K3 polytopes and their quartic surfaces

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SFB–TRR 195 Annual Meeting
Tübingen - September 25, 2018
Tropical hypersurfaces

K3 polytopes

Stability of quartic surfaces
Tropical polynomials

We work over the tropical semiring \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\). Tropical operations are defined as follows:

\[
a \oplus b = \min\{a, b\} \quad \text{and} \quad a \odot b = a + b.
\]

Let \(x_1, x_2, \ldots, x_n\) be variables representing elements in the tropical semiring. A **monomial** is any product of variables:

\[
x_1^{i_1} \odot x_2^{i_2} \odot \cdots \odot x_n^{i_n}.
\]

A **polynomial** is a finite linear combinations of monomials:

\[
f(x_1, x_2, \ldots, x_n) = a_i \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus a_j \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots.
\]
Tropical hypersurfaces

Given a polynomial \( f \), we define the \textbf{hypersurface} \( T(f) \) of \( f \) as the set of points \( x \in \mathbb{R}^n \) at which \textit{the minimum is attained at least twice}.

\textbf{Example:}
\[
f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3
\]
\[
\min\{3+3x, 1+2x+y, 1+x+2y, 3+3y, 1+2x, x+y, 1+2y, 1+x, 1+y, 3\}
\]
Tropical and classical hypersurfaces

Let $K$ be an algebraically closed field with non-trivial non-Archimedean valuation (e.g. $K = \mathbb{C}\{\{t\}\}$). Let $f$ be an Laurent polynomial

$$f = \sum_{u=\langle u_1, \ldots, u_n \rangle \in \mathbb{Z}^n} c_u x_1^{u_1} \cdots x_n^{u_n}, \text{ with } c_u \in K.$$ 

We define its tropicalization $\text{trop}(f)$ as

$$\text{trop}(f) = \min_{u \in \mathbb{Z}^n} \left\{ \text{val}(c_u) + \sum_{i \leq n} u_i x_i \right\}.$$

**Theorem (Kapranov’s theorem)**

The following sets coincide:

1. $\{ w \in \mathbb{R}^n \mid \text{the min in } \text{trop}(f)(w) \text{ is attained at least twice} \}$;
2. the closure of $\{(\text{val}(y_1), \ldots, \text{val}(y_n)) \mid (y_1, \ldots, y_n) \in V(f)\}$.
Theorem (Structure theorem)

The tropical hypersurface $T(f)$ is the support of a pure rational polyhedral complex of dimension $n - 1$.

The closure of the connected components of the complement of a tropical hypersurface $T(f)$ are called regions of $T(f)$. They are convex polyhedra.
Let $T(f)$ be a smooth tropical surface of degree 4 in $\mathbb{R}^3$. The surface $T(f)$ cuts at most one bounded region out.

**Definition**
A 3-dimensional polytope $\mathcal{P}$ is a **K3 polytope** if it arises as the closure of the bounded region in the complement of a smooth tropical surface of degree 4.
Example

Consider the tropical quartic surface defined by the polynomial:

$$f = 5(x^4 \oplus y^4 \oplus z^4) \oplus 3(x^3 y \oplus x^3 z \oplus xy^3 \oplus y^3 z \oplus xz^3 \oplus yz^3)$$
$$\oplus 2(x^2 y^2 \oplus x^2 z^2 \oplus y^2 z^2) \oplus 0(x^2 yz \oplus xy^2 z \oplus xyz^2) \oplus 3(x^3 \oplus y^3 \oplus z^3)$$
$$\oplus 0(x^2 y \oplus x^2 z \oplus xy^2 \oplus y^2 z \oplus xz^2 \oplus yz^2) \oplus 2(x^2 \oplus y^2 \oplus z^2)$$
$$\oplus 0(xy \oplus xz \oplus yz) \oplus 3(x \oplus y \oplus z) \oplus (-9xyz) \oplus 5.$$
Let’s look at the K3 polytope defined by $T(f)$

