A tropical construction of a family of real reducible curves.

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Abstract

We give a constructive proof using tropical modifications of the existence of a family of real algebraic plane curves with asymptotically maximal numbers of even ovals.

1 Introduction

Let $A$ be a non-singular real algebraic plane curve in $\mathbb{CP}^2$. Its real part, denoted by $R_A$, is a disjoint union of embedded circles in $\mathbb{RP}^2$. A component of $R_A$ is called an oval if it divides $\mathbb{RP}^2$ in two connected components. If the degree of the curve $A$ is even, then the real part $R_A$ is a disjoint union of ovals. An oval of $R_A$ is called even (resp., odd) if it is contained inside an even (resp., odd) number of ovals. Denote by $p$ (resp., $n$) the number of even (resp., odd) ovals of $R_A$.

Petrovsky inequalities: For any real algebraic plane curves of degree $2k$, one has

$$-\frac{3}{2}k(k-1) \leq p - n \leq \frac{3}{2}k(k-1) + 1.$$  

One can deduce upper bounds for $p$ and $n$ from Petrovsky inequalities and Harnack theorem (which gives an upper bound for the number of components of a real algebraic curves with respect to its genus):

$$p \leq \frac{7}{4}k^2 - \frac{9}{4}k + \frac{3}{2},$$

and

$$n \leq \frac{7}{4}k^2 - \frac{9}{4}k + 1.$$  

In 1906, V. Ragsdale formulated the following conjecture.

Conjecture. (Ragsdale)

For any real algebraic plane curve of degree $2k$, one has

$$p \leq \frac{3}{2}k(k-1) + 1,$$

and

$$n \leq \frac{3}{2}k(k-1).$$
In 1993, I. Itenberg used combinatorial patchworking to construct, for any \( k \geq 5 \), a real algebraic plane curve of degree \( 2k \) with
\[
p = \frac{3}{2} k(k - 1) + 1 + \left\lfloor \frac{(k - 3)^2 + 4}{8} \right\rfloor,
\]
and a real algebraic plane curve of degree \( 2k \) with
\[
n = \frac{3}{2} k(k - 1) + \left\lfloor \frac{(k - 3)^2 + 4}{8} \right\rfloor,
\]
see [Ite95]. This construction was improved by B. Haas (see [Haa95]) then by Itenberg (see [Ite01]) and finally by E. Brugallé (see [Bru06]). Brugallé constructed a family of real algebraic plane curves with
\[
\lim_{k \to +\infty} \frac{p}{k^2} = \frac{7}{4},
\]
and a family of real algebraic plane curves with
\[
\lim_{k \to +\infty} \frac{n}{k^2} = \frac{7}{4}.
\]
In order to construct such families, Brugallé proved the existence of a family of real reducible curves \( D_n \cup C_n \) in \( \Sigma_n \), the \( n \)th Hirzebruch surface. The curve \( D_n \) has Newton polytope
\[
\Delta_n = \text{Conv} \left( (0, 0), (n, 0), (0, 1) \right),
\]
the curve \( C_n \) has Newton polytope
\[
\Theta_n = \text{Conv} \left( (0, 0), (n, 0), (0, 2), (n, 1) \right),
\]
and the chart of \( D_n \cup C_n \) is homeomorphic to the one depicted in Figure 1.

Brugallé’s construction of this family of real reducible curves used so-called real rational graphs theoretical method, based on Riemann existence theorem (see [Bru06] and [Ore03]). In particular, this method is not constructive. In
this note, we give a constructive method to get such a family using tropical modifications and combinatorial patchworking for complete intersections (see Theorem 3 and [Stu94]). In Section 2, we recall the notion of amoebas, the approximation of tropical hypersurfaces by amoebas and the notion of a real phase on a tropical hypersurface. In Section 3, we remind the notion of a tropical modification along a rational function. In Section 4 we give our strategy to construct the family $D_n \cup C_n$ and in Sections 5 and 6, we explain the details of our construction.

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2 Amoebas and patchworking

In this section, we present a tropical formulation of the combinatorial patchworking theorem for nonsingular tropical hypersurfaces and complete intersections of nonsingular hypersurfaces. Amoebas appear as a fundamental link between classical algebraic geometry and tropical geometry.

Definition 1. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety. Its amoeba (see [GKZ08]) is the set
$$A = \text{Log}(V) \subset \mathbb{R}^n,$$
where
$$\text{Log} (z_1, \cdots, z_n) = (\log |z_1|, \cdots, \log |z_n|).$$
Similarly, we may consider the map
$$\text{Log}_t : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n (z_1 \cdots, z_n) \mapsto \left( \frac{\log |z_1|}{\log t}, \cdots, \frac{\log |z_n|}{\log t} \right),$$
for $t > 1$.

Definition 2. Let $X$ and $Y$ be two non-empty compact subsets of a metric space $(M, d)$. Define their Hausdorff distance $d_H(X, Y)$ by
$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}.$$

Theorem 1. (Mikhalkin [Mik04], Rullgard [Rul01]) Let $P(x) = \sum_{i \in \Delta \cap \mathbb{Z}^n} a_i x^i$ be a tropical polynomial in $n$ variables. Let
$$f_t = \sum_{i \in \Delta \cap \mathbb{Z}^n} A_i(t)^{a_i} z^i,$$
be a family of complex polynomials and suppose that $A_i(t) \sim \gamma_i$ when $t$ goes to $+\infty$ with $\gamma_i \in \mathbb{C}^*$. Denote by $Z(f_t)$ the zero-set of $f_t$ in $(\mathbb{C}^*)^n$ and by $V(P)$ the tropical hypersurface associated to $P$. Then for any compact $K \subset \mathbb{R}^n$,
$$\lim_{t \to +\infty} \text{Log}_t(Z(f_t)) \cap K = V(P) \cap K,$$
with respect to the Hausdorff distance. We say that the family $Z(f_t)$ is an approximating family of the hypersurface $V(P)$.

Remark 1. Consider $A_i(t) \in \{\pm 1\}$, for $i \in \Delta \cap \mathbb{Z}^n$. Then, the polynomial $f_t$ is a Viro polynomial (see for example [Ite95] for the definition of a Viro polynomial).
Figure 2: A real tropical line. Figure 3: The real part of the real tropical line depicted in Figure 2.

To give a tropical formulation of the combinatorial patchworking theorem for nonsingular tropical hypersurfaces, remind first the notion of a real phase for a nonsingular tropical hypersurface in $\mathbb{R}^n$.

