$\boxed{1}$ $\underline{n=1:}$ links: $\sum_{\nu=1}^{1} \frac{1}{\nu(\nu+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$

rechts: $\frac{1}{1+1} = \frac{1}{2}$ o.k.

$\underline{n \to n+1:}$

$$\sum_{\nu=1}^{n+1} \frac{1}{\nu(\nu+1)} = \sum_{\nu=1}^{n} \frac{1}{\nu(\nu+1)} + \frac{1}{(n+1)(n+2)}$$

$$\underset{\text{I.V.}}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{1}{n+1}\left(n + \frac{1}{n+2}\right)$$

$$= \frac{1}{n+1} \frac{n^2+2n+1}{n+2} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} \quad \square$$

$\boxed{2}$ a)

$$\sum_{\nu=2}^{n} x^{n+\nu} = x^n \sum_{\nu=2}^{n} x^\nu = x^n \left(\sum_{\nu=0}^{n} x^\nu - x - 1 \right)$$

$$\underset{x \neq 1}{=} x^n \left(\frac{x^{n+1}-1}{x-1} - x - 1 \right) = x^n \frac{x^{n+1}-1-(x^2-1)}{x-1}$$

$$= x^n \frac{x^{n+1}-x^2}{x-1}$$

b)

$$\sum_{\nu=0}^{n} \sum_{\mu=\nu}^{n} \frac{x^\mu}{\mu+1} = \sum_{\mu=0}^{n} \sum_{\nu=0}^{\mu} \frac{x^\mu}{\mu+1}$$

$$= \sum_{\mu=0}^{n} \frac{x^\mu}{\mu+1} \underbrace{\sum_{\nu=0}^{\mu} 1}_{=\mu+1} = \sum_{\mu=0}^{n} x^\mu \underset{x \neq 1}{=} \frac{x^{n+1}-1}{x-1}$$

$\boxed{2}$ c)

$$\sum_{\nu=0}^{n} \sin(\nu x) = \operatorname{Im} \sum_{\nu=0}^{n} e^{i\nu x}$$

$$= \operatorname{Im} \frac{e^{i(n+1)x} - 1}{e^{ix} - 1}$$

$x \in (0, 2\pi)$

$$= \operatorname{Im} \frac{e^{i\frac{n+1}{2}x}\left(e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}\right)}{e^{i\frac{x}{2}}\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)}$$

$$= \operatorname{Im} e^{i\frac{n}{2}x} \frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}$$

$\boxed{3}$ a)

e-Reihe

$$\lim_{x\to 0} \frac{(e^x - x - 1)^3}{4x^8 - 3x^6} = \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + \ldots - x - 1\right)^3}{4x^8 - 3x^6}$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^2}{2} + \ldots\right)^3}{4x^8 - 3x^6} = \lim_{x\to 0} \frac{\frac{x^6}{8} + \ldots}{4x^8 - 3x^6}$$

$$= -\frac{1}{24}$$

b) $$\lim_{n\to\infty} \left(n - \sqrt{n^2 - n}\right) \frac{n + \sqrt{n^2 - n}}{n + \sqrt{n^2 - n}}$$

$$= \lim_{n\to\infty} \frac{n^2 - (n^2 - n)}{n + \sqrt{n^2 - n}} = \lim_{n\to\infty} \frac{n}{n + \sqrt{n^2 - n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 + \sqrt{1 - \frac{1}{n}}} = \frac{1}{2}$$

c) $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{\frac{n}{2}-5}$

$= \left(\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{-5}\right)\cdot\left(\underbrace{\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{n}}_{=e^{-2}}\right)^{\frac{1}{2}}$

$= e^{-1} = \frac{1}{e}$

d) $\lim_{n\to\infty}\left((-1)^n+(-1)^{n+1}\right) = \lim_{n\to\infty}\left((-1)^n-(-1)^n\right)=0$

$\boxed{4}$ $f(x) = 7^x = e^{\log(7^x)} = e^{x\log 7}$

$f'(x) = \log 7\cdot e^{x\log 7} = \log 7\cdot 7^x$

$g(x) = \sqrt{x\sqrt{x\sqrt{x}}} = x^{\frac{1}{2}}\,x^{\frac{1}{4}}\,x^{\frac{1}{8}} = x^{7/8}$

