An Experimental Classification of Maximal Mediated Sets

Oguzhan Yürük

Technische Universität Berlin

yuruk@math.tu-berlin.de

Joint work with Timo de Wolff and Olivia Röhrig

SFB-TRR 195, Tübingen

September 28, 2018
A polynomial $f \in \mathbb{R}[x]$ is called nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.
A polynomial $f \in \mathbb{R}[x]$ is called nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

How to certify a polynomial $f \in \mathbb{R}[x]$ is nonnegative?
A polynomial \( f \in \mathbb{R}[x] \) is called nonnegative if \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

**How to certify a polynomial \( f \in \mathbb{R}[x] \) is nonnegative?**

- This is an NP-hard problem.
A polynomial $f \in \mathbb{R}[x]$ is called nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

How to certify a polynomial $f \in \mathbb{R}[x]$ is nonnegative?

- This is an NP-hard problem.
- One way is to check whether $f$ is sum of squares of polynomials (SOS).
A polynomial $f \in \mathbb{R}[x]$ is called nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

How to certify a polynomial $f \in \mathbb{R}[x]$ is nonnegative?

- This is an NP-hard problem.
- One way is to check whether $f$ is sum of squares of polynomials (SOS).
- Hilbert (1888) showed that there exists nonnegative polynomials that cannot be represented as sum of squares.
A polynomial $f \in \mathbb{R}[x]$ is called nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

How to certify a polynomial $f \in \mathbb{R}[x]$ is nonnegative?

- This is an NP-hard problem.
- One way is to check whether $f$ is sum of squares of polynomials (SOS).
- Hilbert(1888) showed that there exists nonnegative polynomials that cannot be represented as sum of squares.
- The AM-GM inequality can be used to check the nonnegativity the circuit polynomials.
Motzkin Polynomial (1967): Consider the Motzkin polynomial

\[ f(x, y) = x^4 y^2 + x^2 y^4 + 1 - 3x^2 y^2 \]
Motzkin Polynomial (1967): Consider the Motzkin polynomial

\[ f(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \]

\[ f(x, y) \geq 0 \] due to the classical AM-GM inequality.
A set $L \subseteq \mathbb{Z}^n$ is called $\Delta$-mediated if every point in $L - \Delta$ is midpoint of two distinct points in $L \cap (2\mathbb{Z})^n$. 

Theorem (Reznick (1989)) There exists the maximal $\Delta^*$-mediated set (MMS), $\Delta^*$, that contains every $\Delta$-mediated set.

Maximal mediated set tells us when a nonnegative circuit polynomial is SOS. 

Theorem (Reznick (1989), de Wolff, Iliman (2014)) A nonnegative circuit polynomial $f$ is SOS if and only if "inner term" is in MMS.
A set $L \subseteq \mathbb{Z}^n$ is called $\Delta$-mediated if every point in $L - \Delta$ is midpoint of two distinct points in $L \cap (2\mathbb{Z})^n$.

**Theorem (Reznick (1989))**

There exists the maximal $\Delta$-mediated set (MMS), $\Delta^*$, that contains every $\Delta$-mediated set.
Maximal Mediated Sets

A set $L \subseteq \mathbb{Z}^n$ is called $\Delta$-mediated if every point in $L - \Delta$ is midpoint of two distinct points in $L \cap (2\mathbb{Z})^n$.

**Theorem (Reznick (1989))**

There exists the maximal $\Delta$-mediated set (MMS), $\Delta^*$, that contains every $\Delta$-mediated set.

Maximal mediated set tells us when a nonnegative circuit polynomial is SOS.
A set $L \subseteq \mathbb{Z}^n$ is called $\triangle$-mediated if every point in $L - \triangle$ is midpoint of two distinct points in $L \cap (2\mathbb{Z})^n$.

**Theorem (Reznick (1989))**

There exists the maximal $\triangle$-mediated set (MMS), $\triangle^*$, that contains every $\triangle$-mediated set.

Maximal mediated set tells us when a nonnegative circuit polynomial is SOS.

