A First Expedition to Tropical Geometry

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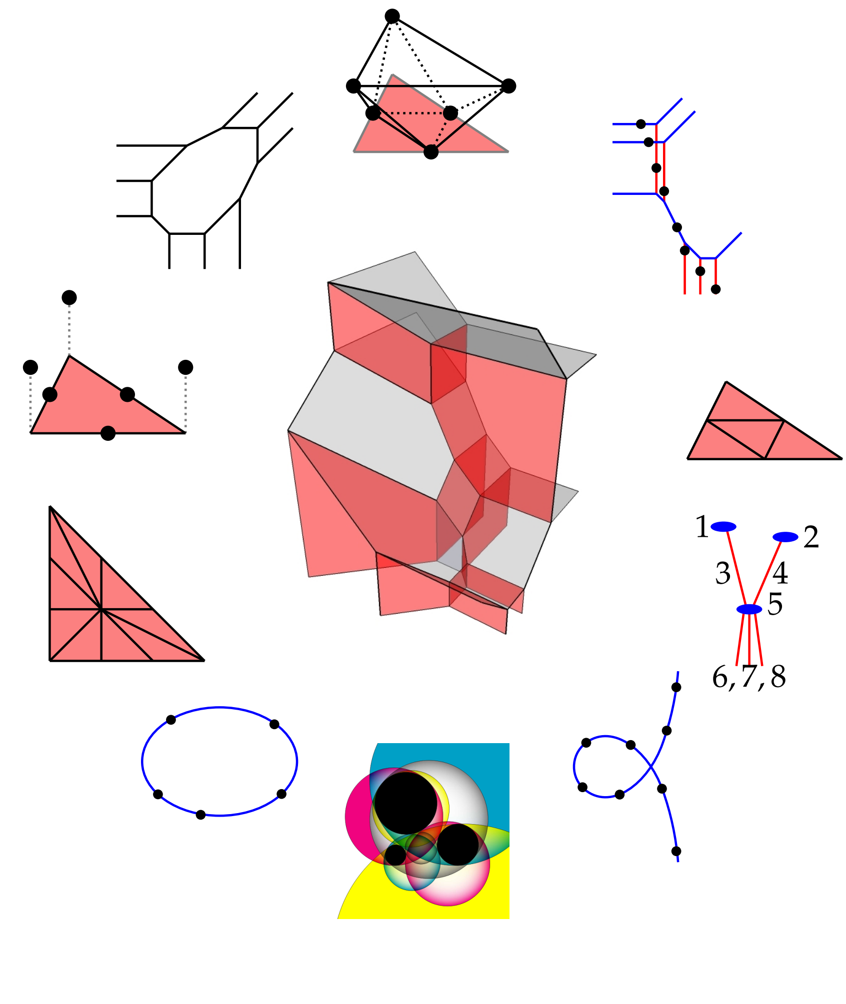


1 The Basics

About these Notes

These notes are based on a three hours mini course I gave at Tehran university. The course was designed for undergraduate students without prior knowledge to tropical geometry (nor necessarily algebraic geometry). The notes consist of two chapters covering the the first two lectures of the mini course. The first chapter is an introduction to the basics of tropical geometry, in particular tropical curves in $\mathbf{R}^2$. The second chapter contains applications of tropical geometry to enumerative geometry in the plane (Correspondence theorem). The third lecture on matroids, phylogenetics, and tropical Grassmannians is not contained in these notes (yet ;).

What is tropical geometry?

Tropical geometry is a new, combinatorial approach to geometry (algebraic, symplectic, arithmetic). It is related and originated from various sources, for example

- Viro's patchworking method, amoeba theory,
- toric geometry, Gröbner bases,
- Berkovich theory/non-archimedean geometry.

If you are acquainted with some of these topics, you might encounter familiar ideas and constructions here.

Before we get started, let us answer the question of all questions: Why is tropical geometry called tropical? To make it short, there is no deeper meaning to it! The adjective "tropical" appeared in the context of the algebraic structures underlying tropical geometry, which we will learn more about in the next section. It was apparently coined in this context (and before the advent of tropical geometry) by some French colleagues to honor the pioneering contributions of the Hungarian-born Brazilian mathematician Imre Simon (who worked in São Paolo near the Tropic of Capricorn)! It seems that the exact details of how this terminology developed are not even clear (cf. this Mathoverflow discussion).

How to start?

To get started, I am going to present tropical geometry as a new type of algebraic geometry. What is algebraic geometry about? You take one (or several)

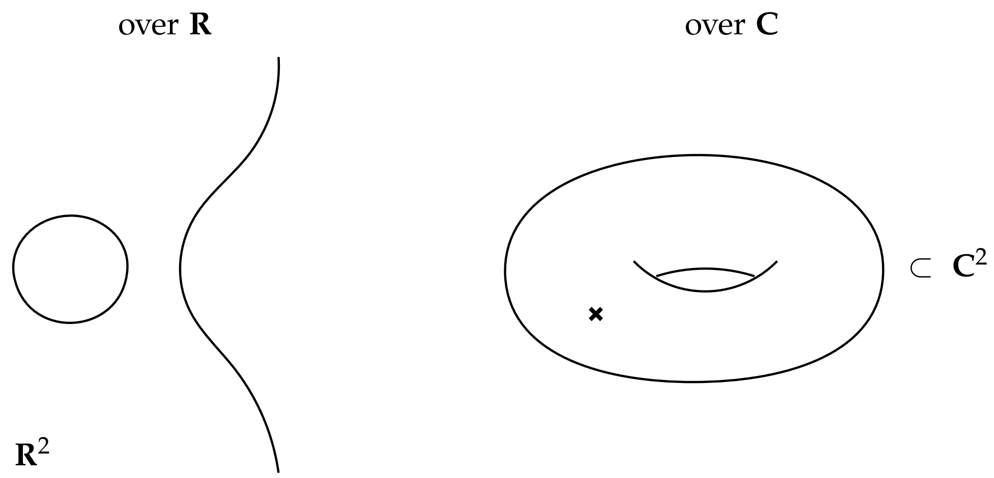


Figure 1 The set of solutions of $w^2 = z^3 - 3z^2 + 2z$, schematically. The real solutions (on the left hand side) form a 1-dimensional manifold with two component. The complex solutions (on the right hand side) form a torus surface minus one point in $\mathbf{C}^2$.

polynomial equation(s) and want to study the structure of the solution set. Let us take for example an equation in two variables.

$$w^2 = z^3 - 3z^2 + 2z$$

One of the great features of algebraic geometry is that you can choose where to look for solutions. More precisely, for any field $\mathbf{K}$ containing the coefficients of your equation, you can study the set of solutions with coordinates in this field. In our example, one natural choice would be $\mathbf{K} = \mathbf{R}$. This is the case we can actually draw. For this specific equation, we get a one-dimensional subset of $\mathbf{R}^2$ consisting of one closed and one open components (see Figure 1). Another choice would be $\mathbf{K} = \mathbf{C}$, in which case we obtain a two-dimensional surface in $\mathbf{C}^2$. Again, in this particular example one can check that this (Riemann) surface is homeomorphic to a torus with a single point removed (see Figure 1 again). If we start to change the coefficients of the equation, say to $w^2 = z^3 - 2z^2 + z - 2$, then the $\mathbf{R}$-picture may change as well (in the given example the two connected components of the curve get "glued" to a single one), while the $\mathbf{C}$-picture stays essentially (i.e. topologically) the same. The power of algebraic geometry stems from the fact that these two pictures, which look and behave so differently, show nevertheless some strong similarities from an algebraic point of view and can be treated by the same algebraic methods.

We now go one step further and replace real numbers or complex numbers (or any other field) by *tropical numbers*. To some extent tropical geometry is nothing else but algebraic geometry over tropical numbers. Alright, then what are tropical numbers?

1.1 Tropical numbers

We define two new operations on the set $\mathbf{T} = \{-\infty\} \cup \mathbf{R}$ called tropical addition and tropical multiplication. In order to distinguish them from classical addition and multiplication, we put them in quotation marks. The two operations are defined by

$$\begin{aligned} \text{“}x + y\text{”} &:= \max\{x, y\}, \\ \text{“}x \cdot y\text{”} &:= x + y. \end{aligned}$$

Regarding $-\infty$, we use the convention $\max\{-\infty, y\} = y$ and $-\infty + y = -\infty$. Hence, tropical addition is ordinary maximum, and tropical multiplication is ordinary plus. This might seem a little strange on first sight, so let us practice a little bit to get a feeling.

$$\text{“}1+1\text{”} = 1, \; \text{“}1 \cdot 1\text{”} = 2,$$
$$x = \text{“}0 \cdot x\text{”} \neq \text{“}1 \cdot x\text{”} = x + 1.$$

Note that both operations have neutral elements, namely $-\infty$ for “ $+$ ” and 0 for “ $\cdot$ ”. (I will refrain from writing things like $0_{\mathbf{T}}$ and $1_{\mathbf{T}}$ since this can cause (even more) confusion). Moreover, both operations are commutative and the distributivity law

$$\text{“}x(y+z)\text{”} = \text{“}xy + xz\text{”}$$

holds (Check this!). Finally, any element $x \in \mathbf{T} \setminus \{-\infty\} = \mathbf{R}$ has an inverse with respect to “ $\cdot$ ” (namely $-x$), in other words, $(\mathbf{T} \setminus \{-\infty\}, \text{“} \cdot \text{”})$ is a group. So, $\mathbf{T}$ is a field, right? No, of course! The big defect of tropical arithmetics is that tropical addition “ $+$ ” does not allow inverses!! Indeed, tropical addition is *idempotent*, which is to say

$$\text{“}x + x\text{”} = x.$$

In some sense, we as far away from having inverses as we can get (in particular, not even the cancellation property holds). Nevertheless, these operations are the basis of tropical geometry.

Definition 1.1 The set $\mathbf{T}$ equipped with operations " $+$ " and " $\cdot$ " is called the *semifield of tropical numbers*.

Tropical arithmetic can be understood as a limit of classical arithmetic under logarithm. This is sometimes called *Maslov dequantization*. Do the calculations yourself in the following exercise.

Exercise 1 Fix a real number $t > 1$ and define arithmetic operations on $\mathbf{R} \cup \{-\infty\}$ by

$$\begin{aligned} \text{“}x +_t y\text{”} &= \log_t(t^x + t^y), \\ \text{“}x \cdot_t y\text{”} &= \log_t(t^x \cdot t^y). \end{aligned}$$

Show that these operations converge to tropical arithmetic " $+$ " and " $\cdot$ " for $t \to \infty$.

Following the idea from the first paragraphs, we will now attempt to do algebraic geometry over $\mathbf{T}$. However, since $\mathbf{T}$ is not a field (not even a ring), we are bound to leave the framework of classical algebraic geometry and be ready to encounter a quite different type of geometry in the next sections.

1.2 Tropical polynomials

We just emphasized that $\mathbf{T}$ is not a field. But what do we actually need in order to do algebraic geometry? As explained in the first paragraphs, to get started we just need polynomials, and fortunately polynomials only involve addition and multiplication, no inverses. We can therefore write down *tropical polynomials* without any difficulties, e.g. a univariate polynomial like that:

$$f(x) = \text{“}\sum_{i=1}^{n} a_i x^i\text{”} = \max_{i=1,\dots,n} \{a_i + ix\}.$$

Here, the coefficients are tropical numbers $a_i \in \mathbf{T}$. Look at this carefully. The first expression is that of a usual polynomial, just using tropical operations. The second expression just reformulates what that means in ordinary terms. The terms $a_i + ix$ are ordinary affine linear functions, and taking the maximum leads to a *convex, piecewise linear function* $f : \mathbf{T} \to \mathbf{T}$. An example is depicted in Figure 2.

Let us look at examples with more variables. In the following, we will mostly

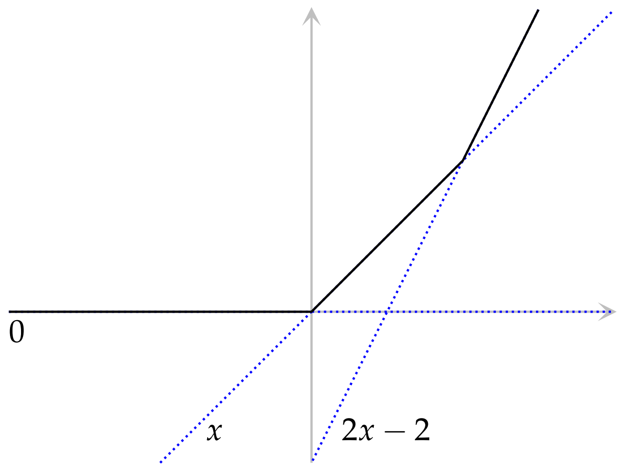


Figure 2 The graph of the tropical polynomial $f(x) = “0 + x + (-2)x^2”$. The three terms turn into three affine linear functions 0, x and $2x - 2$. Taking the maximum we obtain a convex, piecewise linear function.

work with two variables. Such polynomials look as follows.

$$f(x_1, x_2) = “\sum_{(i,j)\in S} a_{ij} x_1^i x_2^j” = \max_{(i,j)\in S} \{a_{ij} + ix + jx\}$$

Here, S is a finite subset of $\mathbf{N}^2$ (or, for Laurent polynomials, $S \subset \mathbf{Z}^2$). For the sake of simpler notation, we set $a_{ij} = -\infty$ if $(i, j) \notin S$. It is instructive (and useful when we add even more variables) to use multi index notation. For any $I \in \mathbf{Z}^2$, using the notation $x^I = x_1^{I_1} x_2^{I_2}$, we get

$$“x^I” = \langle I, x\rangle,$$

where the brackets on the right hand side denote the standard scalar product in $\mathbf{R}^2$. Again, we see that tropical arithmetics turns monomials into (affine) linear functions. In general, the multi index notation gives

$$f(x) = “\sum_{I\in S} a_I x^I” = \max_{I\in S}\{a_I + \langle I, x\rangle\}.$$

Two examples are given in Figure 3.

The tropical semifield $\mathbf{T}$ is algebraically closed in the sense that any tropical polynomial in one variable can be factorized into linear factors. Check out the details in the following exercise.