$g'(x) = \frac{7}{8}x^{-\frac{1}{8}} = \frac{7}{8\sqrt[8]{x}}$

$h(x) = \int_x^{x^2}\sin(t^2)\,dt$

$h'(x) = \sin(x^4)\cdot 2x - \sin(x^2)$

$\boxed{5}$ a)

$\int_{-3}^{5}\frac{x}{1+x^2}\,dx$ $\qquad u = 1+x^2$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$= \int_{1+(-3)^2}^{1+5^2}\frac{du}{2u} = \frac{1}{2}\log u\Big|_{10}^{26} = \frac{1}{2}\log\frac{13}{5}$

$\boxed{5}$ b)

$$\underbrace{\int_0^{2\pi} \cos^2 x \, dx}_{} \underset{\text{p.I.}}{=} \underbrace{\sin x \cos x \Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} \sin^2 x \, dx$$

$$= \int_0^{2\pi} (1 - \cos^2 x) \, dx = \underbrace{\int_0^{2\pi} 1 \, dx}_{= x \big|_0^{2\pi} = 2\pi} - \int_0^{2\pi} \cos^2 x \, dx$$

$$\Rightarrow \; = \frac{1}{2} \cdot 2\pi = \pi$$

c) $$\int_e^{e^e} \frac{dx}{x \log x} \qquad u = \log x \,, \quad du = \frac{dx}{x}$$

$$= \int_{\log e}^{\log e^e} \frac{du}{u} = \int_1^e \frac{du}{u} = \log u \Big|_1^e = 1$$

$\boxed{6}$ a) Definitionsbereich: $\mathbb{D} = \mathbb{R} \setminus \{-1, 1\}$

Nennernullstellen

$\underline{x = -1}$: Pol mit VZW, nicht stetig fortsetzbar

$\underline{x = 1}$: "$\frac{0}{0}$"

$$\lim_{x \to 1} f(x) \underset{\text{l'Hospital}}{=} \lim_{x \to 1} \frac{3x^2 - 2x + 1}{2x} = 1$$

stetig fortsetzbar in $x = 1$ durch $f(1) = 1$

$\boxed{6}$ b) senkrechte Asymptote bei $x = -1$ (Pol s.o.)

keine waagerechten Asymptoten, da
Zählergrad $\neq$ Nennergrad

$$(x^3 - x^2 + x - 1) : (x^2 - 1) = x - 1 + \frac{2(x-1)}{x^2-1}$$

$$\begin{array}{l} x^3 \qquad - x \\ \hline -x^2 + 2x - 1 \\ -x^2 \qquad + 1 \\ \hline 2x - 2 \end{array}$$

also $f(x) = x - 1 + \dfrac{2}{x+1}$

mit schiefer Asymptote $y = x - 1$

c) $f(x) = 0 \iff x - 1 = -\dfrac{2}{x+1}$

$\iff$ $x^2 - 1 = -2 \iff x^2 = -1$

$x = -1$

also keine reellen Nullstellen

$$f'(x) = 1 - \frac{2}{(x+1)^2}$$

$$f''(x) = \frac{4}{(x+1)^3}$$

$$f'(x) = 0 \iff (x+1)^2 = 2 \iff x+1 = \pm\sqrt{2}$$

$$\iff x = -1 \pm \sqrt{2}$$

$$f''(-1\pm\sqrt{2}) = \pm \frac{4}{2\sqrt{2}} = \pm \frac{2}{\sqrt{2}} \gtrless 0$$