**Definition 3.** A real phase on a nonsingular tropical hypersurface $S$ in $\mathbb{R}^n$ is the data for every facet $F$ of $S$ of $2^{n-1}$ $n$-uplet of signs $\varphi_{F,i} = (\varphi^1_{F,i}, \ldots, \varphi^n_{F,i})$, $1 \leq i \leq 2^{n-1}$ satisfying to the following properties:

1. If $1 \leq i \leq 2^{n-1}$ and $v = (v_1, \ldots, v_n)$ is an integer vector in the direction of $F$, then there exists $1 \leq j \leq 2^{n-1}$ such that $(-1)^{v_k} \varphi^k_{F,i} = \varphi^k_{F,j}$, for $1 \leq k \leq n$.

2. Let $H$ be a codimension 1 face of $S$. Then for any facet $F$ adjacent to $H$ and any $1 \leq i \leq 2^{n-1}$, there exists a unique face $G \neq F$ adjacent to $H$ and $1 \leq j \leq 2^{n-1}$ such that $\varphi_{G,j} = \varphi_{F,i}$.

A nonsingular tropical hypersurface equipped with a real phase is called a nonsingular real tropical hypersurface.

**Example 1.** In Figure 2, we depicted a real tropical line.

**Remark 2.** In the case of nonsingular tropical curves in $\mathbb{R}^2$, a real phase can also be described in terms of a ribbon structure (see [BIMS]).

**Definition 4.** For any rational convex polyhedron $F$ in $\mathbb{R}^n$ defined by $N$ inequalities

$$< j_1, x > \leq c_1, \ldots, < j_N, x > \leq c_N,$$
where \( j_1 \cdots j_N \in \mathbb{Z}^N \) and \( c_1, \cdots, c_N \in \mathbb{R} \), denote by \( F^{\exp} \) the rational convex polyhedron in \((\mathbb{R}^*_+)^n\) defined by the inequalities

\[
< j_1, x > \leq \exp(c_1), \cdots, < j_N, x > \leq \exp(c_N).
\]

Reciprocally for any rational convex polyhedron \( F \) in \((\mathbb{R}^*_+)^n\) defined by the inequalities

\[
< k_1, x > \leq d_1, \cdots, < k_N, x > \leq d_N,
\]

where \( k_1 \cdots k_N \in \mathbb{Z}^N \) and \( d_1, \cdots, d_N \in \mathbb{R}^*_+ \), denote by \( F^{\log} \) the rational convex polyhedron in \( \mathbb{R}^n \) defined by the inequalities

\[
< k_1, x > \leq \log(d_1), \cdots, < k_N, x > \leq \log(d_N).
\]

Extend these definitions to rational polyhedral complexes.

For \( \varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n \), denote by \( s_\varepsilon \) the symmetry of \( \mathbb{R}^n \) defined by

\[
s_\varepsilon(u_1, \cdots, u_n) = ((-1)^{\varepsilon_1}u_1, \cdots, (-1)^{\varepsilon_n}u_n).
\]

Let \((S, \varphi)\) be a nonsingular real tropical hypersurface. Denote by \( F(S) \) the set of all facets of \( S \).

**Definition 5.** The real part of \((S, \varphi)\) is

\[
\mathbb{R}^nS_\varphi = \bigcup_{F \in F(S)} \bigcup_{1 \leq i \leq 2^{n-1}} s_{\varphi,F,i}(F^{\exp}).
\]

**Example 2.** In Figure 3, we depicted the real part of the real tropical line from Example 1.

Let \( S \) be a nonsingular tropical hypersurface in \( \mathbb{R}^n \) given by a tropical polynomial \( P \), and let \( \varphi \) be a real structure on \( S \). Denote by \( \tau \) the dual subdivision of \( S \).

**Definition 6.** A distribution of signs \( \delta \) at the vertices of \( \tau \) is called compatible with \( \varphi \) if for any vertex \( v \) of \( \tau \), the following compatibility condition is satisfied.

- For any vertex \( w \) of \( \tau \) adjacent to \( v \), one has \( \delta_v \neq \delta_w \) if and only if there exists \( 1 \leq i \leq 2^{n-1} \) such that \( \varphi_{F,i} = (+, \cdots, +) \), where \( F \) denotes the facet of \( S \) dual to the edge connecting \( v \) and \( w \).

**Lemma 1.** For any real phase \( \varphi \) on \( S \), there exist exactly two distributions of signs \( \delta \) at the vertices of \( \tau \) compatible with \( \varphi \). Reciprocally, given any distribution of signs \( \delta \) at the vertices of \( \tau \), there exists a unique real phase \( \varphi \) on \( S \) such that \( \delta \) is compatible with \( \varphi \).

**Proof.** Let \( \varphi \) be a real phase on \( S \). Choose an arbitrary vertex \( v \) of \( \tau \) and put an arbitrary sign \( \varepsilon \) at \( v \). Given a vertex of \( \tau \) equipped with a sign, define a sign at all adjacent vertices by using the compatibility condition in Definition 6. It gives a distribution of signs \( \delta \) at the vertices of \( \tau \) compatible with \( \varphi \) such that \( \delta_v = \varepsilon \). In fact, let \( G \) be a face of \( S \) of codimension 1 and denote by \( F_1, F_2, F_3 \) the facets of \( S \) adjacent to \( G \). It follows from the definition of a real phase that...
either \( \varphi_{F_i} \neq (+, \cdots, +) \) for all \( 1 \leq i \leq 3 \) and \( 1 \leq k \leq 2^{n-1} \), or that there exist exactly two indices \( 1 \leq i, j \leq 3 \) and such that
\[
\varphi_{F_i, k_i} = \varphi_{F_j, k_j} = (+, \cdots, +),
\]
where \( k_i \in \{1, \cdots, 2^{n-1}\} \) and \( k_j \in \{1, \cdots, 2^{n-1}\} \). This means exactly that going over any cycle \( \Gamma \) made of edges of \( \tau \), the signs at the vertices of \( \Gamma \) change an even number of times, and the distribution of signs \( \delta \) is well defined. The other distribution of signs at the vertices of \( \tau \) compatible with \( \varphi \) is the distribution \( \delta' \) defined by \( \delta'(v) = -\delta(v) \), for all vertices \( v \) of \( \tau \).