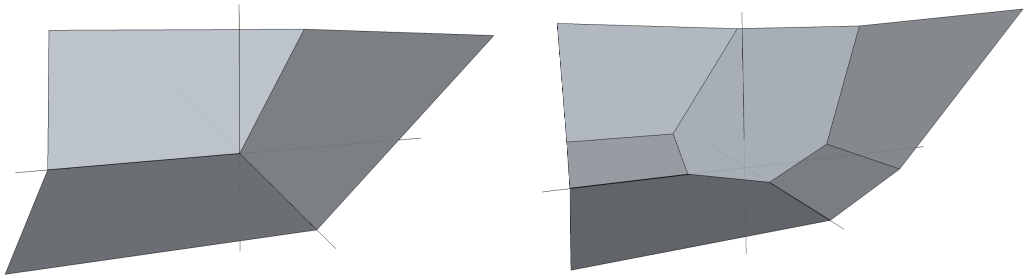


Figure 3 The graphs of two polynomials in two variables. The plot on the left shows the linear polynomial $f = \text{“}0 + x + y\text{”}$. On the right hand side, we see the graph of the quadratic polynomial $f = \text{“}0 + x + y + (-1)x^2 + 1xy + (-1)y^2\text{”}$.

Exercise 2 Let $f \in \mathbf{T}[x]$ be a univariate tropical polynomial of degree d and with non-vanishing constant term (i.e. $\mathrm{NP}(f) = [0, d] \subset \mathbf{R}$). Show that there exist $c \in \mathbf{R}$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ (unique up to reordering) such that

$$f(x) = \text{“}c \prod_{i=1}^{n} (x + \alpha_i)\text{”}$$

for all $x \in \mathbf{R}$ (Watch out: This is an equality of functions $\mathbf{R} \to \mathbf{R}$, not of polynomials).

1.3 Zero Sets

OK, we have polynomials, but in algebraic geometry we are interested in the zero sets of polynomials, like

$$V(F) = \{z \in \mathbf{K}^n : F(z) = 0\} \subset \mathbf{K}^n,$$

for some $F \in \mathbf{K}[z_1, \ldots, z_n]$. This is where the peculiarities of tropical geometry start. Recall from section 1.1 that the tropical counterpart of zero is $-\infty$. But look at our examples of tropical polynomial. None of these functions *ever* attain the value $-\infty$. This is not a peculiarity of the chosen polynomials but rather the normal behavior, due to the idempotency of tropical addition (note that $\text{“}x + y\text{”} = -\infty$ if and only if $x = y = -\infty$). We therefore need a different definition of zero sets for tropical polynomials. At this point, I will just present this alternative definition without much motivation. Some justification for the definition will appear gradually as we proceed.

Definition 1.2 Let $f \in \mathbf{T}[x_1, \dots, x_n]$ be a tropical polynomial. The *tropical hypersurface/tropical zero set* of f is defined to be

$$\begin{aligned} V(f) :=& \{x \in \mathbf{R}^n \mid f : \mathbf{R}^n \to \mathbf{R} \text{ is } \textit{not} \text{ differentiable at } x\} \\ =& \{x \in \mathbf{R}^n \mid \text{at least two terms in } f(x) \text{ attain the maximum}\} \\ =& \{x \in \mathbf{R}^n \mid \exists I \neq J \in \mathbf{Z}^n \text{ such that } f(x) = \text{“}a_I x^I\text{”} = \text{“}a_J x^J\text{”}\}. \end{aligned}$$

Please, convince yourself that the equality signs are justified.

Remark 1.3 In these notes, we will restrict our attention to tropical geometry in $\mathbf{R}^n$, instead of $\mathbf{T}^n$. In classical terms, this corresponds to varieties in $(\mathbf{K}^*)^n$ instead of $\mathbf{K}^n$ (so-called very affine varieties). This is also why we may allow Laurent polynomials, i.e. negative exponents. This restriction to finite coordinates is mostly for simplicity — some care is needed when extending the subsequent constructions to infinite coordinates (so-called points of higher sedentarity).

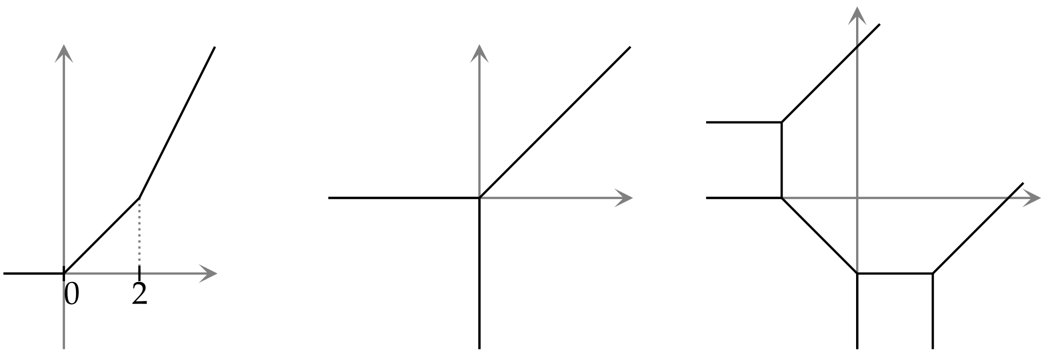


Figure 4 The tropical zero sets/hypersurfaces associated to the polynomials “$0 + x + (-2)x^2$”, “$0 + x + y$” and “$0 + x + y + (-1)x^2 + 1xy + (-1)y^2$”. In the univariate case, we obtain a finite collection of points. For two variables, we obtain a graph in $\mathbf{R}^2$.

Example 1.4 Let us have a look at the polynomials which we encountered in the previous figures. The univariate polynomial $f(x) = $ “$0 + x + (-2)x^2$” breaks at two points (with the constant and linear resp. linear and and quadratic term attaining the maximum). Hence, the tropical zeros of f are $V(f) = \{0, 2\}$. In the case of two variables, the break loci of our polynomials are one-dimensional graphs embedded in $\mathbf{R}^2$. In Figure 4, we depict the “zero sets” of the two polynomials from the previous figure.

Example 1.5 Even though in these notes we will mostly deal with curves, it might be instructive to see some examples of higher-dimensional tropical varieties. You can find a hyperplane and a surface of degree 2 in Figure 5 and Figure 6.

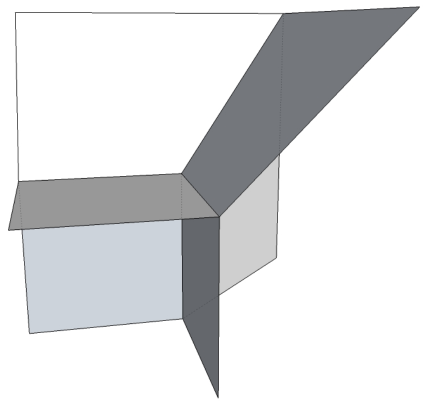


Figure 5 The tropical hyperplane $V(“0 + x_1 + x_2 + x_3”)$.

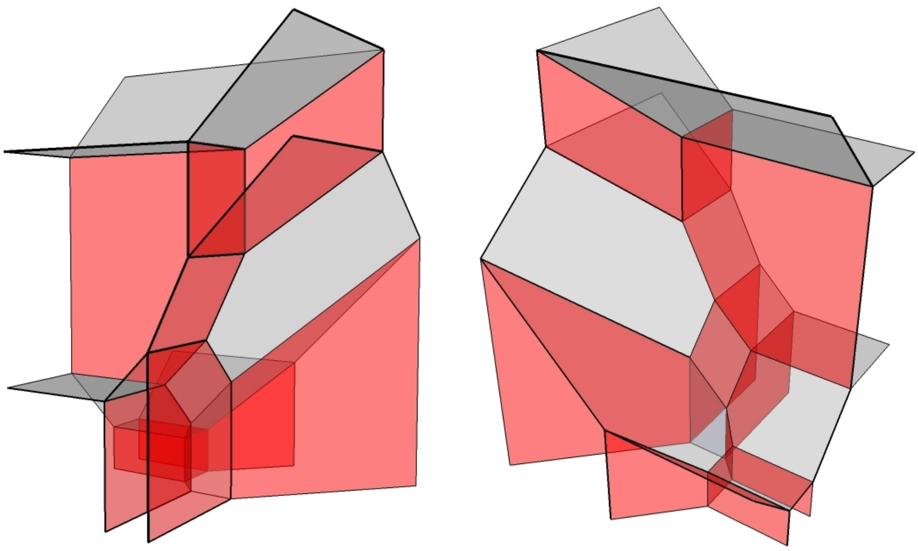


Figure 6 Two views of a tropical conic surface.

1.4 Dual subdivisions

In the previous examples we saw that tropical hypersurfaces are objects from polyhedral geometry. Their computation belongs to the realm of linear programming and is arguably much simpler than in the classical case. Still, for more complicated polynomials it can be a very tedious task to compute a tropical hypersurface. A useful tool in this context is the so-called *dual subdivision*. It is a subdivision of the Newton polytope of f which encodes at least the combinatorial structure of $V(f)$ (neglecting the lengths, areas, volumes of cells). We need a few definitions first. Given a tropical polynomial $f \in \mathbf{T}[x_1, \ldots, x_n]$, we define

- the *support* $\operatorname{supp}(f) := \{I \in \mathbf{Z}^n \mid a_I \neq -\infty\}$,
- the *Newton polytope* $\operatorname{NP}(f) := \operatorname{ConvHull}(\operatorname{supp}(f))$,
- the *lifted support* $\mathcal{L}\operatorname{supp}(f) := \{(I, -a_I) \mid I \in \operatorname{supp}(f)\} \subset \mathbf{Z}^n \times \mathbf{R}$,
- the *lifted Newton polytope* $\mathcal{L}\operatorname{NP}(f) := \operatorname{ConvHull}(\mathcal{L}\operatorname{supp}(f))$.

In the following, a *subdivision* $\mathcal{S}$ of a polytope P is defined to be a collection of polytopes such that

(a) $P = \bigcup_{Q \in \mathcal{S}} Q$,

(b) if $\mathcal{S}$ contains Q, then it also contains all faces of Q,

(c) if $Q, Q' \in \mathcal{S}$, then the intersection $Q \cap Q'$ is a common face.

Definition 1.6 The *dual subdivision* $\operatorname{SD}(f)$ is the subdivision of $\operatorname{NP}(f)$ obtained from projection of the *lower faces* of $\mathcal{L}\operatorname{NP}(f)$ to $\operatorname{NP}(f)$ (along $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$).

Instead of giving a careful definition of lower faces, let us look at an example.

Example 1.7 Figure 7 depicts the computation of the dual subdivision of

$$f = \text{“}(-1) + x + y + xy + (-1)x^2 + (-1)y^2\text{”}.$$

It turns out that $\operatorname{SD}(f)$ consists of 4 (minimal) triangles, 9 edges, and 6 vertices (every lattice point in $\operatorname{NP}(f)$ is used).

The important feature of $\operatorname{SD}(f)$ is that it is dual to $V(f)$ in a quite straightforward sense. To keep the exposition simpler, in the following statement we only consider the case $n = 2$ which means that $V(f)$ is just a graph, with some open ends, linearly embedded in $\mathbf{R}^2$.

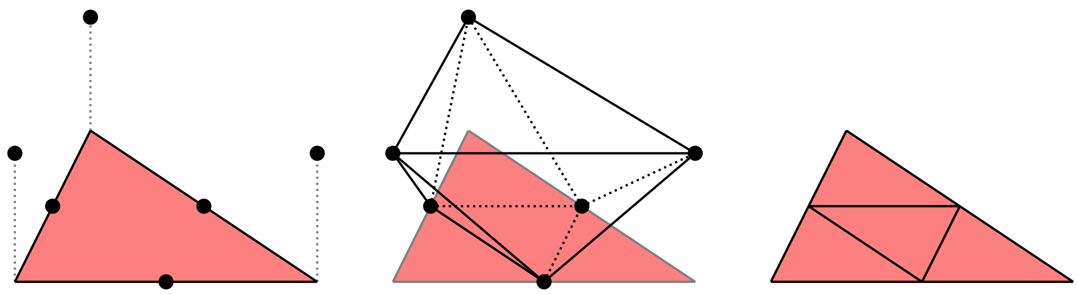


Figure 7 The computation of the dual subdivision of $f = ``(-1) + x + y + xy + (-1)x^2 + (-1)y^2"$ in three steps. To the left, the Newton polytope $\mathrm{NP}(f)$ and the lifted support $\mathcal{L}\mathrm{supp}(f)$. In the middle, the lifted Newton polytope $\mathcal{L}\mathrm{NP}(f)$. Projecting the lower faces back to $\mathrm{NP}(f)$, we obtain the dual subdivision $\mathrm{SD}(f)$.

Proposition 1.8 Let $f \in \mathbf{T}[x_1, x_2]$ be a non-trivial tropical polynomial. Then $V(f)$ and $\mathrm{SD}(f)$ are dual in the following sense. There exist bijections

$$\begin{aligned} \{\text{vertices of } V(f)\} &\longleftrightarrow \{\text{maximal cells of } \mathrm{SD}(f)\}, \\ \{\text{edges of } V(f)\} &\longleftrightarrow \{\text{edges of } \mathrm{SD}(f)\}, \\ \{\text{connected components of } \mathbf{R}^2 \setminus V(f)\} &\longleftrightarrow \{\text{vertices of } \mathrm{SD}(f)\}, \end{aligned}$$

such that

- all inclusion relations are inverted,
- dual edges are orthogonal to each other.

The punchline of this statement is that the dual subdivision encodes the *combinatorial structure* of $V(f)$ (i.e. the combinatorics of the graph and the directions/slopes of the edges in $\mathbf{R}^2$) while it forgets about metric information (the lengths of the edges and the global position of $V(f)$ in $\mathbf{R}^2$).

Example 1.9 In Figure 8, you can find some conics with associated subdivisions. The first two examples correspond to cases which appeared before. In Figure 9 we depicted two cubic curves. Note that in the curve on the right hand side, two edges intersect transversely in a 4-valent vertex, which corresponds to a parallelogram in the dual subdivision. Finally, in Figure 10 you can find the three-dimensional subdivision corresponding to the conic surface from Figure 6.