**Theorem (Reznick (1989), de Wolff, Iliman (2014))**

A nonnegative circuit polynomial $f$ is SOS if and only if “inner term” is in MMS.
Given a set of points $L \subset \mathbb{Z}^n$, we define a set of averages:
\[
\overline{A}(L) = \left\{ \frac{s + t}{2} | s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

Reznick’s MMS algorithm (1989):
\hspace{1cm} \textbf{Input:} $\Delta$: finite set of points in $(2\mathbb{Z})^n$
\hspace{1cm} \textbf{Output:} $\Delta^*$: the $\Delta$-mediated subset of $\mathbb{Z}^n$
\hspace{1cm} that contains every $\Delta$-mediated set

1: $\Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n$
2: repeat
3: \hspace{0.5cm} $\Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta$
4: until $P^n = P^{n-1}$
5: $\Delta^* \leftarrow \Delta^n$
MMS Algorithm

Given a set of points $L \subset \mathbb{Z}^n$, we define a set of averages:

$$\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}$$

Reznick’s MMS algorithm(1989):

**Input:** $\Delta$: finite set of points in $(2\mathbb{Z})^n$

**Output:** $\Delta^*$: the $\Delta$-mediated subset of $\mathbb{Z}^n$ that contains every $\Delta$-mediated set

1: $\Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n$
2: repeat
3: $\Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta$
4: until $P^n = P^{n-1}$
5: $\Delta^* \leftarrow \Delta^n$
MMS Algorithm

Given a set of points \( L \subset \mathbb{Z}^n \), we define a set of averages:

\[
\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

Reznick's MMS algorithm(1989):

**Input:** \( \Delta \): finite set of points in \((2\mathbb{Z})^n\)

**Output:** \( \Delta^* \): the \( \Delta \)-mediated subset of \( \mathbb{Z}^n \) that contains every \( \Delta \)-mediated set

1: \( \Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n \)
2: repeat
3: \( \Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta \)
4: until \( P^n = P^{n-1} \)
5: \( \Delta^* \leftarrow \Delta^n \)
MMS Algorithm

Given a set of points $L \subset \mathbb{Z}^n$, we define a set of averages:

$$\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}$$

**Reznick’s MMS algorithm (1989):**

**Input:** $\Delta$: finite set of points in $(2\mathbb{Z})^n$

**Output:** $\Delta^*$: the $\Delta$-mediated subset of $\mathbb{Z}^n$ that contains every $\Delta$-mediated set

1: $\Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n$
2: repeat
3: $\Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta$
4: until $P^n = P^{n-1}$
5: $\Delta^* \leftarrow \Delta^n$
Given a set of points $L \subset \mathbb{Z}^n$, we define a set of averages:

$$\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}$$

Reznick’s MMS algorithm (1989):

**Input:** $\Delta$: finite set of points in $(2\mathbb{Z})^n$

**Output:** $\Delta^*$: the $\Delta$-mediated subset of $\mathbb{Z}^n$ that contains every $\Delta$-mediated set

1: $\Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n$
2: repeat
3: $\Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta$
4: until $P^n = P^{n-1}$
5: $\Delta^* \leftarrow \Delta^n$
MMS Algorithm

Given a set of points \( L \subset \mathbb{Z}^n \), we define a set of averages:

\[
\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

Reznick’s MMS algorithm (1989):

**Input:** \( \Delta \): finite set of points in \((2\mathbb{Z})^n\)

**Output:** \( \Delta^* \): the \( \Delta \)-mediated subset of \( \mathbb{Z}^n \) that contains every \( \Delta \)-mediated set

1: \( \Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n 
2: \text{repeat}
3: \quad \Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta
4: \quad \text{until } P^n = P^{n-1}
5: \Delta^* \leftarrow \Delta^n

Fact:
\( \Delta \cup \overline{A}(\Delta) \subseteq \Delta^* \subseteq \text{conv}(\Delta) \cap \mathbb{Z}^n \)

Task:
Decide how dense \( \Delta^* \) is in \( \text{conv}(\Delta) \cap \mathbb{Z}^n \), so we define the h-ratio:

\[
H(\Delta) = \frac{|\Delta^* - (\Delta \cup \overline{A}(\Delta))|}{|\text{conv}(\Delta) \cap \mathbb{Z}^n - (\Delta \cup \overline{A}(\Delta))|}
\]
MMS Algorithm