**Definition 7.** Let \( \Delta \) be a 2-dimensional polytope in \( \mathbb{R}^2_+ \), and let \( \tau \) be a primitive triangulation of \( \Delta \). The Harnack distribution of signs at the vertices of \( \tau \) is defined as follows. If \( v \) is a vertex of \( \tau \) with both coordinates even, put \( \delta_v = - \), otherwise put \( \delta_v = + \). The real phase compatible with \( \delta \) is called the Harnack phase. A T-curve associated to any primitive triangulation with a Harnack distribution of signs is a so-called simple Harnack curve (see for example [Itz05] for the notion of a T-curve). Simple Harnack curves have some very particular properties (see [Mik00]).

For any nonsingular real tropical hypersurface \( (S, \varphi) \) and any \( \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n \), put
\[
\mathbb{R}S^\varepsilon = s_\varepsilon \left( \mathbb{R}S \cap \left( (\mathbb{R}^+_n) \right) \right).
\]
The set \( \mathbb{R}S^\varepsilon \) is a finite rational polyhedral complex in \( (\mathbb{R}^+_n) \). The following theorem is a corollary of Theorem 1.

**Theorem 2.** Let \( S \) be a nonsingular tropical hypersurface given by the tropical polynomial \( P(x) = \sum_{i \in \Delta \cap \mathbb{Z}^n} a_i x^i \), and let \( \tau \) be the dual subdivision of \( P \). Let \( \varphi \) be a real phase on \( S \) and let \( \delta \) be a distribution of signs at the vertices of \( \tau \) compatible with \( \delta \). Put \( f_\varepsilon = \sum_{i \in \Delta \cap \mathbb{Z}^n} \delta_i t^{a_i} z^i \). Then, for every \( \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n \) and for every compact \( K \subset \mathbb{R}^n \), one has
\[
\lim_{t \to +\infty} \log_t \left( Z(f_\varepsilon) \cap s_\varepsilon \left( (\mathbb{R}^+_n) \right) \right) \cap K = (\mathbb{R}S^\varepsilon) \log \cap K.
\]
We say that the family \( Z(f_\varepsilon) \) is an approximating family of \( (S, \varphi) \).

**Definition 8.** Let \( S_1, \cdots, S_k \) be \( k \) tropical varieties in \( \mathbb{R}^n \). We say that the varieties \( S_i, 1 \leq i \leq k, \) intersect transversely if every top-dimensional cell of \( S_1 \cap \cdots \cap S_k \) is a transverse intersection \( \cap_{i=1}^k F_i \), where \( F_i \) is a facet of \( S_i \).

The next theorem is a tropical reformulation of the combinatorial patchworking theorem for complete intersections (see [Stu94]).

**Theorem 3.** Let \( S_1, \cdots, S_k \) be \( k \) tropical hypersurfaces in \( \mathbb{R}^n \) such that \( S_j \) is given by the tropical polynomial \( P_j(x) = \sum_{i \in \Delta \cap \mathbb{Z}^n} a_i^j x^i \), for \( 1 \leq j \leq k \). Let \( \tau^j \) be the dual subdivision of \( P_j \), for \( 1 \leq j \leq k \). Assume that the \( S_1, \cdots, S_k \) intersect transversely. Let \( \varphi^j \) be a real phase on \( S_j \) and \( \delta^j \) be a distribution of signs at the vertices of \( \tau^j \) compatible with \( \delta^j \), for \( 1 \leq j \leq k \). Put \( f_{\varepsilon}^j = \sum_{i \in \Delta \cap \mathbb{Z}^n} \delta_i^j t^{-a_i^j} z^i \). Then, for every \( \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n \) and for every compact \( K \subset \mathbb{R}^n \), one has
\[
\lim_{t \to +\infty} \log_t \left( Z(f_{\varepsilon}^1) \cap \cdots \cap Z(f_{\varepsilon}^k) \cap s_\varepsilon ( (\mathbb{R}^+_n) ) \right) \cap K = (\mathbb{R}S_{1,\varepsilon}^{\varepsilon}) \log \cap \cdots \cap (\mathbb{R}S_{k,\varepsilon}^{\varepsilon}) \log \cap K.
\]
We say that the family \( (Z(f_1), \ldots, Z(f_k)) \) is an approximating family of 
\[ ((S_1, \ldots, S_k), (\varphi_1, \ldots, \varphi_k)) \].

3 Tropical modifications of \( \mathbb{R}^n \)

Tropical modifications were introduced by Mikhalkin in [Mik06]. We recall in this section the definition of a tropical modification of \( \mathbb{R}^n \) along a rational function. More details can be found in [Mik06], [BLdM12], [Sha11] and [BIMS].

Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) be two tropical polynomials. One may consider the rational tropical function \( h = \frac{f}{g} \). Denote by \( V(f) \) the tropical hypersurface associated to \( f \) and by \( V(g) \) the tropical hypersurface associated to \( g \).

**Definition 9.** The tropical modification of \( \mathbb{R}^n \) along \( h \), denoted by \( \mathbb{R}^n_h \), is the tropical hypersurface of \( \mathbb{R}^{n+1} \) defined by “\( x_{n+1} g(x) + f(x) \)”.

We may also describe \( \mathbb{R}^n_h \) in a more geometrical way. Consider the graph \( \Gamma_h \) of the piecewise linear function \( h \). It is a polyhedral complex in \( \mathbb{R}^{n+1} \). Equip the graph \( \Gamma_h \) with the constant weight function equal to 1 on it facets. In general, this graph is not a tropical hypersurface of \( \mathbb{R}^{n+1} \) as it is not balanced at faces \( F \) of codimension one. At every codimension one face \( F \) of \( \Gamma_h \) which fails to satisfy the balancing condition, add a new facet as follows. Denote by \( w \) the integer number such that the balancing condition is satisfied at \( F \) if we attach to \( F \) a facet \( F^{-1} \) in the \((0, \ldots, 0, -1)\)-direction equipped with the weight \( w \). If \( w > 0 \), attach \( F^{-1} \) (equipped with \( w \)) to \( F \) and if \( w < 0 \), attach to \( F \) a facet \( F^{+1} \) in the \((0, \ldots, 0, 1)\)-direction equipped with the weight \(-w\). If \( V(f) \) and \( V(g) \) intersect transversely, then the tropical modification \( \mathbb{R}^n_h \) is obtained by attaching to the graph \( \Gamma_h \) the intervals
\[ [(x, -\infty), (x, h(x))] \]
for all \( x \) in the hypersurface \( V(f) \), and
\[ [(x, h(x)), (x, +\infty)] \]
for all \( x \) in the hypersurface \( V(g) \) and by equipping each new facet with the unique weight so that the balancing condition is satisfied.