$\boxed{6}$ c) $\Rightarrow$ Minimum bei $x = -1 + \sqrt{2}$

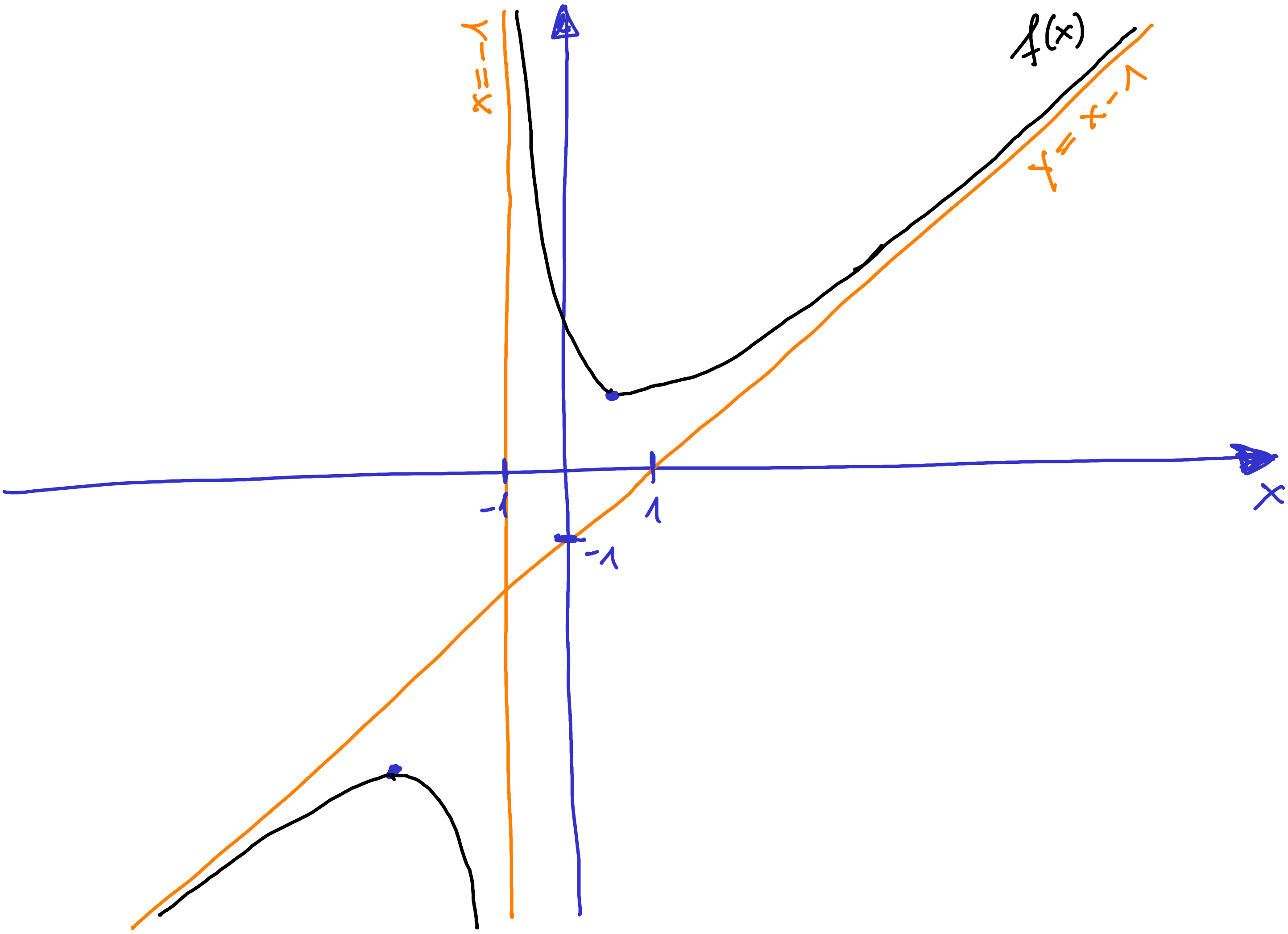
Maximum bei $x = -1 - \sqrt{2}$

$$f(-1 \pm \sqrt{2}) = -2 \pm \sqrt{2} + \frac{2}{\pm\sqrt{2}}$$

$$= -2 \pm \sqrt{2} \pm \sqrt{2} = -2 \pm 2\sqrt{2}$$

also Tiefpunkt $(-1+\sqrt{2}, -2+2\sqrt{2})$

und Hochpunkt $(-1-\sqrt{2}, -2-2\sqrt{2})$



7 a)

$$\det\begin{pmatrix} 1 & 0 & 2 & \alpha \\ -1 & 1 & 1 & 0 \\ 0 & -1 & \alpha & 3 \\ 1 & 1 & 3 & \alpha \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & 2 & \alpha \\ 0 & 1 & 3 & \alpha \\ 0 & -1 & \alpha & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 3 & \alpha \\ 0 & 3+\alpha & 3+\alpha \\ 0 & -2 & -\alpha \end{pmatrix} = (3+\alpha)(-\alpha) - (-2)(3+\alpha)$$

$$= (3+\alpha)(2-\alpha)$$

b) Die Vektoren sind l. a.

