$\boxed{1}$

a) $$\int_0^{2\pi} \sin(x)\sinh(x)\,dx = \underbrace{\Big[\sin(x)\cosh(x)\Big]_0^{2\pi}}_{=0} - \int_0^{2\pi}\cos(x)\cosh(x)\,dx$$

$$= \underbrace{\Big[-\cos(x)\sinh(x)\Big]_0^{2\pi}}_{=-\sinh(2\pi)+\sinh(0)=-\sinh(2\pi)} - \int_0^{2\pi}\sin(x)\sinh(x)\,dx$$

$$\Rightarrow \int_0^{2\pi}\sin(x)\sinh(x)\,dx = -\frac{1}{2}\sinh(2\pi)$$

b) Nennernullstellen:

$$x^4 - 5x^2 + 4 = 0$$

$$\Leftrightarrow \quad x^2 = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5\pm 3}{2} = \begin{cases} 4 \\ 1 \end{cases}$$

also $1, -1, 2, -2$

$$\frac{1+x^2}{x^4-5x^2+4} = \frac{1+x^2}{(x^2-1)(x^2-4)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$$

mal $x-1$, dann $x \to 1$: $A = \frac{2}{2\cdot(-3)} = -\frac{1}{3}$

etc. $B = \frac{2}{(-2)(-3)} = \frac{1}{3}$

$$C = \frac{5}{3\cdot 4} = \frac{5}{12}, \quad D = \frac{5}{3\cdot(-4)} = -\frac{5}{12}$$

also $\frac{1+x^2}{x^4-5x^2+4} = \frac{-1/3}{x-1} + \frac{1/3}{x+1} + \frac{5/12}{x-2} - \frac{5/12}{x+2}$

c) $$\int_3^\infty \frac{1+x^2}{x^4-5x^2+4}\,dx$$

$$= \left[-\frac{1}{3}\log(x-1) + \frac{1}{3}\log(x+1) + \frac{5}{12}\log(x-2) - \frac{5}{12}\log(x+2)\right]_3^\infty$$

$$= \left[\frac{1}{3}\log\frac{x+1}{x-1} + \frac{5}{12}\log\frac{x-2}{x+2}\right]_3^\infty = -\frac{1}{3}\log\frac{4}{2} - \frac{5}{12}\log\frac{1}{5}$$

$$= -\frac{1}{3}\log 2 + \frac{5}{12}\log 5$$

$\boxed{2}$

a) $\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (5-\lambda)(\lambda^2-1)$

Eigenwerte $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = -1$

zugehörige Eigenvektoren:

zu λ_1: $\left(\begin{array}{ccc|c} -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right) \rightsquigarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ und Vielfache

zu λ_2: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ und Vielfache

zu λ_3: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ und Vielfache

b) $B = \frac{d}{dx} e^{Bx}\Big|_{x=0} = \begin{pmatrix} 3\sinh(3x) & 0 & 3\cosh(3x) \\ 0 & 7e^{7x} & 0 \\ 3\cosh(3x) & 0 & 3\sinh(3x) \end{pmatrix}\Bigg|_{x=0}$

$= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 7 & 0 \\ 3 & 0 & 0 \end{pmatrix} \Rightarrow \det B = -63$

$\boxed{3}$ $y' + y^2(x-1)^3 = 0$, $y(1) = 4$

$\Leftrightarrow \frac{dy}{y^2} = -(x-1)^3\,dx$, $y(1) = 4$

$\Leftrightarrow \int_4^y \frac{d\tilde{y}}{\tilde{y}^2} = -\int_1^x (\tilde{x}-1)^3\,d\tilde{x}$

$\Leftrightarrow -\frac{1}{y} + \frac{1}{4} = -\frac{1}{4}(x-1)^4$

$\Leftrightarrow \frac{1}{y} = \frac{1}{4}\left(1+(x-1)^4\right)$

$\Leftrightarrow y = \frac{4}{1+(x-1)^4}$

$\boxed{4}$

a) charakteristisches Polynom: $\lambda(x) = x^2 + 6x + 9 = (x+3)^2$

doppelte Nullstelle: $\lambda = -3$

allg. Lösung: $y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$, $c_{1,2} \in \mathbb{R}$

b) $y(0) = c_1 \overset{!}{=} 0$

$y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$

$y'(0) = -3c_1 + c_2 \underset{c_1=0}{=} c_2 \overset{!}{=} 1$

Lösung des AWP: $y(x) = x e^{-3x}$

c) Ansatz: $y(x) = A + Be^{-x}$

$y'(x) = -Be^{-x}$

$y''(x) = Be^{-x}$

in DGL: $Be^{-x} - 6Be^{-x} + 9A + 9Be^{-x} = 1 - e^{-x}$

$\Rightarrow A = \frac{1}{9}, \quad 4B = -1 \Leftrightarrow B = -\frac{1}{4}$

Also löst z.B.

$$y(x) = \frac{1}{9} - \frac{1}{4} e^{-x}$$

die inhomogene DGL.