**Definition 10.** The principal contraction
\[ \delta_h : \mathbb{R}^n_h \to \mathbb{R}^n \]
associated to \( h \) is the projection of \( \mathbb{R}^n_h \) onto \( \mathbb{R}^n \).

**Example 3.** In Figure 4, we depicted the tropical modification of \( \mathbb{R}^2 \) along the tropical line given by the tropical polynomial “\( x + y + 0 \)”. It is the tropical plane in \( \mathbb{R}^3 \) given by the tropical polynomial “\( x + y + z + 0 \)”. In Figure 5, we depicted the tropical modification of \( \mathbb{R}^2 \) along “\( \frac{P}{Q} \)”, where “\( P = x + y + 0 \)” and “\( Q = y + (1) \)”.

The tropical curves \( V(P) \) and \( V(Q) \) intersect transversely. In Figure 6, we depicted the tropical modification of \( \mathbb{R}^2 \) along “\( \frac{P_1}{Q_1} \)”, where “\( Q_1 = y + 0 \)”.

The tropical curves \( V(P) \) and \( V(Q) \) do not intersect transversely.
Figure 4: The tropical modification of $\mathbb{R}^2$ along the tropical line “$x + y + 0$”.

Figure 5: The tropical modification of $\mathbb{R}^2$ along “$\frac{P}{Q}$”, where “$P = x + y + 0$” and “$Q = y + (-1)$”.

Figure 6: The tropical modification of $\mathbb{R}^2$ along “$\frac{P}{Q_1}$”, where “$P = x + y + 0$” and “$Q_1 = y + 0$”.
4 Strategy of the construction

Let \( n \geq 1 \). We construct the curve \( D_n \) (resp., \( C_n \)) in a 1-parameter family of curves \( D_{n,t} \) (resp., \( C_{n,t} \)). To construct such families of curves, we construct a tropical curve \( D_n \) with Newton polytope \( \Delta_n \) (see Figure 7 for the case \( n = 3 \)) and a tropical curve \( C_n \) with Newton polytope \( \Theta_n \) (see Figure 8 for the case \( n = 3 \)). The family of curves \( D_{n,t} \) (resp., \( C_{n,t} \)) then appears as an approximating family of the tropical curve \( D_n \) (resp., \( C_n \)). It turns out that the tropical curves \( C_n \) and \( D_n \) do not intersect transversely (see Figure 9), so we can not use directly combinatorial patchworking to determine the mutual position of the curves \( D_{n,t} \) and \( C_{n,t} \). We consider then the tropical modification \( X_n \) of \( \mathbb{R}^2 \) along \( P_n \), where \( P_n \) is a tropical polynomial defining \( D_n \). In this new model, the curve \( D_n \) is the boundary in the vertical direction of the compactification of \( X_n \) in \( \mathbb{T}^n \), and if \( \hat{C}_n \) is a lifting of \( C_n \) in \( X_n \) (see Definition 12), then the compactification of \( \hat{C}_n \) in \( \mathbb{T}^n \) intersects \( D_n \) transversely. Then, we show that the curve \( \hat{C}_n \) is the transverse intersection of \( X_n \) with \( Y_n \), a tropical modification of \( \mathbb{R}^2 \) along a tropical rational function (see Definition 8). We define real phases \( \varphi_{D_n} \) on \( D_n \) and real phases \( \varphi_{C_n} \) on \( C_n \) (see Figure 15 and Figure 16 for the case \( n = 3 \)) and we construct real phases \( \varphi_{X_n} \) on \( X_n \) and real phases \( \varphi_{Y_n} \) on \( Y_n \) satisfying compatibility conditions with \( \varphi_{D_n} \) and \( \varphi_{C_n} \) (see Lemma 5). It follows from Theorem 2 that there exists a family of real polynomials \( P_{n,t} \) with Newton polytopes \( \Delta_n \) such that if we put

\[
D_{n,t} = \{ P_{n,t}(x, y) = 0 \},
\]

and

\[
X_{n,t} = \{ z + P_{n,t}(x, y) = 0 \},
\]

one has

\[
\lim_{t \to +\infty} \log \left( \mathbb{R} D_{n,t} \cap s_z (\mathbb{R}^2_n) \right) \cap V = (\mathbb{R} D_n^\epsilon) \log \cap V,
\]

and

\[
\lim_{t \to +\infty} \log \left( \mathbb{R} X_{n,t} \cap s_z (\mathbb{R}^3_n) \right) \cap W = (\mathbb{R} X_n^\epsilon) \log \cap W,
\]

for any \( \epsilon \in (\mathbb{Z}/2\mathbb{Z})^2, \) any \( n \in (\mathbb{Z}/2\mathbb{Z})^3, \) any compact \( V \subset \mathbb{R}^2 \) and any compact \( W \subset \mathbb{R}^3 \). It follows from Theorem 3 that there exists a family of surfaces \( Y_{n,t} \) such that for any \( \epsilon \in (\mathbb{Z}/2\mathbb{Z})^3 \) and any compact \( V \subset \mathbb{R}^3 \), one has

\[
\lim_{t \to +\infty} \log \left( \mathbb{R} X_{n,t} \cap \mathbb{R} Y_{n,t} \cap s_z (\mathbb{R}^3_n) \right) \cap V = (\mathbb{R} X_n^\epsilon) \log \cap (\mathbb{R} Y_n^\epsilon) \log \cap V.
\]

Consider the projection \( \pi^\mathbb{C} : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2 \) forgetting the last coordinate. For every \( t \), put

\[
C_n = \pi^\mathbb{C}(X_{n,t} \cap Y_{n,t}).
\]

Then, the Newton polytope of \( C_{n,t} \) is \( \Theta_n \) and we show that for \( t \) large enough, the chart of \( D_{n,t} \cup C_{n,t} \) is homeomorphic to the chart depicted in Figure 1. Thus, we put

\[ D_n = D_{n,t}, \]

and

\[ C_n = C_{n,t} \]

for \( t \) large enough.
5 Construction of the surfaces $X_n$ and $Y_n$, and of the curve $\tilde{C}_n$