This is the smooth tropical quartic surface $T(f)$:
Let’s look at the K3 polytope defined by $T(f)$

This is the K3 polytope:

Its $f$-vector is $(64, 96, 34)$. 
Newton polytopes

Given a tropical polynomial $f = \bigoplus_{\nu \in \mathbb{Z}^n} a_\nu x^\nu$, we define the Newton polytope $\text{Newt}(f)$ as the polytope

$$\text{Newt}(f) = \text{conv}(\nu : a_\nu \neq \infty).$$

Example: $f = 3x^3 \oplus 1x^2 y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$
Newton polytopes

We consider the convex hull in $\mathbb{R}^{n+1}$ of the points $(v, a_v)$. The projection of the lower faces on $\text{Newt}(f)$ induces a subdivision of the Newton polytope.

Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$
Newton polytopes and tropical hypersurfaces

Tropical hypersurfaces are dual to the regular subdivision of their Newton polytopes induced by the coefficients.

Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$
Smooth tropical hypersurfaces

A tropical hypersurface is smooth if the regular subdivision induced by its coefficients is a unimodular triangulation, i.e., cells in the subdivision are simplices of minimal volume $\frac{1}{n!}$.

Examples:

Unimodular

Not unimodular
Regions of tropical cubic curves

Exercise 13, Section 1.3 of Maclagan–Sturmfels: show that the unique bounded region of a smooth cubic curve in the plane is an $m$-gon with $m \in \{3, 4, 5, 6, 7, 8, 9\}$.

The boundary of the $m$-gon carries the group structure of the tropical elliptic curve (Vigeland ’09) and its lattice length is the tropical $j$-invariant (Katz-Markwig-Markwig ’08).
Regions of a tropical quartic surface

Let $4\Delta_3$ be the 4-th dilatation of the standard simplex,

$$4\Delta_3 = \text{conv} \left( (0, 0, 0), (4, 0, 0), (0, 4, 0), (0, 0, 4) \right)$$

The point $p = (1, 1, 1)$ is the unique interior lattice point of $4\Delta_3$. 

The Newton polytope of a quartic surface is contained in $4\Delta_3$. If the Newton polytope contains $p$ in its relative interior, then a smooth surface will determine a K3 polytopes.

We switch our attention to regular unimodular triangulations of polytopes contained in $4\Delta_3$ containing $p = (1, 1, 1)$ in their relative interior. They are canonical polytopes.
The hunt for K3 polytopes

**Warning:** we are talking about a lot of triangulations!

- $2\Delta_3$ has 10 lattice points and 15 regular triangulations,
- $3\Delta_3$ has 20 lattice points and 21,125,102 regular triangulations.

Triangulations of $3\Delta_3$ were computed by Jordan, Joswig and Kastner with MPTOPCOM.
Central triangulations

We define the central part of a triangulation $\mathcal{T}$ as the subset of $\mathcal{T}$ given by the simplices of $\mathcal{T}$ containing $p$. If $\mathcal{T}$ coincides with its central part, then we say that $\mathcal{T}$ is central.

The K3 polytope is uniquely determined by the central part of the triangulation $\mathcal{T}$.

It is enough to consider central triangulations!
Central triangulations of canonical polytopes

How can we construct central triangulations of a canonical polytope?

\{ \mathcal{T} \text{ central triangulation of } P \} \leftrightarrow \{ \mathcal{T} \text{ triangulation of } \partial P \}

We are interested in polytopes $P$ which admit at least one unimodular central triangulation.