5

a) $\underline{n=0}$: links: $\int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$

rechts: $0! = 1$ ☺

$\underline{n \to n+1}$:

$$\int_0^\infty t^{n+1} e^{-t}\,dt \overset{\text{P.I.}}{=} \underbrace{-t^{n+1} e^{-t}\Big|_0^\infty}_{=0} + \int_0^\infty (n+1)\, t^n e^{-t}\,dt$$

$$\underset{\text{I.V.}}{=} (n+1)\cdot n! = (n+1)! \qquad \square$$

b) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z^2 e^{-\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2\cos^2\theta\; e^{-r}\; r^2 \sin\theta\; dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi} d\phi \cdot \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \int_0^{\infty} r^4 e^{-r}\,dr$$

$$= 2\pi \cdot \underbrace{\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi}}_{=\frac{2}{3}} \cdot 4! \quad \leftarrow \text{laut (a)}$$

$$= 32\pi$$

$\boxed{6}$

a) $$\nabla F = -\frac{3}{2}\frac{(-z)}{(x^2+y^2+z^2)^{5/2}}\begin{pmatrix}2x\\2y\\2z\end{pmatrix} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \frac{1}{(x^2+y^2+z^2)^{5/2}}\begin{pmatrix}3xz\\3yz\\3z^2-(x^2+y^2+z^2)\end{pmatrix}$$

b) $$\int_{\mathcal{C}}(\nabla F)\,d\vec{x} = F(\vec{x}(\pi)) - F(\vec{x}(0))$$

$$= F(0,0,-1) - F(0,0,1)$$

$$= 2$$

c) $$\vec{x}_\theta \times \vec{x}_\phi = R\begin{pmatrix}\cos\theta\cos\phi\\\cos\theta\sin\phi\\-\sin\theta\end{pmatrix} \times R\begin{pmatrix}\sin\theta(-\sin\phi)\\\sin\theta\cos\phi\\0\end{pmatrix}$$

$$= R^2\begin{pmatrix}\sin^2\theta\cos\phi\\\sin^2\theta\sin\phi\\\sin\theta\cos\theta\underbrace{(\cos^2\phi+\sin^2\phi)}_{=1}\end{pmatrix}$$

d.h. $$d\vec{O} = R^2\begin{pmatrix}\sin^2\theta\cos\phi\\\sin^2\theta\sin\phi\\\sin\theta\cos\theta\end{pmatrix}d\theta\,d\phi$$

d) $$\int_{\mathcal{F}}\nabla F\cdot d\vec{O} = \int_0^{2\pi}\int_0^{\pi/2}\frac{1}{R^5}\begin{pmatrix}3R^2\sin\theta\cos\theta\cos\phi\\3R^2\sin\theta\cos\theta\sin\phi\\3R^2\cos^2\theta-R^2\end{pmatrix}R^2\begin{pmatrix}\sin^2\theta\cos\phi\\\sin^2\theta\sin\phi\\\sin\theta\cos\theta\end{pmatrix}d\theta\,d\phi$$

$$= \frac{1}{R}\int_0^{2\pi}\int_0^{\pi/2}\Big(\underbrace{3\sin^3\theta\cos\theta\underbrace{(\cos^2\phi+\sin^2\phi)}_{=1}+3\sin\theta\cos^3\theta}_{=3\sin\theta\cos\theta=\frac{3}{2}\sin(2\theta)}-\underbrace{\sin\theta\cos\theta}_{=\frac{1}{2}\sin(2\theta)}\Big)d\theta\,d\phi$$

$$= \frac{1}{R}\cdot2\pi\cdot\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2} = \frac{2\pi}{R}\left(\frac{1}{2}-\left(-\frac{1}{2}\right)\right)$$

$$= \frac{2\pi}{R}$$

$\boxed{7}$

a) $f_x = \left(2x - 2x\left(\frac{3}{4} + x^2\right)\right) e^{-(x^2+y^2)}$

$= 2x\left(\frac{1}{4} - x^2\right) e^{-(x^2+y^2)}$

$f_y = -2y\left(\frac{3}{4} + x^2\right) e^{-(x^2+y^2)}$

$f_x = 0 \iff x = 0$ oder $x = \frac{1}{2}$ oder $x = -\frac{1}{2}$

$f_y = 0 \iff y = 0$

also drei kritische Punkte: $(0,0)$, $(\frac{1}{2},0)$, $(-\frac{1}{2},0)$

b)

$$f''(x,y) = \begin{pmatrix} \frac{1}{2} - 6x^2 - 4x^2\left(\frac{1}{4} - x^2\right) & -4xy\left(\frac{1}{4} - x^2\right) \\ -4xy\left(\frac{1}{4} - x^2\right) & (4y^2 - 2)\left(\frac{3}{4} + x^2\right) \end{pmatrix} e^{-(x^2+y^2)}$$

$f''(0,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}$ indefinit, also Sattel

$f''(\frac{1}{2},0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} e^{-1/4}$ negativ definit, also Maximum

$f''(-\frac{1}{2},0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} e^{-1/4}$ negativ definit, also Maximum

$\boxed{8}$

$$f'(x,y) = \begin{pmatrix} \sinh x \cos y & -\cosh x \sin y \\ \cosh x \sin y & \sinh x \cos y \end{pmatrix}$$

$$\det(f'(x,y)) = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x \cos^2 y + \sin^2 y + \sinh^2 x \sin^2 y$$
$$= \sinh^2 x + \sin^2 y$$

$$\det(f'(0,\tfrac{\pi}{2})) = 0 + 1^2 = 1 \neq 0$$ also dort lokal umkehrbar

$$f(0,\tfrac{\pi}{2}) = \begin{pmatrix} 1 \cdot 0 \\ 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f^{-1\,\prime}(0,0) = [f'(0,\tfrac{\pi}{2})]^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\boxed{9}$

a) $P(A) = 40\%$

$P(B) = 60\%$

$P(Q|A) = 80\%$

$P(Q|B) = 30\%$

b) $$P(B|Q) = \frac{P(Q|B)P(B)}{P(Q|B)P(B) + P(Q|A)P(A)}$$

$$= \frac{30\% \cdot 60\%}{30\% \cdot 60\% + 80\% \cdot 40\%} = \frac{18}{50} = \frac{9}{25}$$