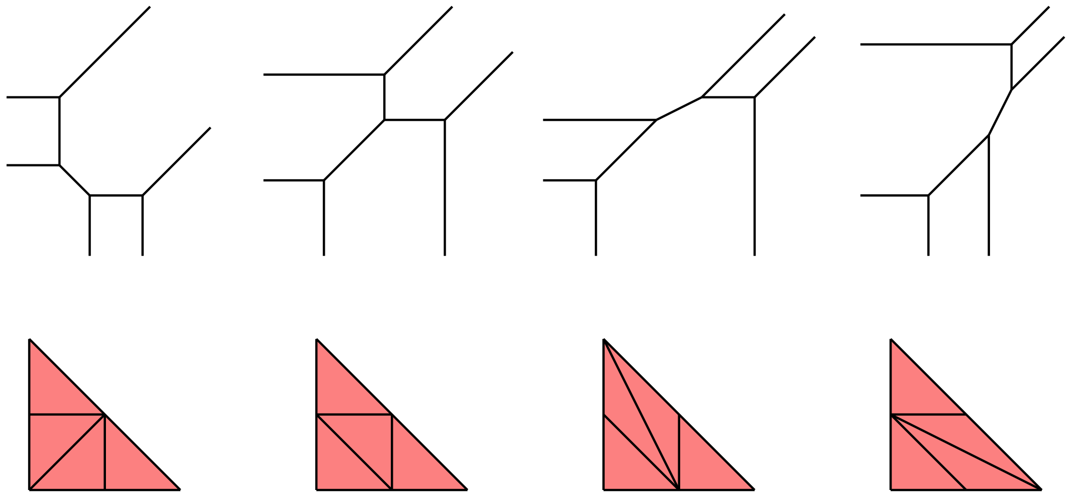


Figure 8 Some tropical conics with their dual subdivisions. The first two curves appeared in the examples before. Note that edges in the curve and in the subdivision are orthogonal to each other.

Exercise 3 Let $f_1, f_2, f_3 \in \mathbf{T}[x,y]$ be the tropical polynomials given by

$$f_1 = \text{“}(-1) + x + 1y + x^2 + xy + (-1)y^2\text{”},$$
$$f_2 = \text{“}(-2) + (-2)x^3 + (-2)y^3 + x + y + x^2 + y^2 + x^2y + xy^2 + 1xy\text{”},$$
$$f_3 = \text{“}0 + (-1)x + (-3)x^2 + y + 1xy + x^2y + (-2)y^2 + xy^2 + x^2y^2\text{”}.$$

In order to compute the associated tropical curves $V(f_i)$, proceed as follows.

(a) Compute the Newton polytopes $\mathrm{NP}(f_i)$ and the dual subdivisions $\mathrm{SD}(f_i)$.

(b) Compute (some of) the vertices of $V(f_i)$. (Each triangle in $\mathrm{SD}(f_i)$ singles out three terms of f_i. The corresponding vertex is the point where these three terms attain the maximum simultaneously.)

(c) Draw the curves $V(f_i) \subset \mathbf{R}^2$ by adding the edges.

Let $\Delta \subset \mathbf{R}^2$ be a (convex) polytope. The subdivisions $\mathcal{S}$ of Δ which are of the form $\mathcal{S} = \mathrm{SD}(f)$ for some tropical polynomial with $\mathrm{NP}(f) = \Delta$ are called *regular* or *convex subdivisions* (since they are induced by a convex function on Δ whose graph is the lower hull of $\mathcal{L}\mathrm{NP}(f)$).

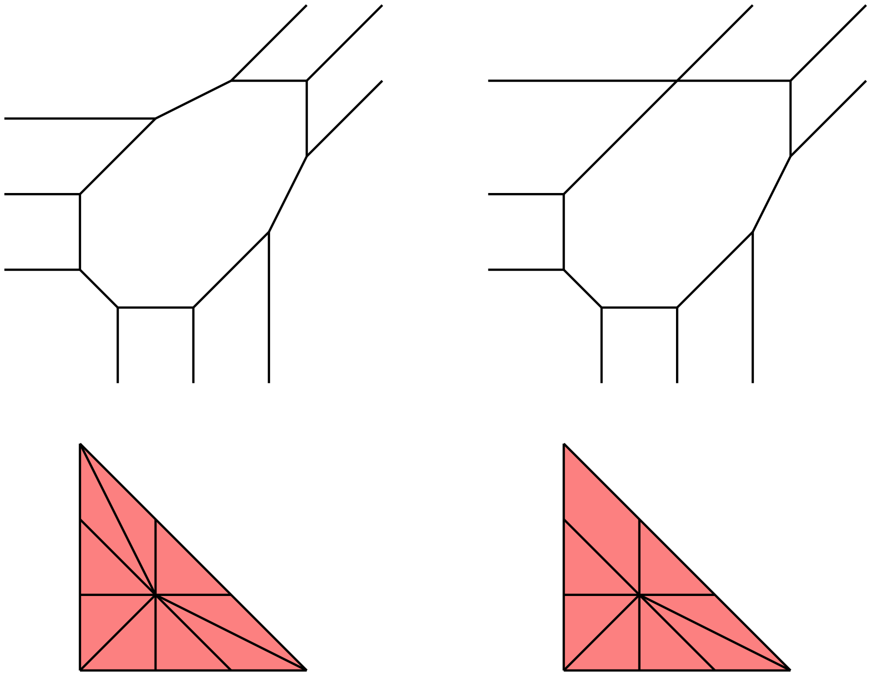


Figure 9 Two cubic curves with dual subdivisions. Both curves contain a cycle now. The right hand side curve also contains two edges intersecting each other transversely, corresponding to a parallelogram in the dual subdivision.

Exercise 4 Show that the subdivision in Figure 11 is *not* a regular subdivision (in particular, there does not exist a tropical curve dual to it).

1.5 Fundamental theorem

In the previous sections, we have made our first steps in the bizarre dreamland of tropical curves. Let us wake up for a second and search for connections to reality. What do these these peculiar graphs/polyhedral complexes have to do with the real world of classical algebraic geometry? It all starts with logarithm. We set

$$\begin{aligned} \mathrm{Log}: \quad & (\mathbf{C}^*)^n \to \mathbf{R}^n, \\ & (z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|). \end{aligned}$$

You might be puzzled, since this map does not at all belong to the world of algebraic geometry. It is therefore no surprise that images of algebraic varieties

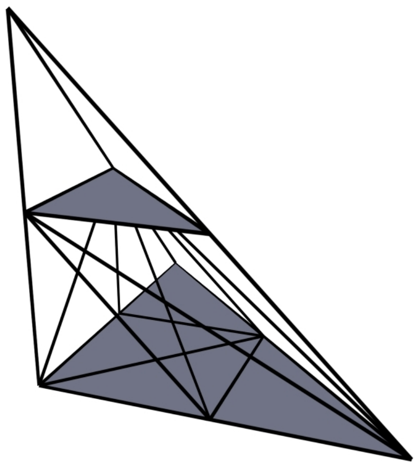


Figure 10 The dual subdivision of a three-dimensional Newton polytope corresponding to the conic surface depicted in Figure 6.

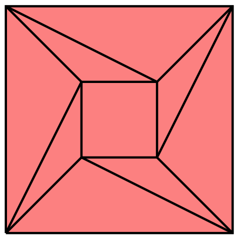


Figure 11 A non-regular subdivision

under Log look rather strange. Such images have been baptized *amoebas* — a name which is hopefully explained by the following examples. We restrict to hypersurfaces here and use the following notation. For any classical polynomial $F \in \mathbf{C}[z_1, \ldots, z_n]$, we set

$$\mathcal{A}(F) := \mathrm{Log}(V(F) \cap (\mathbf{C}^*)^n) \subset \mathbf{R}^n.$$

Amoebas have been studied before tropical geometry and are of independent interest. Here, however, instead of saying more about their properties, we content ourselves with some examples. In Figure 12 and Figure 13 you find the amoebas of a line (namely $\mathcal{A}(1 + z + w)$) and a cubic curve.

The pictures have a certain resemblance with tropical hypersurfaces, but we are certainly not there yet. The second step is a certain limit process. The logarithm is always taken with respect to a base (say, above we used logarithm

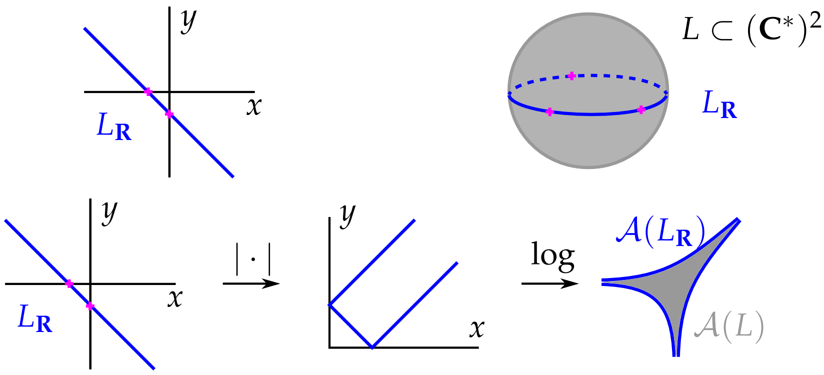


Figure 12 The amoeba of line (courtesy of Lionel Lang). In the top row, the real resp. complex solutions to the linear equation $1 + z + w$ are shown. The second row shows the consecutive effects of the absolute value resp. logarithm map on the set of real solutions. In fact, one can show that the image of $L_{\mathbf{R}}$ forms the boundary of $\mathcal{A}(1 + z + w)$, while each interior point is the image of a pair of complex conjugated points in $L \setminus L_{\mathbf{R}}$.

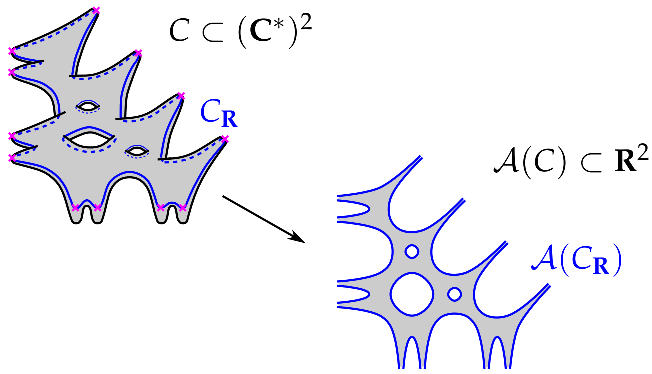


Figure 13 A more complicated amoeba (courtesy of Lionel Lang). This one comes from a quartic curve (and is still of very special type, in fact).

with base 10). We are now going to let this base go to infinity. Here are the necessary changes. We start with a *family* of polynomials depending on some parameter $t \in \mathbf{R}$. We can write this as

$$F_t = \sum \alpha_I(t) z^I,$$

where the coefficients are now functions of t. Let Log_t be the same coordinatewise logarithm as before, but now taken with respect to base t (we assume $t > 1$ throughout the following). By the standard rules of logarithm, this just means that we rescale the previous map by

$$\operatorname{Log}_t = \frac{1}{\log t} \operatorname{Log}.$$

Again, we set

$$\mathcal{A}_t(F_t) := \operatorname{Log}_t(V(F_t) \cap (\mathbf{C}^*)^n) \subset \mathbf{R}^n.$$

Now, letting t go to infinity, we get $\log t \to +\infty$, so everything gets shrunk towards the origin and the amoebas get thinner and thinner: These poor things really starve to death until only their bones are left! The *skeletons* that remain, surprising as it may be on first sight, turn out to be tropical hypersurfaces!!! (The violent analogy to biology is flawed since real world amoebas do not have skeleta! ;) Check out Figure 14 for an example.

Figure 14 The starving of a malnourished amoeba, down to its tropical skeleton.

Here is the precise statement (for hypersurfaces), which is sometimes referred to as the fundamental theorem of tropical geometry.

Theorem 1.10 — Fundamental theorem of tropical hypersurfaces. Let $F_t = \sum \alpha_I(t) z^I$ be a family of complex polynomials as before. We assume that

- for $t \gg 1$, the support $\operatorname{supp}(F_t)$ stabilizes (say, to $S \subset \mathbf{Z}^n$),
- for all $I \in S$, there exist $b_I \in \mathbf{C}^*$ and $a_I \in \mathbf{R}$ such that for $t \to \infty$

$$\alpha_I(t) \sim b_I t^{a_I}.$$

Then the limit of amoebas converges to

$$\lim_{t \to \infty} \mathcal{A}_t(F_t) = V(f) \subset \mathbf{R}^n,$$

where $V(f)$ is the tropical hypersurface associated to the polynomial $f = \sum a_I x^I$. (The coefficients of f are the orders of growth of $\alpha_I(t)$ from above).

In other words (and more general):

Tropical varieties appear as limits of amoebas of families of classical algebraic varieties.

Remark 1.11 In the above theorem, we are taking the limit of subsets in $\mathbf{R}^n$, the meaning of which is possibly ambiguous. One possibility for making this precise relies on the notion of Hausdorff metric (on compact subsets of $\mathbf{R}^n$). We are not going into the details here.

Remark 1.12 Why do we need families of hypersurfaces? Let us look at the special case when $F_t \equiv F$ is constant. As explained above, $\mathrm{Log}_t = \frac{1}{\log t} \mathrm{Log}$ is just a rescaling. So all that happens in this case is that we shrink the amoeba $\mathcal{A}(F)$ to the origin, with only asymptotic directions of $\mathcal{A}(F)$ surviving. In other words, the limit $\lim_{t\to\infty} \mathcal{A}_t(F)$ is a fan in $\mathbf{R}^n$ centered at the origin. These fans have been studied some time before the advent of tropical geometry under the names *Bergman fans* and *logarithmic limit sets*. Note that the coefficients of the tropical polynomial on the right hand side are all 0, so $\mathrm{SD}(f)$ is just the undivided Newton polytope and $V(f)$ is indeed a fan, in accordance with the statement. It is only by allowing families of hypersurfaces that we obtain more interesting limits like the tropical curves shown in the examples.