Consider the subdivision of $\Delta_n$ given by the triangles

$$\Delta^k_n = \text{Conv}((k,0),(0,1),(k+1,0)),$$

for $0 \leq k \leq n-1$. Consider the subdivision of $\Theta_n$ given by the triangles

- $K^k_n = \text{Conv}((k,0),(k,1),(k+1,0))$,
- $L^k_n = \text{Conv}((k,1),(k+1,0),(k+1,1))$ and
- $M^k_n = \text{Conv}((k,1),(k,2),(k+1,1))$,

for $0 \leq k \leq n-1$. Consider a tropical curve $D_n$ dual to the subdivision $(\Delta^k_n)_{0 \leq k \leq n-1}$ of $\Delta_n$ (see Figure 7 for the case $n = 3$), a tropical curve $C_n$ dual to the subdivision $(K^k_n,L^k_n,M^k_n)_{0 \leq k \leq n-1}$ of $\Theta_n$ (see Figure 8 for the case $n = 3$), and $2n$ marked points $x_1, \cdots, x_{2n}$ on $C_n$, such that $D_n$, $C_n$ and $x_1, \cdots, x_{2n}$ satisfy the following conditions.

1. For any $0 \leq k \leq n-1$, the coordinates of the vertex of $D_n$ dual to $\Delta^k_n$ are equal to the coordinates of the vertex of $C_n$ dual to $M^k_n$. 

Figure 7: Tropical curve $D_3$.  

Figure 8: Tropical curve $C_3$. 

2. For any $1 \leq k \leq n$ the first coordinate of $x_k$ is equal to the first coordinate of the vertex dual to $K_n^{k-1}$.

3. For any $n + 1 \leq k \leq 2n$, the marked point $x_k$ is on the edge of $C_n$ dual to the edge $[(0, 2), (n, 1)]$ of $\Theta_n$.

For each marked point $x_i$, $1 \leq i \leq 2n$, refine the edge of $C_n$ containing $x_i$ by considering the marked point $x_i$ as a vertex of $C_n$. In Figure 9, we draw the tropical curves $D_3$ and $C_3$ on the same picture. Denote by $P_n$ a tropical polynomial defining the tropical curve $D_n$ and put $X_n = \mathbb{R}^2_{P_n}$.

**Definition 11.** Let $C \subset \mathbb{R}^2$ be a tropical curve with $k$ vertices of valence 2 denoted by $x_1, \ldots, x_k$ (called the marked points of $C$). We say that a tropical curve $\tilde{C} \subset \mathbb{R}^3$ is a lift of $(C, x_1, \ldots, x_k)$ if the following conditions are satisfied.

- $\pi^\mathbb{R}(\tilde{C}) = C$, where $\pi^\mathbb{R} : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the vertical projection on the first two coordinates.
- Any infinite vertical edge of $\tilde{C}$ is of the form $[(x, -\infty), (x, r)]$, with $x \in \mathbb{R}^2$ and $r \in \mathbb{R}$.
- An edge $e$ of $\tilde{C}$ is an infinite vertical edge if and only if $\pi^\mathbb{R}(e) \subset \{x_1, \ldots, x_k\}$.
For any point \( x \) in the interior of an edge \( e \) of \( C \) such that \( (π^R)^{-1}(x) \cap \tilde{C} \) is finite, one has

\[
w(e) = \sum_{i=1}^{l} w(f_i) [Λ_e : Λ_{f_i}],
\]

where \( f_1, \ldots, f_l \) are the edges of \( \tilde{C} \) containing the preimages of \( x \), the weight of \( e \) (resp., \( f_i \)) is denoted by \( w(e) \) (resp., \( w(f_i) \)) and \( Λ_e \) (resp., \( Λ_{f_i} \)) denotes the sublattice of \( \mathbb{Z}^3 \) generated by a primitive vector in the direction of \( e \) (resp., \( f_i \)) and by the vector \((0,0,1)\).

Remark 3. Let \( C \) be a nonsingular tropical curve in \( \mathbb{R}^2 \) with \( k \) marked points \( x_1, \ldots, x_k \) such that there exists a lifting \( \tilde{C} \) of \((C, x_1, \ldots, x_k)\) in \( \mathbb{R}^3 \). Then, it follows from Definition 11 that for any edge \( e \) of \( C \), there exists a unique edge \( f \) of \( \tilde{C} \) such that \( π^R(f) = e \). Moreover, one has \( w(f) = 1 \). It follows from the balancing condition that at any trivalent vertex \( v \) of \( C \), the directions of the lifts of any two edges adjacent to \( v \) determine the direction of the lift of the third edge adjacent to \( v \). At any marked point \( x_i \), the direction of the lift of an edge adjacent to \( x_i \) and the weight of the infinite vertical edge of \( \tilde{C} \) associated to \( x_i \) determine the direction of the lift of the other edge adjacent to \( x_i \).

Definition 12. Let \( h \) be a tropical rational function on \( \mathbb{R}^2 \) and let \( \mathbb{R}^2_h \) be the tropical modification of \( \mathbb{R}^2 \) along \( h \). Let \( C \subset \mathbb{R}^2 \) be a tropical curve with \( k \) marked points \( x_1, \ldots, x_k \). We say that the marked tropical curve \((C, x_1, \ldots, x_k)\) can be lifted to \( \mathbb{R}^2_h \) if there exists a lifting \( \tilde{C} \) of \((C, x_1, \ldots, x_k)\) in \( \mathbb{R}^3 \) such that \( C \subset \tilde{C} \subset \mathbb{R}^2_h \).

Remark 4. Assume that a trivalent marked tropical curve \((C, x_1, \ldots, x_k)\) can be lifted to some tropical modification \( \mathbb{R}^2_h \), where \( h = \frac{f}{g} \) is some tropical rational function. It follows from the definition of a tropical modification that outside of \( V(f) \cup V(g) \), the lift of an edge of \( C \) to \( \mathbb{R}^2_h \) is uniquely determined.

Lemma 2. There exists a unique lifting of the marked tropical curve \((C_n, x_1, \ldots, x_{2n})\) to \( X_n \).

