Beispiel Potenzmenge

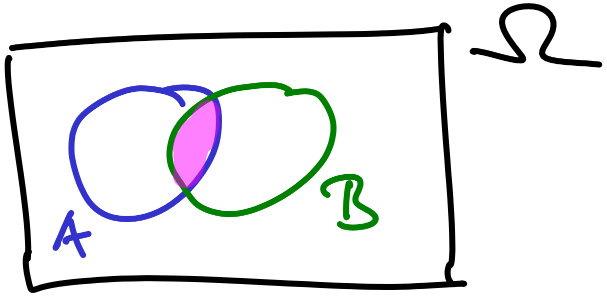
$$\Omega = \{x, y, z\}$$

$$\mathcal{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \Omega\}$$

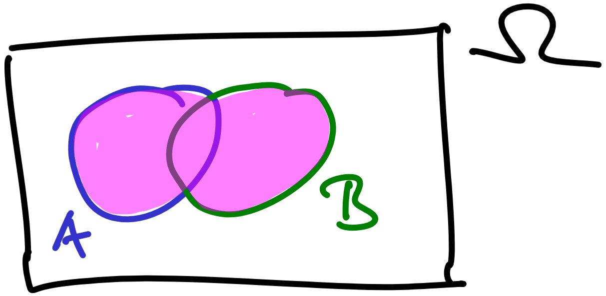
leere Menge

$\emptyset = \{\,\}$

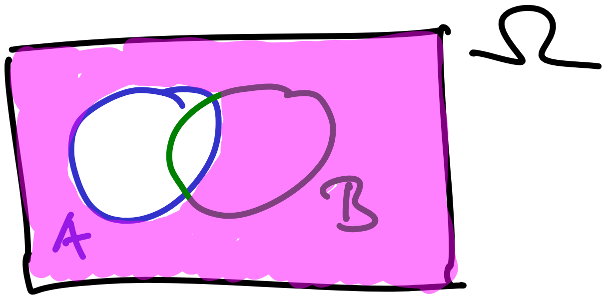
$= \{x, y, z\}$



$A \cap B$

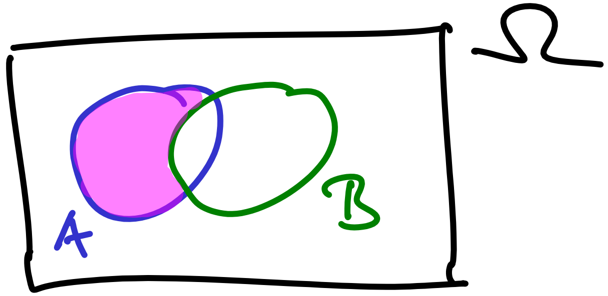


$A \cup B$



$A^c = \{\omega \in \Omega \mid \omega \notin A\}$

Komplement, Gegenereignis



$A \setminus B = \{\omega \in A$ <u>und</u> $\omega \notin B\}$

$A \setminus B = A \cap B^c$

Laplacescher W'Raum

W'keit für jedes Elementarereignis gleich groß

$\omega_1, \omega_2 \in \Omega$: $\mathbb{P}[\omega_1] = \mathbb{P}[\omega_2]$

Ereignis $A \subset \Omega$

$$\mathbb{P}[A] = \frac{\# A}{\# \Omega} = \frac{\text{Anzahl der Elemente von } A}{\text{Anzahl der Elemente von } \Omega}$$

$$= \frac{\text{"Anzahl günstiger Fälle"}}{\text{"Anzahl möglicher Fälle"}}$$

Würfel

$$\Omega = \{1,2,3,4,5,6\}$$

$A = \{1,3,5\}$, $B = \{3,4,5,6\}$

- Laplace-Würfel (fairer Würfel)

$$P[A] = \frac{3}{6} = \frac{1}{2}, \quad P[B] = \frac{4}{6} = \frac{2}{3}$$

- "unfairer" Würfel (kein Laplace'scher W'Raum)

z.B. $P[\{1\}] = \frac{1}{12}$, $P[\{2\}] = \frac{3}{12} = \frac{1}{4}$

$$P[\{j\}] = \frac{1}{6}, \quad j = 3,4,5,6$$

$$P[A] = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$$

$$P[B] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

Beweise für Folgerungen

(i) $$\mathbb{P}[A \cup A^c] = \mathbb{P}[\Omega] \underset{(1)}{=} 1$$

$$\underset{(2)}{=} \mathbb{P}[A] + \mathbb{P}[A^c]$$

$$\Rightarrow \mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

(ii) $$\mathbb{P}[\emptyset] \underset{(i)}{=} 1 - \mathbb{P}[\Omega] \underset{(1)}{=} 0$$

(iii) $$\mathbb{P}[A \cup B] = \mathbb{P}[A \cup (B \setminus A)]$$

$$\underset{(2)}{=} \mathbb{P}[A] + \underbrace{\mathbb{P}[B \setminus A] + \mathbb{P}[A \cap B]} - \mathbb{P}[A \cap B]$$

$$= \mathbb{P}[A] + \mathbb{P}[\underbrace{(B \setminus A) \cup (A \cap B)}_{= B}] - \mathbb{P}[A \cap B]$$