A glimpse of proof. For concreteness, let us restrict to the case $n = 2$. We only want to give the main idea for one of the inclusions, namely

$$\lim_{t\to\infty} \mathcal{A}_t(F_t) \subset V(f).$$

Let $(z_1(t), z_2(t))$ be a family of points in $(\mathbf{C}^*)^2$, depending on the parameter t. We are interested in the dominant terms of the functions $z_i(t)$ for $t \to \infty$. Assume the highest order of t in $z_i(t)$ is $x_i \in \mathbf{R}$, i.e. we can write

$$z_i(t) = c_i t^{x_i} + \text{ lower order terms.}$$

Note that this means $\lim_{t\to\infty} \mathrm{Log}_t(z_1(t), z_2(t)) = (x_1, x_2)$. Plugging $(z_1(t), z_2(t))$ into F_t, for each term of F_t we get

$$\alpha_{ij}(t) z_1(t)^i z_2(t)^j = (b_{ij} c_1^i c_2^j) \cdot t^{a_{ij} + i x_1 + j x_2} + \text{ lower order terms.}$$

The expression in the exponent is equal to the corresponding term in the tropical polynomials. Hence we conclude

$$F_t(z_1(t), z_2(t)) = \text{const} \cdot t^{f(x_1,x_2)} + \text{ lower order terms.}$$

Let us now assume that $(z_1(t), z_2(t)) \in V(F_t)$ for all t (in which case $(x_1, x_2) \in \lim_{t\to\infty} \mathcal{A}_t(F_t)$). Of course, this implies $F_t(z_1(t), z_2(t)) \equiv 0$ and hence the constant appearing in the above expression must be zero. Since for each single term, the coefficient $b_{ij}c_1^i c_2^j$ is non-zero, this can only hold true if at least two terms of $F_t(z_1(t), z_2(t))$ contribute to the maximal order $t^{f(x_1,x_2)}$. But this is equivalent to the maximum in $f(x_1, x_2)$ being attained at least twice. Hence $(x_1, x_2) \in V(f)$. ■

The proof shows that tropical arithmetics somehow capture the behavior of the leading exponents of power series. We can now understand better the motivation behind the definition of $V(f)$/tropical zeros: *A sum of functions in t can only be zero if the maximal leading order in t occurs in at least two of the summands.*

1.6 The Balancing Condition

In the previous sections we learned that planar tropical curves have the structure of graphs embedded piecewise linearly in $\mathbf{R}^2$. In the set of all such graphs, tropical curves are distinguished by certain properties which are somehow encoded in the fact that they admit dual subdivisions. The goal of this section is to make these properties more explicit. While the material in this section will not be used much in these notes, it plays an important role in the general theory of tropical varieties (which are not necessarily hypersurfaces).

A *polyhedron* in $\mathbf{R}^n$ is a (not necessarily bounded) intersection of finitely many affine halfspaces in $\mathbf{R}^n$. Similar to the notion of subdivision of a polytope, we define a *polyhedral complex* $\mathcal{P}$ in $\mathbf{R}^n$ to be a collection of polyhedra such that

(a) if $\mathcal{P}$ contains Q, then it also contains all faces of Q,

(b) if $Q, Q' \in \mathcal{P}$, then the intersection $Q \cap Q'$ is a common face.

We will often abuse notation and use the same letter for $\mathcal{P}$ and its support

$$|\mathcal{P}| = \bigcup_{Q\in\mathcal{P}} Q.$$

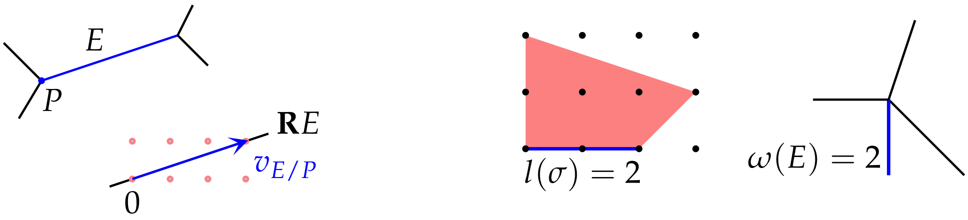


Figure 15 A tropical curve as a *rational* graph: All edges E have rational slope, i.e. the linear space $\mathbf{R}E$ is generated by an integer vector. On the left hand side, a primitive generator $v_{E/P}$ with respect to the endpoint P is shown. On the right hand side, the weight $\omega(E)$ of an edge is calculated as the integer length $l(\sigma)$ of the dual edge σ.

For any $f \in \mathbf{T}[x,y]$ the tropical curve $C = V(f)$ is a polyhedral complex of pure dimension 1 in $\mathbf{R}^2$. One immediate consequence of the existence of dual subdivisions is that the edges of C have rational slope. More precisely, let $\mathbf{R}E$ denote the *linear* space spanned by the edge E in C. Then there exists an integer vector $v_E \in \mathbf{Z}^2$ which spans $\mathbf{R}E$,

$$\mathbf{R}E = \langle v_E \rangle_{\mathbf{R}}, \quad v_E \in \mathbf{Z}^2.$$

See Figure 15. We say that C is a *rational* polyhedral complex. Moreover, the edges of C carry natural integer *weights* as follows. Let E be an edge of C and let $\sigma \in \mathrm{SD}(f)$ be the dual edge in the dual subdivision. We set the weight of E to be the *integer length* of σ,

$$\omega(E) := l(\sigma) := \#(\sigma \cap \mathbf{Z}^2) - 1.$$

A polyhedral complex with weights on the maximal cells is called a *weighted* polyhedral complex.

An integer vector $v \in \mathbf{Z}^2$ is called *primitive* if it is minimal among all integer vectors in this direction, i.e.

$$w = \lambda v, \lambda \in \mathbf{R}, w \in \mathbf{Z}^2 \implies \lambda \in \mathbf{Z}.$$

See Figure 15. Let E be an edge of C and let $P \in E$ be an endpoint of E (hence a vertex of C). The unique primitive vector of $\mathbf{R}E$ pointing from P in the direction of E is called *primitive generator* of E modulo P and denoted by

$$v_{E/P} \in \mathbf{Z}^2.$$

We are now ready to formulate the so-called balancing condition (see Figure 16).

Proposition 1.13 Let $C = V(f)$ be a tropical curve. Then for any vertex $P \in C$ the *balancing condition*

$$\sum_{E \ni P} \omega(E) v_{E/P} = 0$$

is satisfied (where the sum runs through all edges of C containing P).

We say that C is a *balanced* (weighted, rational) polyhedral complex.

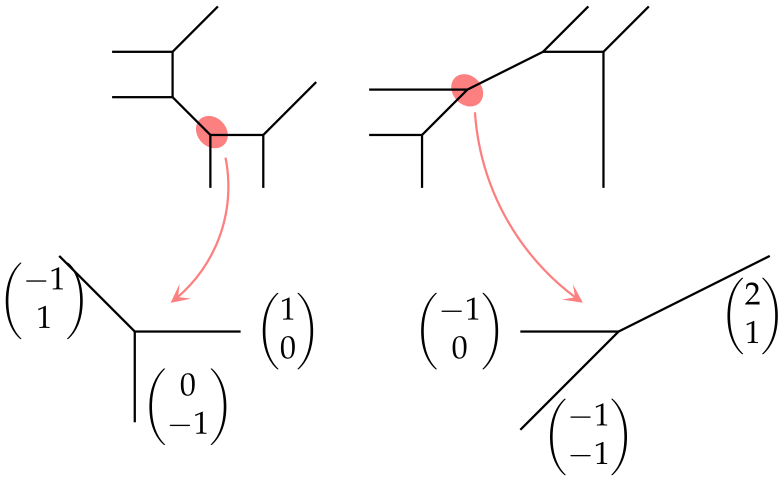


Figure 16 The balancing condition in two examples. Here, all weights are 1.

Proof. In fact, we just manufactured the definitions in such a way for the statement to hold: The vector $\omega(E)v_{E/P}$ rotated by 90 degrees is equal to the vector that connects the two endpoints of the dual edge σ of E in $\mathrm{SD}(f)$. Let $E_1, \ldots, E_n$ be a counterclockwise ordering of the edges around P. Then the path $\omega(E_1)v_{E_1/P}, \ldots, \omega(E_n)v_{E_n/P}$ retraces the boundary of the 2-cell τ in $\mathrm{SD}(f)$ dual to P (rotated by 90 degrees again). The balancing condition is therefore just a reformulation of the fact that the boundary of a polygon closes up. ∎

The importance of the balancing condition stems from the fact that it characterizes tropical hypersurfaces completely.

Proposition 1.14 Let $\Gamma \subset \mathbf{R}^2$ be a weighted rational polyhedral complex of dimension 1 such that the balancing condition holds at each vertex of Γ. Then there exists a tropical polynomial $f \in \mathbf{T}[x, y]$ such that

$$\Gamma = V(f).$$

Moreover, f is unique up to rescaling to "λf" $= \lambda + f$, $\lambda \in \mathbf{R}$.

Exercise 5 Can you prove this?

Let us briefly mention how to generalize the definitions and statements in higher dimensions. Let $f \in \mathbf{T}[x_1, \ldots, x_n]$ be a general polynomial and let $V = V(f)$ be the associated tropical hypersurface. Again, V is a *rational* polyhedral complex which this time means that for any cell F of V the linear space $\mathbf{R}F$ spanned by F admits a basis consisting of integer vectors only. Such a basis is called primitive if any integer vector in the span can be written as a linear combination with *integer* coefficients. The maximal cells of V are called *facets*. The dual cells of facets in V are still edges (one-dimensional), hence we can use the same definition to assign positive integer weights $\omega(F)$ to the facets of V. A cell of codimension 1 is called a *ridge*. Let $Q \subset F$ be a ridge contained in a facet. A *primitive generator* of F modulo Q is an integer vector pointing from Q in the direction of F which can be extended to a primitive basis of $\mathbf{R}F$ by adding a primitive basis of $\mathbf{R}Q$. The balancing condition is now a condition for any ridge and reads as

$$\sum_{F \supset Q} \omega(F) v_{F/Q} \in \mathbf{R}Q.$$

We have the following statement.

Proposition 1.15 Let $\Gamma \subset \mathbf{R}^n$ be a weighted rational polyhedral complex of dimension $n-1$. Then Γ satisfies the balancing condition at each ridge if and only if Γ is a tropical hypersurface $\Gamma = V(f)$. In this case, the tropical polynomial f is unique up to rescaling "λf" $= \lambda + f$, $\lambda \in \mathbf{R}$.

Exercise 6 Try to prove this as well.

In fact, we are slightly cheating here. All the statements here are in fact statements about the supports of the polyhedral complexes, or about polyhedral complexes up to refinements. For example, the equality $\Gamma = V(f)$ means that the supports of the two polyhedral complexes are the same and that the weights on overlapping facets agree.

By what we have said in this section, it is natural to use the balancing condition in order to characterize arbitrary tropical varieties.

Definition 1.16 A *tropical variety* of dimension m in $\mathbf{R}^n$ (a *a very affine tropical variety* is a balanced (weighted, rational) polyhedral complex $\Gamma \subset \mathbf{R}^n$ of pure dimension m.

1.7 Further exercises

Let $\Delta_d := \operatorname{ConvHull}\{(0,0),(d,0),(0,d)\}$ be the triangle of size d. Let $f \in \mathbf{T}[x,y]$ be a tropical polynomial and let $C = V(f)$ be the associated tropical curve.

Definition 1.17 We define the *(arithmetic) genus* of $C = V(f)$ to be the first Betti number $g(C) := b_1(C)$ (i.e. the number of independent cycles in C, as a graph).
Moreover, if $\operatorname{NP}(f) = \Delta_d$ we say that $C = V(f)$ has *(projective) degree* d.

A famous formula of classical algebraic geometry says that the genus of a smooth planar curve of degree d is equal to

$$g = \frac{(d-1)(d-2)}{2}.$$

Here is the tropical version.

Exercise 7 — Genus formula. Let $C = V(f)$ be a tropical curve in $\mathbf{R}^2$. Show that $g(C)$ is equal to the number of vertices of $\operatorname{SD}(f)$ that lie in the interior of $\operatorname{NP}(f)$. Show that if C is of degree d and each integer point in $\Delta_d \cap \mathbf{Z}^2$ occurs as a vertex in $\operatorname{SD}(f)$, then we obtain the classical genus formula from above.

Complex planar (projective) curves satisfy the so-called Bézout theorem: The number of intersection points (counted with multiplicities) of two given curves is equal to the product of their degrees. Here is the tropical analogue.

Let $C_1 = V(f_1)$ and $C_2 = V(f_2)$ be two tropical curves of degree d_1 and d_2. Assume that C_1, C_2 do not intersect in vertices. For any intersection point $p \in C_1 \cap C_2$, let $e_i \subset \operatorname{SD}(f_i)$ denote the edge dual to the edge of C_i containing p. We define the *intersection multiplicity* of p by

$$\operatorname{mult}_p(C_1, C_2) = |\det(v_1, v_2)|,$$

where v_i is a vector connecting start and end point of e_i (orientation is not important, since we take the absolute value).

Exercise 8 — Bézout's theorem. Show that the number of intersection points of C_1 and C_2, counted with this multiplicity, is equal to

$$\sum_{p \in C_1 \cap C_2} \operatorname{mult}_p(C_1, C_2) = d_1 d_2.$$

2 Enumerative Geometry

2.1 Counting things — why not?

When the ancient Greeks did geometry, they often did it in terms of construction problems. Here is a very simple example. Given a line segment $\overline{pq}$ in the plane, how can we construct a equilateral triangle such that $\overline{pq}$ is one of its sides? Answer: Draw two circles with midpoints p and q respectively, both of radius $\overline{pq}$. The intersection point of these circles gives the third vertex of the triangle (see Figure 17). We might generalize this type of problems in the following way. Fix

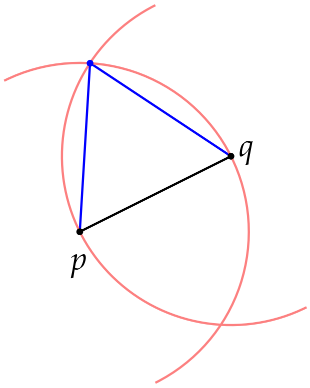
- a class of geometric objects (here, triangles),
- some geometric conditions, (having $\overline{pq}$ as side, being equilateral).