Proof. It follows from Remark 4 that for all edges \( e \) of \( C_n \) not belonging to \( D_n \), the lift of \( e \) in \( \mathbb{X}_n \) is uniquely determined. The lift of a marked point \( x_i \) is an edge \( s_i \) of the form \([x_i, −∞), (x_i, r_i)]\), where \( r_i \in \mathbb{R} \). Denote by \( e_0 \) the edge of \( C_n \) dual to the edge \([0,1), (0,2)]\) and by \( e_{2n} \) the infinite edge of \( C_n \) dual to the edge \([0,2), (n,1)]\) adjacent to \( x_{2n} \) (see Figure 8 for the case \( n = 3 \)). One can see from Remark 3 that if the direction of \( e_0 \) is \((1,0,0)\) then the direction of the lift of \( e_{2n} \) is \((1,0,s+\sum w(s_i))\). Since the edges \( e_0 \) and \( e_{2n} \) are unbounded, one has \( s \geq 0 \) and \( s+\sum w(s_i) \leq n \). Thus, the direction of \( e_0 \) is \((1,0,0)\) and \( w(s_i) = 1 \), for \( 1 \leq i \leq n \). One can see following Remark 3 that in this case the direction of a lift of any edge of \( C_n \) is uniquely determined. The only potential obstructions on the lifts of the edges of \( C_n \) to close up come from the cycles of \( C_n \). Denote by \( Z_k \) the cycle bounding the face dual to the vertex \((k-1,1)\) of \( Θ_n \), for \( 2 \leq k \leq n \) (see Figure 10). Denote by \( e^k_1, \ldots, e^k_k \) the edges of \( Z_k \) as indicated in Figure 10. Orient the edges \( e^k_i \) with the orientation coming from the counterclockwise orientation of the cycle \( Z_k \). Denote by \([e^k_1] \) the lift of the edge \( e^k_1 \) oriented with the orientation coming from the one of \( e^k_1 \). Denote by \( l^k_i \) the primitive vector in the direction of \([e^k_1] \), and denote by \( l^k_i \) the integer length
of $e_i^k$. The lifts of the edges of $Z_k$ close up if and only if the following equation is satisfied:

$$\sum_{i=1}^{6} l_i^k e_i^k = 0.$$  

This equation is equivalent to

$$l_1^k \begin{pmatrix} -1 \\ -1 \end{pmatrix} + l_2^k \begin{pmatrix} -1 \\ 0 \end{pmatrix} + l_3^k \begin{pmatrix} 0 \\ -1 \end{pmatrix} + l_4^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l_5^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + l_6^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$  

This equation is equivalent on the cycle $Z_k$ to

$$\begin{cases} l_2^k = l_4^k, \\ l_1^k = l_6^k. \end{cases}$$  

This is equivalent to say that the first coordinate of the marked point $x_k$ is equal to the first coordinate of the vertex dual to $K_{n-1}^k$, for $2 \leq k \leq n$.  

We construct a tropical rational function $h_n$ such that the curve $\tilde{C}_n$ is the transverse intersection of $X_n$ and $\mathbb{R}_{h_n}^2$. Consider the subdivision of $\Theta_n$ given by the triangles

- $G_n^k = \text{Conv} ((k, 0), (0, 1), (k + 1, 0))$,
- $H_n^k = \text{Conv} ((k, 1), (k + 1, 1), (n, 0))$ and
- $I_n^k = \text{Conv} ((k, 1), (0, 2), (k + 1, 1))$. 

Figure 10: The cycle $Z_k$. 

13
for $0 \leq k \leq n - 1$. Consider a tropical curve $F_n$ dual to the subdivision $(G^k_n, H^k_n, I^k_n)_{0 \leq k \leq n-1}$ and a horizontal line $E$ such that the following conditions are satisfied (see Figure 11 and Figure 12 for the case $n = 3$).

1. The horizontal line $E$ is below any vertex of $C_n$.

2. The marked point $x_k$ is on the edge dual to the edge $[(k-1, 0), (k, 0)]$, for $1 \leq k \leq n$.

3. The edge of $F_n$ dual to the edge $[(1, 1), (n, 0)]$ intersects transversely the edge of $C_n$ dual to the edge $[(0, 2), (n, 1)]$ at the point $x_{n+1}$.

4. The edge of $F_n$ dual to the edge $[(k, 1), (k+1, 1)]$ intersects transversely the edge of $C_n$ dual to the edge $[(0, 2), (n, 1)]$ at the point $x_{n+k+1}$, for $1 \leq k \leq n - 1$.

Denote by $f_n$ a tropical polynomial satisfying $V(f_n) = F_n$ and by $g$ a tropical polynomial satisfying $V(g) = E$. Put $h^0_n = \frac{L_n}{g}$.

**Lemma 3.** There exists a unique lifting of the curve $(C_n, x_1, \cdots, x_{2n})$ to $\mathbb{R}^2_{h^0_n}$. Moreover, there exists $\lambda_0 \in \mathbb{R}$ such that $\tilde{C}_n$ is the lifting of $(C_n, x_1, \cdots, x_{2n})$ to $\mathbb{R}^2_{\lambda_0 h^0_n}$. 

14
Proof. Let $e$ be an edge of $C_n$ not contained in $F_n$. Then the lift of $e$ to $\mathbb{R}^2_{h_0}$ is uniquely determined. Consider the subdivision of $\mathbb{R}^2$ induced by the tropical curve $F_n \cup E$. Denote by $F^k$ the face of this subdivision dual to the point $(k, 1)$, and by $G^k$ the face dual to the point $(k, 2)$, for $0 \leq k \leq n$. The direction of the face of $\mathbb{R}^2_{h_0}$ projecting to $F^k$ is generated by the vectors $(0, 1, -1)$ and $(1, k, 0)$, and the direction of the face of $\mathbb{R}^2_{h_0}$ projecting to $G^k$ is generated by the vectors $(0, 1, 0)$ and $(1, k, k)$. It follows from these computations that the lift to $\mathbb{R}^2_{h_0}$ of an edge $e$ of $C_n$ not contained in $F_n$ has the same direction as the edge of $\tilde{C}_n$ projecting to $e$. From Remark 3, we deduce in this case that $(C_n, x_1, \cdots, x_{2n})$ has a unique lifting to $\mathbb{R}^2_{h_0}$ and that the result is a vertical translation of $\tilde{C}_n$.

Then there exists $\lambda_0 \in \mathbb{R}$ such that the lifting of $(C_n, x_1, \cdots, x_{2n})$ to $\mathbb{R}^2_{\lambda_0 h_0}$ is $\tilde{C}_n$.

Put $h_n = "\lambda_0 h_0"$ and $Y_n = \mathbb{R}^2_{\lambda_0 h_0}$.