Beispiel für (iii) Würfel

$$A = \{1,3,5\}, \quad B = \{3,4,5,6\}$$

$$\mathbb{P}[A \cup B] = \mathbb{P}[\{1,3,4,5,6\}]$$

$$\overset{\text{Laplace-W.}}{=} \frac{5}{6}$$

andererseits

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

$$= \mathbb{P}[\{1,3,5\}] + \mathbb{P}[\{3,4,5,6\}] - \mathbb{P}[\{3,5\}]$$

$$\overset{\text{Laplace-W.}}{=} \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}$$

Beispiel für bedingte W'keiten

Würfel, A, B wie oben

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[\{3,5\}]}{\mathbb{P}[\{3,4,5,6\}]}$$

$$\underset{\text{Lapl. W}}{=} \frac{2/6}{4/6} = \frac{1}{2}$$

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[\{3,5\}]}{\mathbb{P}[\{1,3,5\}]}$$

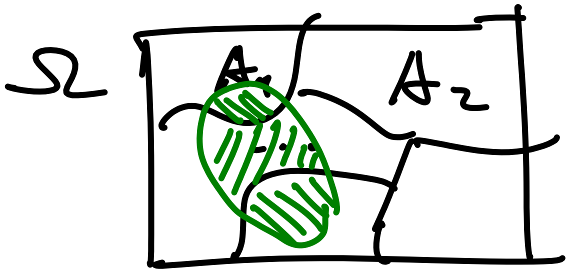
$$\underset{\text{Lapl. W}}{=} \frac{2/6}{3/6} = \frac{2}{3}$$

weiter $A = \{1,3,5\}$ fairer Würfel

$B = \{6\}$

$\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[B] = \frac{1}{6}$

$\mathbb{P}[A \cap B] = 0 \neq \frac{1}{2} \cdot \frac{1}{6} = \mathbb{P}[A] \cdot \mathbb{P}[B]$



$\Omega = \dot{\bigcup_{j=1}} A_j = A_1 \cup A_2 \cup \dots$

$A_1 \cap A_2 = \emptyset$ etc

(Satz von Bayes)

Beweis zum Satz von Bayes

$$\mathbb{P}[A_j \mid B] = \frac{\mathbb{P}[A_j \cap B]}{\mathbb{P}[B]} \cdot \frac{\mathbb{P}[A_j]}{\mathbb{P}[A_j]}$$

$$= \frac{\mathbb{P}[B \mid A_j] \cdot \mathbb{P}[A_j]}{\mathbb{P}[B]} \quad \longleftarrow \text{so auch nützlich}$$

$$= \frac{\mathbb{P}[B \mid A_j] \cdot \mathbb{P}[A_j]}{\mathbb{P}[B \cap \Omega]} = \frac{\mathbb{P}[B \mid A_j] \cdot \mathbb{P}[A_j]}{\mathbb{P}\Big[\underbrace{B \cap \Big(\bigcup_{k=1}^{n} A_k\Big)}_{\bigcup_{k=1}^{n} (B \cap A_k)}\Big]}$$

da disjunkt

$$= \frac{\mathbb{P}[B \mid A_j] \cdot \mathbb{P}[A_j]}{\sum_{k=1}^{n} \mathbb{P}[B \cap A_k]} = \frac{\mathbb{P}[B \mid A_j] \cdot \mathbb{P}[A_j]}{\sum_{k=1}^{n} \mathbb{P}[B \mid A_k] \cdot \mathbb{P}[A_k]}$$

Diagnostischer Test

$$P[A_1 | B] = \,?$$

$$P[A_1] = 0{,}01 \implies P[A_2] = 1 - 0{,}01 = 0{,}99$$

$$P[B | A_1] = 0{,}98$$

$$P[B^c | A_2] = 0{,}95 \implies P[B | A_2] = 0{,}05$$

$$P[A_1 | B] = \frac{P[B|A_1] \cdot P[A_1]}{P[B|A_1] \cdot P[A_1] + P[B|A_2] \cdot P[A_2]}$$

$$= \frac{0{,}98 \cdot 0{,}01}{0{,}98 \cdot 0{,}01 + 0{,}05 \cdot 0{,}99} \approx \frac{1}{6}$$

A_b: Autor wurde begnadigt

analog B_b, C_b

$\cancel{A_g}$: Autor wurde genannt

analog B_g, C_g

$$P[A_b \mid B_g] = ?$$

$$P[A_b] = P[B_b] = P[C_b] = \frac{1}{3}$$

$$P[B_g \mid A_b] = \frac{1}{2} = P[C_g \mid A_b]$$

$$P[B_g \mid B_b] = 0 = P[C_g \mid C_b]$$

$$P[B_g \mid C_b] = 1 = P[C_g \mid B_b]$$

$$P[A_b|B_g] = \frac{P[B_g|A_b] \cdot P[A_b]}{P[B_g|A_b] \cdot P[A_b] + P[B_g|B_g] P[B_b] + P[B_g|C_b] \cdot P[C_s]}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1/2}{1/2 + 1} = \frac{1}{3}$$