A polytope $P$ is reflexive if $Ap - c = 1$, where $Ax \geq c$ are the equations defining $P$. If $P$ is reflexive,

\{ \mathcal{T} \text{ central unimod triang of } P \} \leftrightarrow \{ \mathcal{T} \text{ is a unimod triang of } \partial P \}

A three dimensional canonical lattice polytope $P$ is reflexive if and only if every central fine triangulation of $P$ is unimodular. We need to consider reflexive polytopes!
Theorem (Balletti-P-Sturmfels)

*Up to symmetry there are 15139 possible reflexive polytopes contained in $4\Delta_3$.***
Theorem (Balletti-P-Sturmfels)

The reflexive polytopes of volume \( \leq 30 \) in the theorem above admit a total of 36,297,333 regular unimodular central triangulations. Every K3 polytope with \( \leq 30 \) vertices arises from one of these.
**f-vectors of K3 polytopes**

**Theorem (Balletti-P-Sturmfels)**

Let $\mathcal{P}$ be a K3 polytope obtained from a polytope $P$. Then $\mathcal{P}$ is a simple polytope. The entries of its $f$-vector are

\[
v = \text{vol}(P), \quad e = \frac{3\text{vol}(P)}{2}, \quad f = |P \cap \mathbb{Z}^2| - 1.
\]

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<th>$f$-vect</th>
<th># of $P$</th>
<th>$f$-vect</th>
<th># of $P$</th>
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Moduli space of quartic surfaces

A quartic surface is the variety in \( \mathbb{P}^3 \) defined by a homogeneous polynomial of degree 4 in \( \mathbb{C}[x, y, z, w] \),

\[
f(x, y, z, w) = \sum_{i+j+k \leq 4} c_{ijk} x^i y^j z^k w^{4-i-j-k}.
\]

The 35 coefficients \( c_{ijk} \) parameterize all the quartic surfaces. So we consider the 34-dimensional projective space \( \mathbb{H}S_{4,3} = \mathbb{P}(\mathbb{H}S_{4,3}) \) of quartic surfaces. This gives us a “moduli space of quartic surfaces”.

More precisely, the special linear group \( SL(4) \) acts on \( \mathbb{H}S_{4,3} \), and on the associated polynomial ring \( \mathbb{C}[\mathbb{H}S_{4,3}] \), generated by \( c_{ijk} \). The moduli space of quartic surfaces in \( \mathbb{P}^3 \) is the projective variety determined by \( \text{Proj}(\mathbb{C}[\mathbb{H}S_{4,3}]^{SL(4)}) \).
In the context of Geometric Invariant Theory, we want to restrict to a “nicer” set inside the moduli space. This is the set of stable elements.

More precisely, $f$ is stable if the orbit $O(f)^{SL(4)}$ is closed and the stabilizer $\text{stab}(f)$ is finite.

Determining whether an element is stable is connected to the study of its singular locus.

**Theorem (Shah ’81)**

*If the singular locus of a quartic surface contains at most rational double points, then the surface is stable.*
Arnold’s classification ’72

\[ A_k: x^2 + y^2 + z^{k+1} \]

\[ D_k: x^2 + y^2z + z^{k-1} \]

Pictures from Greuel-Lossen-Shustin “Introduction to Singularities and Deformation”.
Arnold’s classification ’72

\[ E_6: x^2 + y^3 + z^4, \quad E_7: x^2 + y^3 + yz^3, \quad \text{and} \quad E_8: x^2 + y^3 + z^5 \]

Pictures from Greuel-Lossen-Shustin “Introduction to Singularities and Deformation”.

A combinatorial criterion

Theorem (Mumford ’77)

A point $f$ in $\text{HS}_{4,3}$ is stable if and only if, for every choice of coordinates, and for all planes $H$ through $p$, each open halfspace of $H$ contains a monomial of $f$.

A reflexive lattice polytope $P$ contained in $4\Delta_3$ is called minimal if it does not properly contain any reflexive polytope.

There are precisely 115 minimal reflexive polytopes in $4\Delta_3$. 
Stability

Theorem (Balletti-P-Sturmfels)

Let \( f \in \mathbb{C}[x, y, z, w] \) be a generic homogeneous quartic surface whose Newton polytope arises from a smooth tropical surface. Then the quartic surface \( V(f) \) in \( \mathbb{P}^3 \) is stable.

- We show the stability of surfaces having a minimal polytope as Newton polytope by studying their singular locus.
- We use Mumford’s criterion to conclude that also generic surfaces with Newton polytope containing a minimal one are stable.
Thank you!