The question is: Can you find a construction algorithm which produces a specific element in the class of objects which satisfies all the conditions?

But wait, here is something funny: In our above example, we did not construct one but, in fact, two triangles satisfying the conditions. Obviously, the two circles intersect in two points and we can use both of them as third vertex for our triangle. The ancient mathematicians sometimes noticed that a construction problem could be solved in several ways, but it was usually not worth more than a side remark to them. Here is a more interesting example for this multiplicity of solutions, known as *Apollonius's problem* or *Circles of Apollonius*. Among other things, Apollonius of Perga (~ 200 BC) posed the following problem. Fix three circles in the plane (say non-intersecting and not contained one in another). Construct a circle which is tangent to each of the three given circles! Beyond finding explicit construction methods for this problem, Apollonius (and later researchers) noticed something very interesting: There are exactly 8 circles solving the problem, i.e. being tangent to the three given circles (see Figure 17).

Exercise 9 Can you find an (heuristic) argument why there are exactly 8 circles tangent to three given circles?

Note (again) that when moving the three initial circles around, the solution circles obviously change. However, the *number* of solutions, here 8, does *not* change. It is really an invariant number associated to the abstract geometric problem, not to the specific instance of the problem once we fix the three initial circles. Hence we may shift the focus from



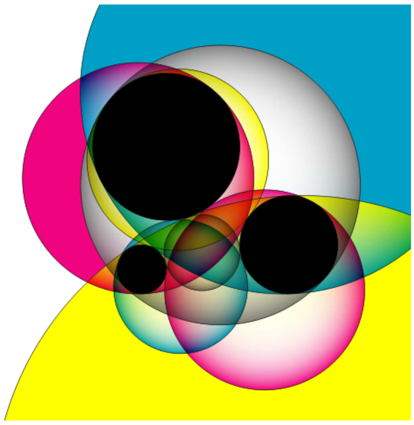


Figure 17 Some geometric construction problems. To the left, the construction of an equilateral triangle. Note that the second intersection point of the circles gives a second solution. To the right, a beautiful demonstration of the famous Circles of Apollonius (by Melchoir – Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=4056136). Given the three initial circles (in black), there exist 8 circles (in shaded colors) which are tangent to each of the black circles. Can you see why?

- How to construct a (single) solution of the geometric problem?

to

- How many solutions does the geometric problem have?
- Does this number depend on the specific geometric conditions, or is it an invariant of the (abstract) problem?

This shift of attention from "constructing" to "counting" forms the basis of what is called *enumerative geometry*.

2.2 Counting algebraic curves in the plane

We now make an big step forward from this still rather simple enumerative problem to a whole family of problems which are of very archetypal form in modern enumerative geometry. Instead of circles, we now want to count algebraic curves of given genus g and degree d in the plane. Instead of tangency conditions, we will now pose point conditions, meaning that the curves are supposed to pass through a given collection of points $p_1, \dots, p_N$. You might not have heard of these notions before, so let me give you a quick overview.

We will mostly work over the complex numbers, hence "plane" refers to the complex 2-dimensional (real 4-dimensional) space $\mathbf{C}^2$ (advanced readers can use projective plane $\mathbf{CP}^2$ instead). We already have a good understanding of what an algebraic curve is from Section 1: Any polynomial $F \in \mathbf{C}[z, w]$ defines an *algebraic curve* as its zero set in $\mathbf{C}^2$

$$C = V(F) = \{(z, w) \in \mathbf{C}^2 : F(z, w) = 0\}.$$

In the following, we will always assume that F is an irreducible polynomial, in which case we also call C *irreducible*. The *degree* of C is just the degree of F, i.e. the maximal number of factors (z and w) appearing in a term of F (again, more advanced readers may think of homogeneous polynomials of some degree in three variables, instead).

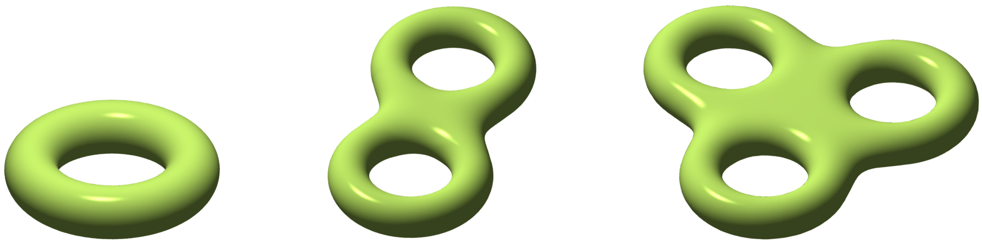


Figure 18 An algebraic curve over $\mathbf{C}$, after removing a finite number of points, can be compactified to a unique compact Riemann surface. The figure depicts compact surfaces of genus 1, 2, and 3 (by Oleg Alexandrov – Own work, MATLAB, Public Domain, https://en.wikipedia.org/wiki/User:Oleg_Alexandrov/Pictures).

For the genus, recall that C is a real 2-dimensional object. In fact, away from finitely many *singular* points, C carries the structure of a Riemann surface (note the irony in calling the same object (algebraic) curve and (Riemann) surface). There exists a unique completion of this Riemann surface to a *compact* Riemann surface (meaning without punctures or open ends). The *(geometric) genus* of C is defined to be the genus of this Riemann surface. This, in turn, is the number of holes of the surface, or in other words, the number of handles one has to attach to a sphere in order to construct the surface. The examples we met in section 1 are elliptic curves (i.e. degree 3, genus 1, see Figure 1) and lines (i.e. degree 1, genus 0, see Figure 12). Higher genus surfaces can be found in Figure 18.

We are now ready to introduce the (family of) enumerative problem which will be our main focus in this section.

Enumerative Problem Fix integers $d > 0$ and $g \geq 0$. Set $N = 3d - 1 + g$. Fix N generic points $p_1, \ldots, p_N$ in $\mathbf{C}^2$. How many irreducible algebraic curves of degree d and genus g pass through all the points $p_1, \ldots, p_N$?

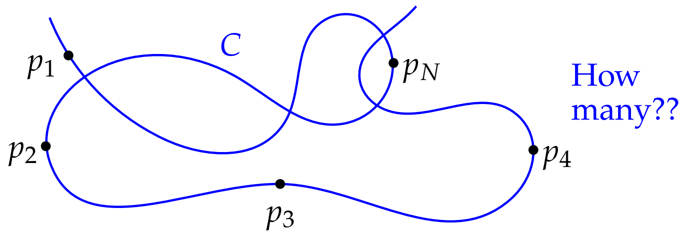


Figure 19 Fix integers $d > 0$ and $g \geq 0$. Set $N = 3d - 1 + g$. Fix N generic points $p_1, \ldots, p_N$. How many irreducible algebraic curves of degree d and genus g pass through all the points $p_1, \ldots, p_N$?

Let us collect a few facts regarding this problem.

- The number N is chosen such that we expect a finite number of solutions to this counting problem (namely, N is equal to the number of parameters for an algebraic curve of degree d and genus g and each point condition imposes one equation on these parameters).
- The term "generic" will not be explained carefully here. You should think of the points as chosen sufficiently randomly, such that no "special" point configurations (like three points on a line) occur. This can be made precise, in such a way that the set of generic point configurations forms an open dense subset in the set of all point configurations. In other words, nearly all point configurations are generic.
- Indeed, one can show that for $g \leq \frac{(d-1)(d-2)}{2}$ and a generic point configuration $p_1, \ldots, p_N$, the number of curves solving the enumerative problem is finite and non-zero. Moreover, this number is independent of the position of the points $p_1, \ldots, p_N$ (as long as they are generic). This is completely analogous to the number 8 in Apollonius's problem being independent of the chosen three circles.
- We denote this number, which only depends on d and g, by

$$N_{d,g}.$$

- The number $\frac{(d-1)(d-2)}{2}$ appearing above is the maximal genus that a curve of degree d can have. Moreover, for any "generic" (again) polynomial F, the compactification of the curve $\overline{C} = \overline{V(F)}$ in $\mathbf{CP}^2$ is a *smooth* curve, in which case its genus is in fact equal to

$$g(d) = \frac{(d-1)(d-2)}{2}.$$

It follows that whenever we set $g < g(d)$ in our enumerative problem, we are actually counting curves with *singularities*. However, one can show that for generic point configurations only the simplest type of singularities, so-called *simple nodes*, appear. They locally look like the transverse intersection of two branches of C (in an analytic neighborhood, they look like the intersection of two coordinate lines $zw = 0$). The occurrence of a node reduces the genus of the curve by one, i.e.

$$g(C) = g(d) - \#\text{nodes}(C).$$

Therefore, in an equivalent reformulation of the enumerative problem we could drop the genus g and instead ask for *nodal* curves (i.e. with at most simple nodes for singularities) of fixed degree and fixed number of nodes.

- At the other end of the genus range, for $g = 0$, we are counting so-called *rational* curves. Their special feature, as the name indicates, is that they can be parametrized by rational functions. More precisely, a rational curve is (the closure of) the image of a map of the form

$$\begin{aligned} \varphi : \mathbf{C} &\dashrightarrow \mathbf{C}^2, \\ u &\mapsto \left(\frac{f(u)}{h(u)}, \frac{g(u)}{h(u)} \right), \end{aligned}$$

where $f, g, h \in \mathbf{C}[u]$ are univariate polynomials of degree (at most) d (the arrow is dashed since φ is not well-defined at the finitely many zeros of the denominator).

Example 2.1 Let us get some practice with these numbers $N_{d,g}$. First, a word of warning: We just emphasized that we work over the complex number here (and it is actually important that our base field is algebraically closed). However, all the following figures depict curves over the real numbers, of course, should be merely regarded as schematic presentations.

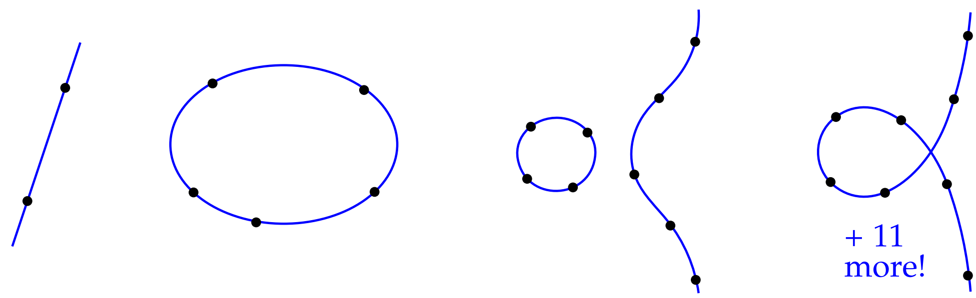


Figure 20 There exist exactly one line through two points and one ellipse through 5 (convexly arranged) points. Schematic representations (over $\mathbf{R}$) of the enumerative problems for $N_{1,0} = N_{2,0} = N_{3,1} = 1$ and $N_{3,0} = 12$.

(a) Through any pair of distinct points p_1, p_2 in the plane, you can draw (exactly) one line. In our notation, we computed

$$N_{1,0} = 1.$$

(b) The next one is already a little tougher. Fix five points in the plane. How many conic curves pass through these five points? In fact, if for example the five points form the vertices of a convex 5-gon, then there exists exactly one ellipse passing through these points. In other words,

$$N_{2,0} = 1.$$

(c) How about cubic curves? In the generic case, such curves have genus $g(3) = 1$ and are called elliptic curves. Indeed, one can show again that there is exactly one elliptic curve passing through 9 fixed generic points in the plane. This is getting boring, right? In fact, in the maximal genus case there is always exactly one solution curve (check out Exercise 19), i.e.

$$N_{d,g(d)} = 1.$$

(d) The first non-trivial case shows up for $d = 3, g = 0$. This means we are searching for rational curves (with a single simple node) passing through 8 fixed points. There are twelve such curves, so

$$N_{3,0} = 12.$$

For a nice non-tropical computation of this number, check out Computation 2.21.

(e) Generalizing the previous case, we can look at the next-to-maximal genus case. One can show that

$$N_{d,g(d)-1} = 3(d-1)^2.$$

This number is just the degree of the discriminant hypersurface of singular curves inside the complete linear system $|\mathcal{O}(d)|$ of all curves of degree d. It can be computed for example via the *incidence variety* (see Exercise 20)

$$\mathcal{I} = \{(C, p) : p \text{ singular point of } C\} \subset |\mathcal{O}(d)| \times \mathbf{CP}^2.$$

(f) In fact, these are pretty much the only examples of d, g for which $N_{d,g}$ can be computed more or less by hand. Only very few other numbers, such as $N_{4,0} = 620$, were known to mathematicians until 30 years ago. A few more numbers are displayed in Figure 21.

$g \setminus d$	1	2	3	4	5	6	7
0	1	1	12	620	87304	26312976	14616808192
1			1	225	87192	57435240	60478511040
2				27	36855	58444767	122824720116
3				1	7915	34435125	153796445095
4					882	12587820	128618514477

Figure 21 The values of $N_{d,g}$ for some choices of d, g.

