
**Application** - Orbispaces & discrete infinite groups

**Theorem:** If \((M, d, m)\) is locally isometric to a quotient of a (locally) QCD*-space then it is an QCD*-space.

**Definition:** A discrete group acts almost effectively if 
\[
\forall x \in M : G_x = \{ g \in G | gx = x^3 \} \text{ is finite}
\]

**Theorem:** If \((M, d, m)\) has (GTB) and \(G\) acts effectively and almost effectively then \(G_x = \{ \text{id} \}\) \(m\)-almost everywhere.

**Corollary:** \((M^*, d^*)\) is an orbispace which is loc. isometric to \(M/G_x\). Furthermore, there is a unique \(m^*\) such that 
\[
m^*/u_x = m^*/V_{x^*} \text{ for all regular } x, \text{ i.e. } G_x = \{ \text{id} \}.
\]
Thank you!