**Lemma 4.** The surfaces $X_n$ and $Y_n$ intersect transversely and $\tilde{C}_n = X_n \cap Y_n$.

**Proof.** It follows from Lemma 3 that $\tilde{C}_n \subset X_n \cap Y_n$. By construction, any edge of $\tilde{C}_n$ is the transverse intersection of a face of $X_n$ with a face of $Y_n$. So $\tilde{C}_n \subset X_n \cdot Y_n$, where $X_n \cdot Y_n$ denotes the stable intersection of $X_n$ and $Y_n$. Let us compute the number (counted with multiplicity) of infinite edges of $X_n \cdot Y_n$ in a given direction. Denote by $\Delta(X_n)$ the Newton polytope of $X_n$, and denote by $\Delta(Y_n)$ the Newton polytope of $Y_n$. One has

$$\Delta(X_n) = Conv \left( (0, 0, 0), (n, 0, 0), (0, 1, 0), (0, 0, 1) \right),$$

and

$$\Delta(Y_n) = Conv \left( (0, 0, 0), (n, 0, 0), (n, 1, 0), (0, 2, 0), (0, 0, 1), (1, 0, 1) \right),$$

see Figure 13 and Figure 14. By considering faces of the Minkowsky sum

$$\Delta(X_n) + \Delta(Y_n) = \{ a + b \mid a \in \Delta(X_n) \text{ and } b \in \Delta(Y_n) \},$$

one can see that $X_n \cdot Y_n$ has two edges (counted with multiplicity) of direction $(-1, 0, 0)$, $n$ edges (counted with multiplicity) of direction $(0, -1, 0)$, $2n$ edges (counted with multiplicity) of direction $(0, 0, -1)$, one edge of direction $(1, 0, n)$
and one edge of direction \((1, n, n)\). Since the curve \(\tilde{C}_n\) has also two edges of direction \((-1, 0, 0), n\) edges of direction \((0, -1, 0), 2n\) edges of direction \((0, 0, -1), \) one edge of direction \((1, 0, n)\) and one edge of direction \((1, n, n)\), we conclude that \(\tilde{C}_n = X_n \cdot Y_n = X_n \cap Y_n\).

\[\Box\]

### 6 Real phases on \(X_n\) and \(Y_n\)

Consider the Harnack phase \(\varphi_{D_n}\) on \(D_n\) and the Harnack phase \(\varphi_{C_n}\) on \(C_n\) (see Definition 7). We depicted the real part of \(D_3\) on Figure 15 and the real part of \(C_3\) on Figure 16. Notice that for the real phase \(\varphi_{C_n}\) on \(C_n\), every edge of \(C_n\) containing a marked point is equipped with the sign \((-,-)\). Consider the \(2n\) marked points \(r_1, \ldots, r_{2n}\) on \(RC_n\), where \(r_i\) is the symmetric copy of \(x_i\) in the quadrant \(R^*_+ \times R^*_+\) (see Figure 16 for the case \(n = 3\)).

**Lemma 5.** There exist a real phase \(\varphi_{X_n}\) on \(X_n\) and a real phase \(\varphi_{Y_n}\) on \(Y_n\) such that the following conditions are satisfied:

1. \(\mathbb{R}D_n\) is the projection of the union of all vertical faces of \(\mathbb{R}X_n\).
2. \(\pi^R(\mathbb{R}X_n \cap \mathbb{R}Y_n) = \mathbb{R}C_n\).
3. An edge \(e\) of \(\mathbb{R}X_n \cap \mathbb{R}Y_n\) is unbounded in direction \((0, 0, -1)\) if and only if \(\pi^R(e) \subset \{r_1, \ldots, r_{2n}\}\).

**Proof.** Denote by \(\delta_{D_n}\) the Harnack distribution of signs at the vertices of the subdivision of \(\Delta_n\) dual to \(D_n\). By definition, \(\varphi_{D_n}\) is compatible with \(\delta_{D_n}\). Complete the distribution of signs \(\delta_{D_n}\) to a distribution of signs \(\delta_{X_n}\) at the vertices of the dual subdivision of \(X_n\) by choosing an arbitrary sign for the vertex \((0,0,1)\). Define \(\varphi_{X_n}\) to be the real phase on \(X_n\) compatible with \(\delta_{X_n}\). By construction, \(\varphi_{X_n}\) satisfies Condition 1 of Lemma 5. Define a real phase on \(Y_n\) as follows. Denote by \(\mathcal{G}\) the set of all faces of \(Y_n\) containing an edge of \(\tilde{C}_n\). We first define a real phase \(\varphi_{Y_n,F}\) on any face \(F \in \mathcal{G}\). Consider, as in the proof
of Lemma 3, the subdivision of $\mathbb{R}^2$ induced by the tropical curve $F_n \cup E$. Denote by $F^k$ the face of this subdivision dual to the point $(k, 1)$, and by $G^k$ the face dual to the point $(k, 2)$, for $0 \leq k \leq n$. Denote by $\tilde{F}^k$ the face of $Y_n$ such that $\pi^R(\tilde{F}^k) = F^k$ and by $\tilde{G}^k$ the face of $Y_n$ such that $\pi^R(\tilde{G}^k) = G^k$, for $0 \leq k \leq n$. Denote by $f_k$ the edge of $F_n$ containing $x_k$ and by $\tilde{f}_k$ the vertical face of $Y_n$ projecting to $f_k$, for $1 \leq k \leq 2n$. One can see that all the edges of $\tilde{C}_n$ are contained in the union of all faces $\tilde{F}^k$, $\tilde{G}^k$ and $\tilde{f}_k$. For any $0 \leq k \leq n$, one can see that the face $\tilde{F}^k$ contains an edge $e$ of $\tilde{C}_n$ such that $e$ is contained in a non-vertical face $F_e$ of $X_n$. Denote by $(\varepsilon_1, \varepsilon_2)$ a component of the real phase $\varphi_{\tilde{C}_n}$ on $\pi^R(e)$. Since $F_e$ is non-vertical, there exists a unique sign $\varepsilon_3$ such that $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a component of the real phase $\varphi_{X_n}$ on $F_e$. Define the real phase $\varphi_{Y_n}$ on $F^k$ to contain $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ as a component. Condition 3 of Lemma 5 determines the real phase $\varphi_{Y_n}$ on $f_k$, for any $1 \leq k \leq 2n$. Since the three faces $\tilde{F}^k$, $\tilde{G}^k$ and $\tilde{f}_{n+1}$ are adjacent, it follows from the definition of a real phase that the phase on $\tilde{G}^k$ is determined from the phase on $\tilde{F}^k$ and the phase on $\tilde{f}_{n+1}$. Since the three faces $\tilde{G}^{k+1}$, $\tilde{G}^k$ and $\tilde{f}_{n+k+1}$ are adjacent, for any $1 \leq k \leq n-1$, it follows from the definition of a real phase that the phase on $\tilde{G}^{k+1}$ is determined from the phase on $\tilde{G}^k$ and the phase on $\tilde{f}_{n+k+1}$. By induction, it determines the real phase $\varphi_{Y_n}$ on $\tilde{G}^k$, for $1 \leq k \leq n$.