Historical remarks

Let me give a brief sketch of the history of the numbers $N_{d,g}$. I mentioned above that only very few of the numbers $N_{d,g}$ could be computed before 1990. In the late 19th century, which is sometimes referred to as the (first) golden era of enumerative geometry, Zeuthen was able to compute $N_{4,0} = 620$ (in 1873). Kock and Vainsecher report that until 1990, the only additional number unveiled for $g = 0$ was $N_{5,0} = 87304$ (in the 1980's). During that period, enumerative geometry was a field for specialists, mostly algebraic geometers. It was used as a nice playground for testing important advances in intersection theory and moduli theory, but the main focus was often somewhere else. This changed drastically when a small miracle happened. In the 1980's, new ideas from theoretical physics (string theory) began to show deep impact on mathematics. In string theory, classical point particles are replaced by small strings (homeomorphic to the circle S^1), and when these strings evolve in time, they trace out a surface called *world sheet*. These world sheets are actually closely related to (can be approximated by) our algebraic curves from above. Using what is now called *Mirror symmetry*, physicists were able to "predict" some enumerative numbers very similar to the $N_{d,g}$ series from above — not just a few numbers, but whole series of numbers! In the years that followed, mathematicians tried hard to justify the physical predictions by mathematical arguments. This lead to some great advances in mathematics (e.g. Gromov-Witten theory as a vast generalization of this kind of enumerative geometry), and in particular the numbers $N_{d,g}$ could finally be computed. In 1994, Kontsevich found his famous recursive formula for the case $g = 0$,

$$N_{d,0} = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1,0}N_{d_2,0} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

In 1998, Caporaso and Harris found a (more complicated) recursive formula for all numbers $N_{d,g}$ (in fact, the formula involves an even larger set of numbers called relative Gromov-Witten invariants).

2.3 The tropical enumerative problem

One of the outstanding early successes of tropical geometry is an alternative computation of the numbers $N_{d,g}$. In fact, what we are going to do is carefully translate our enumerative problem to the tropical world. It then turns out that the classical and tropical problem have the same answer, i.e. the classical and tropical counts produce the same numbers! This is not only a surprising statement, but also simplifies the computation of $N_{d,g}$ drastically, reducing it to a more or less combinatorial problem (which is, however, still quite complicated — I am taking a geometer's standpoint here, i.e. I regard the problem as solved once it is reduced to "pure combinatorics" ;).

The tropical set-up.

In order to reformulate our enumerative problem in the tropical world, we have to translate the notions irreducible algebraic curve, degree and genus. Most of this was done in section 1, but it might be more convenient to collect all the ingredients here again.

- A *(planar) tropical curve* is a set of the form

$$\begin{aligned}\Gamma &= V(f)\\ &= \{x \in \mathbf{R}^2 \mid \text{at least two terms in } f(x) \text{ attain the maximum}\},\end{aligned}$$

  where $f \in \mathbf{T}[x,y]$ is a tropical polynomial. By Proposition 1.14, this equivalent to $\Gamma \subset \mathbf{R}^2$ being a purely 1-dimensional rational polyhedral complex which carries positive weights on the edges and satisfies the balancing condition at each vertex.

- A tropical curve Γ is called *irreducible* if it cannot be written as

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

  for two non-empty tropical curves Γ_1, Γ_2. The union here is understood to take care of the weights as well, i.e. if two edges of Γ_1 and Γ_2 overlap, then the weights on this segment should be added.

- A tropical curve $\Gamma = V(f)$ is said to be of *degree d* if its Newton polytope is $\mathrm{NP}(f) = \Delta_d$, the standard triangle of size d with corners $(0,0)$, $(d,0)$ and $(0,d)$. Equivalently, we can ask Γ to have d ends (counted with weights) in each of the directions $(-1,0)$, $(0,-1)$ and $(1,1)$, and no ends pointing in other directions.

- The genus of a tropical curve appeared in Figure 9 and Exercise 7. The main idea is to consider the first Betti number of Γ,

$$b_1(\Gamma) = \dim H_1(\Gamma, \mathbf{R}),$$

  i.e. the number of independent loops in Γ. However, in analogy to the classical case, we will also have to deal with *singular* tropical curves, and in this case the definition has to be adapted. Namely, we set the *(geometric) genus* of Γ to be the minimal first Betti number of an *abstract* graph (with ends) which parametrizes Γ, i.e.

$$g(\Gamma) := \min\{b_1(\Gamma') : \Gamma' \text{ abstract graph and } \varphi : \Gamma' \to \Gamma \text{ parametrization}\}.$$

  Instead of giving a precise definition of what a tropical "parametrization" is, let us consider the example given in Figure 9. The cubic curve to the left is actually a *smooth* tropical curve and its genus is 1, since it cannot be parametrized by a graph of lower genus. The curve on the right, however, has a node (the 4-valent intersection point of two edges). Abstractly, you can hence unfold this intersection point and parametrize the curve by a tree (a graph with $b_1(\Gamma') = 0$, see Figure 22). Hence the genus of this curve is 0.

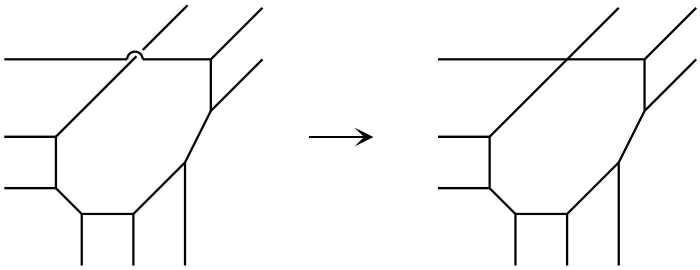


Figure 22 The cubic curve on the right hand side has a node, which can be resolved in a parametrization. The curve is hence of genus 0, since it can be parametrized by the tree graph on the left hand side.

We can now formulate the tropical enumerative problem in complete analogy to the classical case.

Enumerative Problem — tropical version. Fix integers $d > 0$ and $g \geq 0$. Set $N = 3d - 1 + g$. Fix N generic points $p_1, \ldots, p_N$ in $\mathbf{R}^2$. How many irreducible tropical curves of degree d and genus g pass through all the points $p_1, \ldots, p_N$?

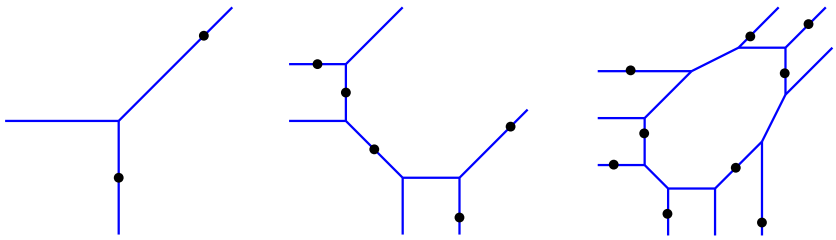


Figure 23 The *unique* tropical line, conic, resp. cubic through two, five, resp. nine given generic points.

Example 2.2 In Figure 23 you can find solutions to the tropical enumerative problem for lines, conics, and smooth cubics. The case of lines can be easily done by hand. For conics and cubics, it is easy to check that the depicted curves are the only ones, among the curves of the *same combinatorial type* (i.e. with the same dual subdivision), which pass through the given points. To show that no curves of *different* combinatorial type occur (for the specific choice of points), is a more tedious task.

Exercise 10 Show that for any two points $p_1, p_2 \in \mathbf{R}^2$, there is a tropical line passing through both of them. Which cases do you have to distinguish? Show that this line is unique if

$$p_2 - p_1 \notin \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle_{\mathbf{R}} \cup \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbf{R}} \cup \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbf{R}}.$$

Multiplicities.

There is a little twist to the tropical version of the story: Tropical curves have to be counted with some multiplicity. The meaning of this multiplicity will become clear in a short while. Let us define it first.

Definition 2.3 Let $\Gamma \subset \mathbf{R}^2$ be a tropical curve and let $P \in \Gamma$ be a 3-valent vertex of Γ. We define the *multiplicity* of P

$$\mathrm{mult}_\Gamma(P) := \omega(E_1)\omega(E_2)|\det(v_{E_1/P}, v_{E_2/P})| = 2\mathrm{Area}(\sigma(P)).$$

Here, E_1, E_2 are two of the three edges adjacent to P, and $\omega(E_i)$ and $v_{E_1/P}$ denote the corresponding weights and primitive generators. In the second expression, $\sigma(P)$ denotes the 2-cell in $\mathrm{SD}(f)$ dual to P, and Area is the standard Euclidean area measure in $\mathbf{R}^2$.

Exercise 11 Show that the first expression for $\mathrm{mult}_C(p)$ does not depend on which two edges you choose, and is equal to the second expression.

Definition 2.4 Let $\Gamma \subset \mathbf{R}^2$ be a tropical curve. We define the *multiplicity* of C

$$\mathrm{mult}(\Gamma) := \prod_{\substack{P \in \Gamma \\ P \text{ 3-valent}}} \mathrm{mult}_\Gamma(P).$$

Note that the product runs through 3-valent vertices of Γ only.

Example 2.5 Note that in the examples of Figure 23, all vertex multiplicities are 1. It follows that the multiplicity of each of the three curves is 1, as well.

Back to our enumerative problem. Fix integers $d > 0$ and $0 \leq g \leq \frac{(d-1)(d-2)}{2}$ and a generic point configuration $\mathcal{P} = (p_1, \ldots, p_N)$ in $\mathbf{R}^2$. Here are a few facts.

- The set $\Sigma^{\text{trop}}(\mathcal{P})$ of tropical curves of degree d and genus g passing through $p_1, \ldots, p_N$ is finite and non-empty.
- All the curves in $\Sigma^{\text{trop}}(\mathcal{P})$ are *nodal*, which means that all vertices are either 3-valent or "nodes". Equivalently, the dual subdivision of such a curve consists of triangles and parallelograms only. The previous definition of multiplicity is well-behaved only for this type of curves.
- We denote the count of curves in $\Sigma^{\text{trop}}(\mathcal{P})$, weighted with their respective multiplicities, by
$$N^{\text{trop}}_{d,g} := \sum_{\Gamma \in \Sigma^{\text{trop}}(\mathcal{P})} \text{mult}(\Gamma).$$
It turns out that this number is, again, independent of the point configuration $\mathcal{P}$ (as long as it is generic). This can be proven "purely tropically", but it also follows from the Correspondence theorem which we are going to state next.

2.4 The Correspondence theorem

We are now ready to present the main theorems of this section.

Theorem 2.6 For any admissible choice of d, g, we have
$$N_{d,g} = N^{\text{trop}}_{d,g}.$$

In fact, the statement follows from a considerably stronger statement due to Mikhalkin (around 2002).

Theorem 2.7 — Correspondence theorem. We fix the following data.

- Fix integers $d > 0$ and $0 \leq g \leq \frac{(d-1)(d-2)}{2}$. Fix a generic point configuration $\mathcal{P} = (p_1, \ldots, p_N)$ in $\mathbf{R}^2$.
- Fix a continuous family of classical generic point configuration $\mathcal{Q}(t) =$

$(q_1(t), \ldots, q_N(t))$ in $(\mathbf{C}^*)^2$ for all $t > 1$ such that

$$\lim_{t \to \infty} \operatorname{Log}_t(\mathcal{Q}(t)) = \mathcal{P},$$

meaning that $\lim_{t\to\infty} \operatorname{Log}_t(q_i(t)) = p_i$ for all $0 \leq i \leq N$.

- Denote by $\Sigma(\mathcal{Q}(t))$ the set of classical algebraic curves of degree d and genus g passing through $\mathcal{Q}(t)$. Analogously, denote by $\Sigma^{\text{trop}}(\mathcal{P})$ the set of tropical curves of degree d and genus g passing through $\mathcal{P}$.

Then the following holds true.

- The amoebas of the classical curves in $\Sigma(\mathcal{Q}(t))$ converge to tropical curves in $\Sigma^{\text{trop}}(\mathcal{P})$ for $t \to \infty$. Slightly abusing notation, we write this as

$$\lim_{t \to \infty} \operatorname{Log}_t(\Sigma(\mathcal{Q}(t))) \subset \Sigma^{\text{trop}}(\mathcal{P}).$$

- In fact, the two sets in the previous equation are equal, and the number of classical curves in $\Sigma(\mathcal{Q}(t))$ whose amoebas converge to a given tropical $\Gamma \in \Sigma^{\text{trop}}(\mathcal{P})$ is equal to $\operatorname{mult}(\Gamma)$,

$$\operatorname{mult}(\Gamma) = \#\{\text{curves in } \Sigma(\mathcal{Q}(t)) \text{ with amoebas converging to } \Gamma\}.$$

We can think of the theorem in terms of a map

$$\operatorname{Trop} : \Sigma(\mathcal{Q}(t)) \to \Sigma^{\text{trop}}(\mathcal{P}),$$

say, for some large t, assigning to each classical curve the limit of its amoeba. The statement then says that this map is surjective and the size of the fibers is

$$\#\operatorname{Trop}^{-1}(\Gamma) = \operatorname{mult}(\Gamma).$$

In some sense, the strength of the theorem relies on the fact that this size of the fibers can be computed so easily and explicitly in terms of the combinatorics of the tropical curves (note, for example, that is independent of how the points in $\mathcal{P}$ are sitting inside of Γ.