Given a set of points \( L \subset \mathbb{Z}^n \), we define a set of averages:
\[
\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

Reznick’s MMS algorithm (1989):

**Input:** \( \Delta \): finite set of points in \((2\mathbb{Z})^n\)

**Output:** \( \Delta^* \): the \( \Delta \)-mediated subset of \( \mathbb{Z}^n \) that contains every \( \Delta \)-mediated set

1: \( \Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n \)
2: repeat
3: \( \Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta \)
4: until \( P^n = P^{n-1} \)
5: \( \Delta^* \leftarrow \Delta^n \)
Given a set of points \( L \subset \mathbb{Z}^n \), we define a set of averages:
\[
\overline{A}(L) = \left\{ \frac{s + t}{2} \mid s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

Reznick's MMS algorithm (1989):

**Input:** \( \Delta \): finite set of points in \((2\mathbb{Z})^n\)

**Output:** \( \Delta^* \): the \( \Delta \)-mediated subset of \( \mathbb{Z}^n \) that contains every \( \Delta \)-mediated set

1. \( \Delta^0 \leftarrow \text{Conv}(\Delta) \cap \mathbb{Z}^n \)
2. repeat
3. \( \Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta \)
4. until \( P^n = P^{n-1} \)
5. \( \Delta^* \leftarrow \Delta^n \)

**Fact:**

\[ \Delta \cup \overline{A}(\Delta) \subseteq \Delta^* \subseteq \text{conv}(\Delta) \cap \mathbb{Z}^n \]
**MMS Algorithm**

Given a set of points \( L \subset \mathbb{Z}^n \), we define a set of averages:

\[
\overline{A}(L) = \left\{ \frac{s + t}{2} | s, t \in L \cap (2\mathbb{Z})^n, s \neq t \right\}
\]

**Reznick’s MMS algorithm (1989):**

**Input:** \( \Delta \): finite set of points in \((2\mathbb{Z})^n\)

**Output:** \( \Delta^* \): the \( \Delta \)-mediated subset of \( \mathbb{Z}^n \) that contains every \( \Delta \)-mediated set

1: \( \Delta^0 \leftarrow Conv(\Delta) \cap \mathbb{Z}^n \)
2: repeat
3: \( \Delta^n \leftarrow \overline{A}(\Delta^{n-1}) \cup \Delta \)
4: until \( P^n = P^{n-1} \)
5: \( \Delta^* \leftarrow \Delta^n \)

**Fact:**

- \( \Delta \cup \overline{A}(\Delta) \subseteq \Delta^* \subseteq Conv(\Delta) \cap \mathbb{Z}^n \)

**Task:** Decide how dense \( \Delta^* \) is in \( Conv(\Delta) \cap \mathbb{Z}^n \), so we define the h-ratio:

\[
\mathcal{H}(\Delta) = \frac{|\Delta^* - (\Delta \cup \overline{A}(\Delta))|}{|(Conv(\Delta) \cap \mathbb{Z}^n) - (\Delta \cup \overline{A}(\Delta))|}
\]
Main Question: What is the distribution on $\mathcal{H}$ when $n > 2$?
Main Question: What is the distribution on $\mathcal{H}$ when $n > 2$?

Observations:

- Even though the algorithm looks easy, there are too many simplices to consider. In dimension 4, maximal degree 8 there are more than 300000 simplices to check. This makes it hard to write to database.
- Thus, we need to get rid of the redundant data. In fact, instead of simplices one can consider the underlying lattice.
\[ \Delta_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\} \]

\[ M_{\Delta_1} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \]

\[ L_{\Delta_1} = \langle (2, 4), (4, 2) \rangle \]

\[ \Delta_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\} \]

\[ M_{\Delta_2} = \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix} \]

\[ L_{\Delta_2} = \langle (2, 4), (0, 6) \rangle \]

\[ T(x) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} x \]
Thank you for your attention!