$\iff$ Die Determinante aus (a) ist gleich Null

$\iff$ $\alpha = -3$ <u>oder</u> $\alpha = 2$

8 a)

$$\frac{1}{7+x} = \frac{1}{7}\,\frac{1}{1+\frac{x}{7}} \underset{\left|\frac{x}{7}\right|<1}{\overset{\text{geom. Reihe}}{=}} \frac{1}{7}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{x^{\nu}}{7^{\nu}}$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu}\,7^{-(\nu+1)}\,x^{\nu} \quad \text{für } |x| < 7$$

$\boxed{8}$ b)

$$e^{-x^2} \underset{\forall x \in \mathbb{R}}{\overset{\text{e-Reihe}}{=}} \sum_{\nu=0}^{\infty} \frac{(-x^2)^\nu}{\nu!} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu x^{2\nu}}{\nu!} \quad \forall x \in \mathbb{R}$$

c)

$$\frac{e^x}{1-x} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \cdot \sum_{\mu=0}^{\infty} x^\mu$$

$n = \nu + \mu : 0 \ldots \infty$

$\nu : 0 \ldots n$

$\mu = n - \nu$

$\left.\begin{array}{l}\text{e-Reihe } \forall x \in \mathbb{R} \\ \text{geom. Reihe für } |x|<1\end{array}\right\}$ gesamt für $|x|<1$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \frac{x^n}{\nu!} = \sum_{n=0}^{\infty} x^n \sum_{\nu=0}^{n} \frac{1}{\nu!} \quad \text{für } |x|<1$$

d)

$$\frac{\sin x}{x} \underset{\forall x \in \mathbb{R}}{\overset{\text{sin-Reihe}}{=}} \frac{1}{x} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} x^{2\nu+1}$$

$$= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} x^{2\nu} \quad \forall x \in \mathbb{R}$$

e)

$$f(x) := \frac{\sin x}{x}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{allg. Taylorreihe}$$

vergleiche $n = 100$ - Term mit $\nu = 50$ - Term aus (d):

$$\frac{f^{(100)}(0)}{100!} = \frac{(-1)^{50}}{101!} = \frac{1}{101!}$$

$$\Rightarrow f^{(100)}(0) = \frac{100!}{101!} = \frac{1}{101}$$

$\boxed{9}$ a)

$$\begin{pmatrix} e^{i\phi} & -1 & 1 \\ 1 & e^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 1 \\ -1 & e^{-i\phi} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & e^{i\phi} - e^{-i\phi} \\ e^{-i\phi} - e^{i\phi} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2i\sin\phi \\ -2i\sin\phi & 2 \end{pmatrix}$$

b) $AB = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \iff \sin\phi = 0$

$\iff \phi = \pi k \;, \; k \in \mathbb{Z}$

$\boxed{10}$ Normalenvektor für E_1:

$$\vec{n}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-1 \\ 2+3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$$

normiert: $\vec{n}_{10} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Gleichung für E_1:

$$\vec{n}_{10} \left[\vec{x} - \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \right] = 0$$

$$\iff \vec{n}_{10}\vec{x} = \frac{1}{\sqrt{3}}(3+1-5) = -\frac{1}{\sqrt{3}}$$

$\boxed{\text{zu } 10}$

Normalenvektor für E_2:

$$\vec{n}_2 = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+21 \\ 28-3 \\ -9-16 \end{pmatrix} = \begin{pmatrix} 25 \\ 25 \\ -25 \end{pmatrix}$$

normiert:

$$\vec{n}_{20} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \vec{n}_{10}$$

Gleichung für E_2:

$$\vec{n}_{10} \left[\vec{x} - \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} \right] = 0$$

$$\Longleftrightarrow \quad \vec{n}_{10} \vec{x} = \frac{1}{\sqrt{3}} (3 + 6 - 10) = -\frac{1}{\sqrt{3}}$$

$$\Longrightarrow \quad E_1 = E_2$$