Example 2.8 In Figure 24 you can find the tropical computation of $N_{3,0}^{\text{trop}} = 12$. For the particular choice of point configuration $\mathcal{P}$, there are nine tropical curves $\Gamma_1, \ldots, \Gamma_9$ of degree 3 and genus 0 passing through $\mathcal{P}$. The first (top left) curve Γ_1 is special, since it contains an edge of weight 2. Thus the two adjacent

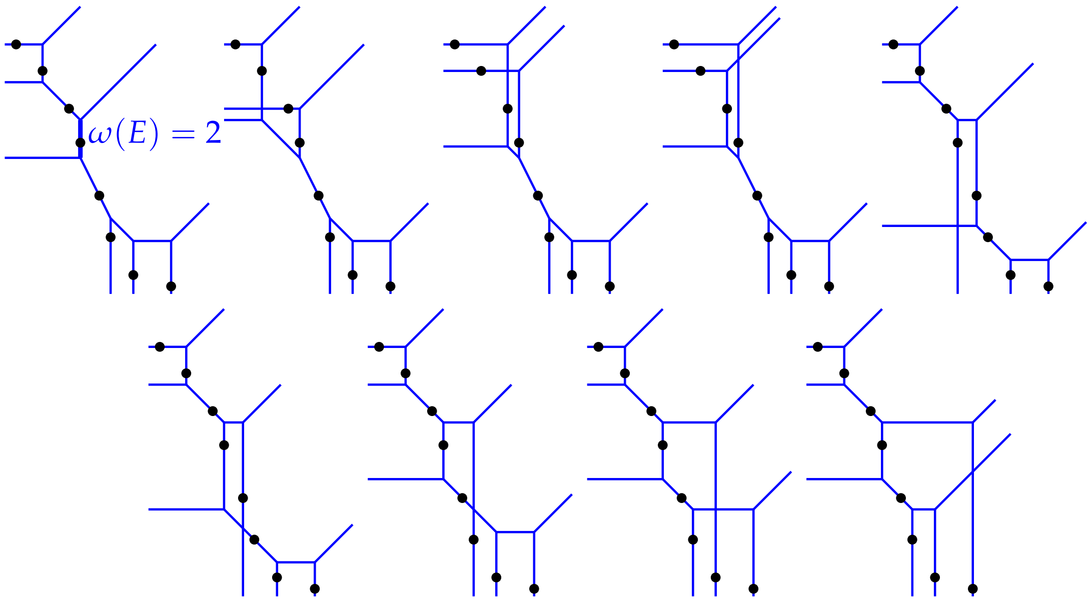


Figure 24 The tropical computation of $N_{3,0} = 12$ as $N_{3,0}^{\text{trop}} = 4 + 1 + \ldots + 1 = 12$. The first curve has multiplicity 4, since the two vertices adjacent to the edge of weight 2 contribute with multiplicity 2 each. All other curves have multiplicity 1.

vertices P, Q have multiplicity $\text{mult}_{\Gamma_1}(P) = \text{mult}_{\Gamma_1}(Q) = 2$. All other vertex multiplicities are 1. Hence we compute

$$\begin{aligned} \text{mult}(\Gamma_1) &= 4, \\ \text{mult}(\Gamma_i) &= 1 \quad \text{for all } i \in \{2, \ldots, 9\}, \\ N_{3,0}^{\text{trop}} &= 4 + 8 \cdot 1 = 12. \end{aligned}$$

Note also that the first curve Γ_1 is a tree per se, while all other curves contain a node which can be "resolved" in order to obtain a genus 0 parametrization.

Summary.
The computation of the numbers $N_{d,g}$ can be reduced to the corresponding tropical enumerative problem and the computation of the numbers $N_{d,g}^{\text{trop}}$. This means we can, instead of counting algebraic curves over $\mathbf{C}$, count certain piecewise linear graphs in $\mathbf{R}^2$. This can still be quite difficult to perform in practice. For example, why are the nine curves displayed in Figure 24 *all* the curves that appear in the the computation of $N_{3,0}^{\text{trop}}$? Nevertheless, the tropical

count is in spirit a finite combinatorial problem and could for example be turned into an (computationally horrible) algorithm along the lines

(a) Enumerate all (finitely many) combinatorial types of tropical curves of degree d and genus g.

(b) Check for each type whether a curve of this type passes through the given points (this is a problem in linear programming).

(c) If you find a curve passing through $\mathcal{P}$, compute its multiplicity and sum up.

Given the history of the numbers $N_{d,g}$ alluded to above, this is a rather remarkable result!

2.5 Floor diagrams

As explained above, the Correspondence theorem reduces the computation of the numbers $N_{d,g}$ to a considerably simpler and almost combinatorial problem. Still, it is very complicated — even the computation for $N^{\text{trop}}_{3,0} = 12$ is difficult to do by hand. Is there some trick to exploit the power of the Correspondence theorem more systematically? Yes, of course!

Stretch your points!

The main idea behind floor diagrams is to use some freedom which we have neglected so far. In analogy to the classical case, the numbers $N^{\text{trop}}_{3,0}$ do not depend on the point configuration $p_1, \dots, p_N$ in $\mathbf{R}^2$, as long as we choose them generically. So why not choose a *special* generic point configuration? This sounds contradictory, but what it really means is just this. Among the abundance of generic point configuration, we can of course choose a certain subclass of configurations with some special properties. In our case, we are interested in *vertically stretched* point configurations.

Definition 2.9 A point configuration $p_1, \dots, p_N$ in $\mathbf{R}^2$ called *vertically stretched* if for any pair of points p_i, p_j, the difference in the y-coordinates is much greater than the difference in the x-coordinates, i.e.

$$(p_j - p_i)_y > C \cdot (p_j - p_i)_x,$$

for $C \gg 0$ (how big one should choose C might depend on d and g). We use the convention to number the points from top to bottom, i.e. p_1 is the top point, p_N is the lowest point.

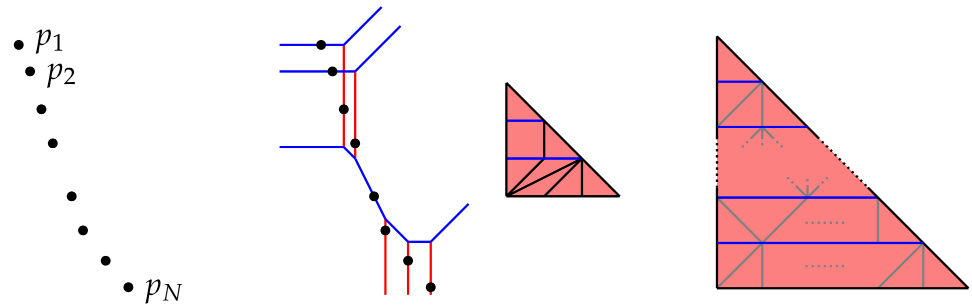


Figure 25 If the points are vertically stretched (like on the left), then all curves passing through these points are floor decomposed. This means that the edges of the dual subdivision cover all horizontal line segments in $\mathrm{NP}(f)$ (the blue segments on the right).

What happens if we count curves passing through this kind of point configuration? Well, the curves which appear will also be of a special kind, in some sense. Here is the precise description (cf. Figure 25).

Definition 2.10 A tropical curve C is called *floor decomposed* if the primitive vector v for any edge has first coordinate $v_x \in \{0, \pm 1\}$, and all non-vertical edges have weight 1. Equivalently, the edges of the dual subdivision $\mathrm{SD}(f)$ cover all the horizontal line segments

$$\mathrm{NP}(f) \cap (\mathbf{R} \times \mathbf{Z}).$$

Exercise 12 Show that the two conditions given in the definition are equivalent.

Proposition 2.11 Let $p_1, \ldots, p_N$ generic vertically stretched point configuration in $\mathbf{R}^2$. Then all the curves of genus g and degree d passing through these points are floor decomposed.

Example 2.12 If you go back to Figure 24, you will find that the given point configuration is indeed stretched vertically — sufficiently enough such that all the nine solution curves are floor decomposed.

Floor diagrams

OK, this all very nice, but how can we actually take advantage of the considerations in the previous paragraph? We need to find a way to encode the structure of a floor decomposed curve in a more combinatorial way, and to understand the possibilities for how such a curve can pass through the given points. To do so, we need some terminology first. Let C be a floor decomposed curve. The vertical edges of C (drawn in red in Picture 25) are called *elevators*. Here, similar to the genus computation, we do not consider transverse intersection points of two edges as actual vertices of the curve (in other words, we are in fact referring to the edges of Γ, where $\Gamma \to C$ is the genus g parametrization of C). After removing the interior of all elevators, we are left with a disconnected graph. The connected components of this graph are called the *floors* of C (drawn in blue in Figure 25).

Construction 2.13 Floor diagrams are abstract graphs which are obtained from a floor decomposed curve by the following construction. Each elevator either connects two floors or starts at one floor and goes of to infinity. We can hence form a abstract graph D (with ends), where each floor represents a vertex, and each elevator e gives an edge of D, attached to the vertices which represent the floors adjacent to e (see Figure 26). Moreover, note that in the example of Figure 25, each floor and each elevator contain exactly one of the chosen points p_i. This is actually no coincidence, but always the case. Hence, we may label each vertex/edge of D by the label i of the point p_i which lies on the corresponding floor/elevator. This gives a labeling of all vertices and edges of D by the numbers $\{1, \ldots, N\}$.

It turns out that we can actually restore the curve C from this construction, so D is the combinatorial object we are looking for and it makes sense to turn this into a rigorous definition first.

Definition 2.14 A *floor diagram* D of degree d and genus g is a connected oriented graph (with ends) with weights $\omega(e) \in \mathbf{N}$ on the edges such that

- the graph D has no oriented cycles, but first Betti number $b_1(D) = g$,
- the graph D consists of d vertices and has d ends, all ends are oriented outwards of D,
- the total loss at each vertex, i.e., the weighted sum of outgoing minus

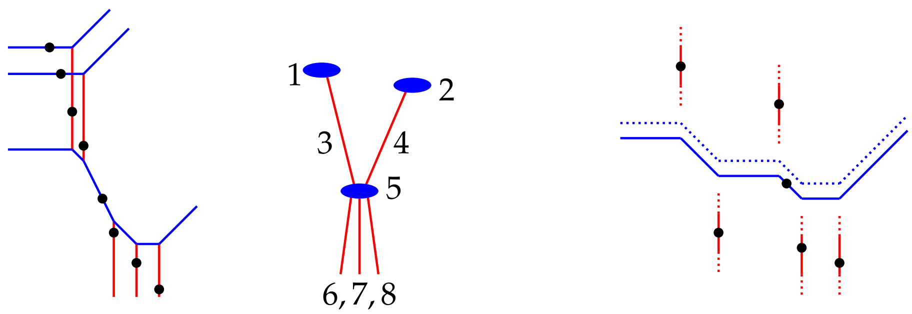


Figure 26 On the left, the construction of a floor diagram. We draw the vertices as flat fat dots and use the convention that all edges are oriented downwards (and that only non-trivial edge weights are shown). The numbers indicate the induced marking encoding which floor/elevator intersects which point p_i. On the right, a sketch of the construction of a floor in the "inverse construction". Note that the vertically stretched property is not correctly displayed.

incoming edges, is 1,

$$\sum_{a \to e} \omega(e) - \sum_{e \to a} \omega(e) = 1.$$

Consider the union $V \cup E$ of all vertices and edges of D. It carries a partial order induced from the orientation of D (where a is smaller than b if b can be reached from a by an oriented path).

Definition 2.15 A bijection

$$m : V \cup E \to \{1, \dots, N\}$$

which satisfies $a < b \Rightarrow m(a) < m(b)$ (i.e., each extension of the partial order to a total order) is called a *marking* of D.

Let $M(D)$ denote the number of distinct markings for D. By distinct we mean that we consider two markings to be the same if they only differ by an (oriented) automorphisms of D. We define the *multiplicity* of D by

$$\operatorname{mult}(D) := M(D) \prod_e \omega(e)^2.$$

Example 2.16 Let us again consider the case $d = 3$ and $g = 0$. If we perform Construction 2.13 for all the nine curves in Figure 24, we end up with the 3 floor diagrams depicted in Figure 27. It is easy to check that all three graphs

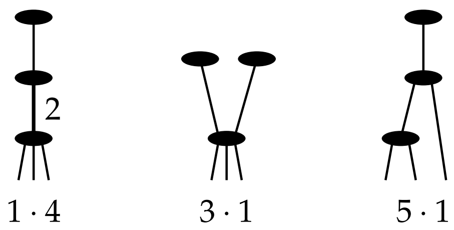


Figure 27 Floor diagrams of degree 3 and genus 0 with multiplicity computation. Note that the number of markings (the first number in each product) is exactly the number of tropical curves from Figure 24 that turn into that floor diagram. In total, we get $N_{3,0} = 4 + 3 + 5 = 12$.

satisfy the properties of Definition 2.14. The multiplicity is shown as a product of the number of markings and the edge weight factor.

Exercise 13 Associate to each tropical curve in Figure 24 the corresponding floor diagram in Figure 27. Check that the number of markings (the first factor in each product) is equal to number of curves that turn into that floor diagram. What about the second factor?

Let us now turn these considerations and examples into statements and proofs. We (still) assume that $\mathcal{P}$ is a vertically stretched point configuration and hence all curves in $\Sigma^{\text{trop}}(\mathcal{P})$ are floor decomposed by Proposition 2.11. Let $\mathcal{D}^{\text{mark}}_{d,g}$ denote the set of all marked floor diagrams of given degree and genus.

Proposition 2.17 Construction 2.13 associates to each floor decomposed curve in $\Sigma^{\text{trop}}(\mathcal{P})$ a well-defined floor diagram in $\mathcal{D}^{\text{mark}}_{d,g}$. The associated map

$$\Sigma^{\text{trop}}(\mathcal{P}) \to \mathcal{D}^{\text{mark}}_{d,g}$$

is a bijection.

Idea of proof. First we should fill the gaps left in Construction 2.13 and show that the map is indeed well-defined. Let C be one of the curves and D the associated floor/elevator graph. With the convention that we orient all elevators from top to bottom (and orient the edges of D accordingly), it is quite clear that D is a floor diagram in the sense of Definition 2.14. In particular, note that each floor corresponds to a horizontal strip of height 1 in the dual subdivision of C. Hence the third condition amounts to the fact that the bottom edge of such a strip is one segment longer than the top edge. To prove that each floor/elevator of C contains exactly one point p_i is a bit more interesting. First, since D has d vertices, d ends and $g + d - 1$ inner edges, we see that the numbers match. The easiest way to continue now is just to assume that we chose our points $\mathcal{P}$ even more specially generic. For example assume the points sit on a line

$$y = -\alpha x$$

with very large irrational slope α. (The minus sign is chosen for better agreement with the pictures). In fact, since the slope is irrational this does not contradict the assumption that the points are generic. Such a line intersects each elevator and floor in at most one point. In the first case that is obvious, in the second it follows from the fact that the slopes appearing when traversing a floor from left to right is bounded absolutely (for example, by the degree d). Hence each floor/elevator contains at most one and hence exactly one point p_i.