We now extend the definition of $\varphi_{Y_n}$ to all faces of $Y_n$. Consider the set of edges $E$ of the dual subdivision of $Y_n$ such that $e \in E$ if and only if $e$ is dual to an element in $G$. Consider $V$ the set of vertices of edges in $E$. As in Lemma 1, one can consider a distribution of signs on $V$ compatible with the real phase on $G$. Extend arbitrarily this distribution to all vertices of the dual subdivision of $Y_n$ and consider the real phase $\varphi_{Y_n}$ on $Y_n$ compatible with the extended distribution of signs. By construction, the real phases $\varphi_{X_n}$ and $\varphi_{Y_n}$ satisfy the Conditions 1, 2 and 3 of Lemma 5. 

Put

$$\mathbb{R}\tilde{C}_n = \mathbb{R}X_n \cap \mathbb{R}Y_n.$$ 

Consider, as explained in Section 4, a family of real algebraic curves $D_{n,t}$ approximating $(D_n, \varphi_{D_n})$, a family of real algebraic surfaces $X_{n,t}$ approximating $(X_n, \varphi_{X_n})$ and a family of real algebraic surfaces $Y_{n,t}$ approximating $(Y_n, \varphi_{Y_n})$. Put $\tilde{C}_{n,t} = X_{n,t} \cap Y_{n,t}$.

For every $t$, put $C_{n,t} = \pi^C(\tilde{C}_{n,t})$. Consider $\mathbb{R}X_{n,t}$ the partial compactification of $\mathbb{R}X_n$ in $(\mathbb{R}^2)^2 \times \mathbb{R}$ and $\mathbb{R}X_{n,t}$ the partial compactification of $\mathbb{R}X_{n,t}$ in $(\mathbb{R}^2)^2 \times \mathbb{R}$, for any $t$. One has $\mathbb{R}X_{n,t} \cap ((\mathbb{R}^2)^2 \times \{0\}) = RD_n$ and $\mathbb{R}X_{n,t} \cap ((\mathbb{R}^2)^2 \times \{0\}) = RD_{n,t}$. Then, it follows from Theorem 3 that for $t$ large enough, one has the following homeomorphism of pairs:

$$\left(\mathbb{R}X_{n,t}, RD_{n,t} \cup \mathbb{R}\tilde{C}_{n,t}\right) \simeq \left(\mathbb{R}X_n, RD_n \cup \mathbb{R}\tilde{C}_n\right). \quad (1)$$

Moreover, the map $\pi^R$ gives by restriction a bijection from $\mathbb{R}X_{n,t}$ to $(\mathbb{R}^2)^2$ fixing $RD_{n,t}$ and sending $\mathbb{R}\tilde{C}_{n,t}$ to $\mathbb{R}C_{n,t}$. The map $\pi^R$ and the homeomorphism (1) give rise to the following homeomorphism of pairs, for $t$ large enough:

$$\left((\mathbb{R}^2)^2, RD_{n,t} \cup \mathbb{R}C_{n,t}\right) \simeq \left(\mathbb{R}X_n, RD_n \cup \mathbb{R}\tilde{C}_n\right). \quad (2)$$
It remains to describe the pair \((\mathbb{R}X_n, \mathbb{R}D_n \cup \mathbb{R}\tilde{C}_n)\). One has \(\pi^\mathbb{R}(\mathbb{R}X_n) = (\mathbb{R}^*)^2\) and for any \(x \in \mathbb{R}D_n\), \((\pi^\mathbb{R})^{-1}(x)\) is an interval of the form \(\{x\} \times [-h, h]\), see Figure 17 for a local picture.

**Remark 5.** The set \((\pi^\mathbb{R})^{-1}(\mathbb{R}D_n)\) can be seen as a tubular neighborhood of \(\mathbb{R}D_n\) in \(\mathbb{R}X_n\).

Outside of \((\pi^\mathbb{R})^{-1}(\mathbb{R}D_n)\), the map \(\pi|_{\mathbb{R}X_n}\) is bijective. Let \(V\) be a small tubular neighborhood of \(\mathbb{R}D_n\) in \((\mathbb{R}^*)^2\). Perturb slightly the pair \((\mathbb{R}X_n, \mathbb{R}\tilde{C}_n)\) inside \((\pi^\mathbb{R})^{-1}(V)\) to produce a pair \((S_n, T_n)\), such that

- \((\mathbb{R}X_n, \mathbb{R}\tilde{C}_n) \simeq (S_n, T_n)\),
- \(\mathbb{R}D_n \subset S_n\),
- \(\pi^\mathbb{R}\) defines an homeomorphism from \(S_n\) to \((\mathbb{R}^*)^2\),

see Figure 17 and Figure 18 for a local picture. One obtains the following homeomorphism of pairs, for \(t\) large enough:

\[
((\mathbb{R}^*)^2, \mathbb{R}D_{n,t} \cup \mathbb{R}C_{n,t}) \simeq ((\mathbb{R}^*)^2, \mathbb{R}D_n \cup \pi^\mathbb{R}(T_n))
\]

By construction, the curve \(\pi^\mathbb{R}(T_n)\) is a small perturbation of \(\mathbb{R}C_n\), intersecting the curve \(\mathbb{R}D_n\) transversely and only at the marked points \(r_1, \ldots, r_{2n}\), see Figure 19 and Figure 20 for the case \(n = 3\). Then, for \(t\) large enough, the chart of the reducible curve \(D_{n,t} \cup C_{n,t}\) is homeomorphic to the chart depicted in Figure 1.

**References**

Figure 19: $\mathbb{R}C_3 \cup \mathbb{R}D_3$.

Figure 20: $\pi(T_3) \cup \mathbb{R}D_3$