It remains to show that there is an inverse map. We will at least sketch this (cf. Figure 26). Let D be marked floor diagram. First note that the information contained in the marking fixes the horizontal position (the x-coordinate) of each elevator. Indeed, the edge marked by i corresponds to an elevator whose x-coordinate agrees with that of p_i. Next, we can construct the shape of a floor from that, up to vertical translations. Say we want to construct the floor associated to the vertex v of D. Think about it as follows. We start on the far left with a horizontal path coming from infinity. Each time that we a pass the y-coordinate of an elevator whose edge in D is adjacent to v, we perform a break in our path — to the top or to the bottom depending on whether the associated edge is an outgoing or an incoming edge at v. Moreover, how much we break the path depends on the weight of the edge (it is equal to the change of slope of the path). This procedure fixes the exact shape of the corresponding floor, except for its vertical position or height in $\mathbf{R}^2$. However, we also have one marked point p_j associated to the floor. Obviously, there is exactly one vertical shift of the the piecewise linear path we constructed which passes through p_j. We can now easily assemble the desired curve C from all its pieces. We construct all the floors in the way we just described and glue in the elevators at the points where the floors break. Since our points are assumed to be vertically stretched, the various floors do not mingle with each other but stay separated and hence the construction really works out. We constructed a curve C whose associated floor diagram is obviously the diagram D we started with. Moreover, from the uniqueness of the construction also the injectivity part follows and we proved the claim. ■

We can now state the main statement of this section. As promised at the beginning, it reduces the computation of the numbers $N_{d,g}$ to the purely combinatorial problem of counting all floor diagrams of given degree and genus, weighted with their multiplicities.

Corollary 2.18 For any choice of $d > 0$ and $g \geq 0$ we have

$$N_{d,g} = \sum_{D \in \mathcal{D}_{d,g}} \mathrm{mult}(D),$$

where $\mathcal{D}_{d,g}$ is the set of all (unmarked) floor diagrams of degree d and g and $\mathrm{mult}(D)$ is the multiplicity defined in Definition 2.15.

Proof. By the Correspondence Theorem 2.6 we know $N_{d,g} = N_{d,g}^{\mathrm{trop}}$. By Proposition 2.17, the concatenated map

$$\Sigma^{\mathrm{trop}}(\mathcal{P}) \to \mathcal{D}_{d,g}^{\mathrm{mark}} \to \mathcal{D}_{d,g}$$

has fibers of cardinality $M(D)$, the number of markings of D. Comparing this with the definition of $\mathrm{mult}(D)$, it remains to show that the multiplicity $\mathrm{mult}(C)$ of a floor decomposed curve is equal to the product of the squares of all elevator edge weights

$$\prod_{e \text{ elev.}} \omega(e)^2.$$

We leave this as an exercise. ∎

Exercise 14 Show that the multiplicity $\mathrm{mult}(C)$ of a floor decomposed nodal curve C with trivial weights on all ends is equal to

$$\prod_{e \text{ elev.}} \omega(e)^2.$$

It is time for more examples! Let us use floor diagrams to (re)compute some of the $N_{d,g}$ numbers.

Example 2.19 In the maximal genus case (Examples 2.1 (a) – (c)) it is easy to check that there is a single floor diagram of multiplicity 1 in each degree — the "cascading" floor diagrams shown in Figure 28. We conclude $N_{d,g(d)} = 1$ if g is maximal.

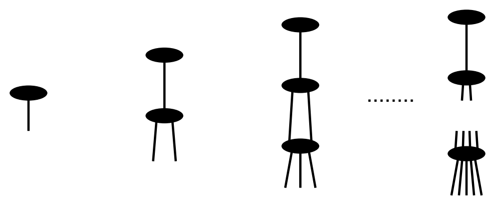


Figure 28 Floor Diagrams of maximal genus

Exercise 15 Check that there there are no other floor diagrams of degree 3 and genus 0 than the ones shown in Figure 27. Conclude that $N_{3,0} = 12$. (Recall that this the first time in these notes that we actually compute this number "rigorously").

Exercise 16 Let us extend the previous exercise to the case of one-nodal curves, i.e. $g = g(d) - 1$ (cf. Examples 2.1 (d) – (e))

(a) Show that starting with a maximal genus floor diagram (see Figure 28), the following two transformations turn it into a "one-nodal" floor diagram (see Figure 27).

  (1) Merge two edges on the same level into a single edge of weight 2.

  (2) For a given vertex, remove an incoming and an outgoing edge. Instead, glue in an edge which starts at the starting point of the incoming edge and ends at the endpoint of the outgoing edge (if this was an end, the the edge will also be an end).

(b) Show that all "one-nodal" floor diagram are obtained uniquely by one of the above transformations.

(c) Sum up the multiplicities in order to show $N_{d,g(d)-1} = 3(d-1)^2$.

Example 2.20 As a final example, let us consider a case which does not belong to one of the special schemes previously discussed, namely $d = 4, g = 1$. Figure 29 shows the 11 floor diagrams which are needed in this calculation. Also the multiplicities are given in the way as before. If we sum up, we obtain $N_{4,1} = 225$. Again, let us emphasize again that this number was not computable by any standard procedure for a long time. Even the afore-mentioned Caporaso-Harris formula is rather tedious for by-hand-computations such as this. It is therefore

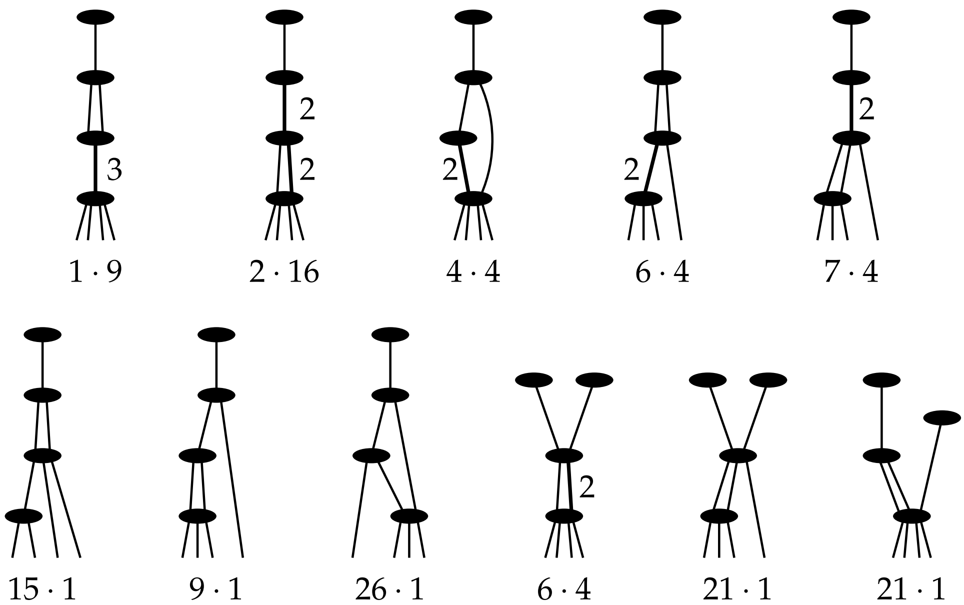


Figure 29 Floor diagrams of degree 4 and genus 1. In total, we get $N_{4,1} = 225$.

remarkable that the tools of tropical geometry make it possible to compute this number by hand!

Exercise 17 Find all floor diagrams of degree 4 and (re)compute the numbers $N_{4,2} = 27$, $N_{4,1} = 225$, and $N_{4,0} = 620$. For the latter number, you need to find 12 floor diagrams.

Exercise 18 If you are on a really boring train ride or an intercontinental flight, go on to compute $N_{5,4} = 882$. Haven't tried it myself yet ;-)

2.6 Further topics/exercises

Computation 2.21 Here is a nice classical justification for $N_{3,0} = 12$ counting rational curves of degree 3 passing through 8 given points The argument assumes that we are working in the projective (compact) setting alluded to above. The 8 points in the plane fix a pencil of cubic curves in the linear system of all cubic curves. Note that the base locus of this pencil (the common intersection of all its fibers) consists of 9 points (adding one extra point to the 8 points). The general fiber of the pencil is a smooth cubic curve (homeomorphic to a torus). Let $\mathcal{P}$ denote the total space of the pencil. If all fibers were smooth, its Euler characteristic would be equal to

$$\chi(\mathbf{CP}^1 \times \text{torus}) = \chi(\mathbf{CP}^1) \cdot \chi(\text{torus}) = 0.$$

But this is not the case, since the pencil contains singular fibers, which are actually exactly the curves we want to count. Indeed, if the eight points are chosen generically, then each singular fiber is a rational curve with exactly one node. We think such a fiber as coming from a torus after contracting one of its meridians to a point (the node). This operation increases the Euler characteristic by one, hence the correct formula is

$$\chi(\mathcal{P}) = \#\text{nodes in } \mathcal{P} = \#\text{singular fibers} = N_{3,0}.$$

Now, there is a second way to compute $\chi(\mathcal{P})$. Note that choosing any point in $\mathbf{CP}^2 \setminus \{\text{base pts}\}$ fixes a unique curve in the pencil, together with a marked point (the point we chose). This can be extended to the base points by choosing a tangent line of the curve at any of these points instead. In other words, we can conclude that $\mathcal{P}$ is equal to the blow up of $\mathbf{CP}^2$ in the 9 base points. Note that $\chi(\mathbf{CP}^2) = 3$ and blowing up a point replaces this point by a $\mathbf{CP}^1$ (a sphere), hence increases the Euler characteristic by one. We conclude

$$\chi(\mathcal{P}) = \chi(\mathrm{Bl}_9\mathbf{CP}^2) = 3 + 9 = 12.$$

Exercise 19 The goal of this exercise is to show classically that $N_{d,g(d)} = 1$. You can proceed as follows.

(a) Show that the set of all curves of degree d can be identified with (a subset of) $\mathbf{CP}^M$ for suitable M, using the coefficients of the polynomial F as homogeneous coordinates. What is M? You may use that two polynomials F, F' describe the same curve if and only if $F = \lambda F'$ for some $\lambda \in \mathbf{C}^*$.

(b) Fix one point $p \in \mathbf{C}^2$. Describe the subset H_p in $\mathbf{CP}^M$ parametrizing curves that pass through p. What kind of subset is it?

(c) Conclude that for generic points $p_1, \dots, p_N$, the subsets $H_{p_1}, \dots, H_{p_N}$

intersect in a unique point.

Exercise 20 Can you also show $N_{d,g(d)-1} = 3(d-1)^2$ classically? You need some more advanced algebraic geometry (and ignore some details about compactification, etc.) to do so.

(a) Similar to before, we identify $\mathbf{CP}^N$ with the space of homogeneous polynomials of fixed degree d, up to rescaling. Show that the set

$$\{(F, p) : \frac{\partial F}{\partial z_1}(p) = 0\} \subset \mathbf{CP}^N \times \mathbf{CP}^2$$

is an algebraic hypersurface and determine its homology class in $\mathbf{CP}^N \times \mathbf{CP}^2$.

(b) Determine the homology class of

$$\mathcal{I} = \overline{\{(C, p) : p \text{ singular point of } C\}} \subset \mathbf{CP}^N \times \mathbf{CP}^2.$$

(c) Let $D \subset \mathbf{CP}^N$ be the discriminant hypersurface corresponding to singular curves. Let $\pi : \mathbf{CP}^N \times \mathbf{CP}^2 \to \mathbf{CP}^N$ be the projection to the first factor. Use $\pi_*[\mathcal{I}] = [D]$ in order to compute the degree of D.

(d) By an (heuristic) argument similar to the previous exercise, convince yourself that the degree of D is equal to $N_{d,g(d)-1}$.

3 References

Since this text is supposed to be a very first outlook to tropical geometry, I did not bother to carefully include references. Instead, I will list here a few books/papers which might serve as good starting points for further reading (and which contain detailed references to the results mentioned here).

The Basics For the basics of tropical geometry including further references to the literature, check out the following two texts. Both of them are suitable for students.

[1] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry.* Providence, RI: American Mathematical Society (AMS), 2015, pp. xii + 363.

[2] Grigory Mikhalkin and Johannes Rau. *Tropical Geometry*. ICM publication, textbook in preparation. URL: https://www.math.uni-tuebingen.de/user/jora/downloads/main.pdf.

Enumerative Geometry More applications of tropical geometry in enumerating planar algebraic curves can be found in the following (more demanding) research papers.

[3] Grigory Mikhalkin. *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$. J. Am. Math. Soc. 18.2 (2005), pp. 313–377. arXiv: math/0312530.

[4] Erwan Brugallé and Grigory Mikhalkin. *Floor decompositions of tropical curves: the planar case.* Proceedings of the 15th Gökova geometry-topology conference, Gökova, Turkey, May 26–31, 2008. Cambridge, MA: International Press, 2009, pp. 64–90. arXiv: 0812.3354.

[5] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin. *Welschinger invariant and enumeration of real rational curves*. Int. Math. Res. Not. 2003.49 (2003), pp. 2639–2653. arXiv: math/0303378.

[6] Hannah Markwig and Johannes Rau. *Tropical descendant Gromov-Witten invariants*. Manuscr. Math. 129.3 (2009), pp. 293–335. arXiv: 0809.1102.

[7] Mark Gross and Bernd Siebert. *Mirror symmetry via logarithmic degeneration data I*. J. Differ. Geom. 72.2 (2006), pp. 169–338. arXiv: math/0309070.

[8] Ilia Itenberg and Grigory Mikhalkin. *On Block-Göttsche multiplicities for planar tropical curves*. Int. Math. Res. Not. 2013.23 (2013), pp. 5289–5320. arXiv: 1201.0451.

Matroids and Phylogenetics Information on matroid fans and tropical Grassmannian (which are not yet covered in this text) can be found here.

[9] Federico Ardila and Caroline J. Klivans. *The Bergman complex of a matroid and phylogenetic trees*. J. Comb. Theory, Ser. B 96.1 (2006), pp. 38–49. arXiv: math/0311370.

[10] David E. Speyer. *Tropical linear spaces*. SIAM J. Discrete Math. 22.4 (2008), pp. 1527–1558. arXiv: math/0410455.

[11] Kristin M. Shaw. *A tropical intersection product in matroidal fans*. SIAM J. Discrete Math. 27.1 (2013), pp. 459–491. arXiv: 1010.3967.

[12] Georges François and Johannes Rau. *The diagonal of tropical matroid varieties and cycle intersections*. Collect. Math. 64.2 (2013), pp. 185–210. arXiv: 1012.3260.

[13] David Speyer and Bernd Sturmfels. *The tropical Grassmannian*. Adv. Geom. 4.3 (2004), pp. 389–411. arXiv: math/0304218.