$\boxed{1}$ zu zeigen: $\sum_{\nu=1}^{n} \nu = \frac{n(n+1)}{2} \quad \forall\, n \in \mathbb{N}$

mit vollständiger Induktion

$\underline{n=1}$: Induktionsanfang

links: $\sum_{\nu=1}^{1} \nu = 1$, rechts: $\frac{1(1+1)}{2} = 1$ ✓

$\underline{n \to n+1}$: Induktionsschritt

$$\sum_{\nu=1}^{n+1} \nu = \sum_{\nu=1}^{n} \nu + n+1 \underset{\text{I.V.}}{=} \frac{n(n+1)}{2} + n+1$$

$$= (n+1) \cdot \left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2} \qquad \square$$

$\boxed{2}$

a) $$\sum_{\nu=0}^{2n} x^{3\nu} \underset{\text{geom. Reihe}}{=} \begin{cases} \dfrac{(x^3)^{2n+1}-1}{x^3-1}, & x^3 \neq 1 \\ 2n+1, & x^3 = 1 \end{cases}$$

$$= \begin{cases} \dfrac{x^{6n+3}-1}{x^3-1}, & x \neq 1 \\ 2n+1, & x = 1 \end{cases}$$

b) $$\sum_{n=1}^{10} \sum_{\nu=1}^{n} \sum_{\mu=1}^{\nu} \frac{1}{n-\mu+1} = \sum_{n=1}^{10} \sum_{\mu=1}^{n} \sum_{\nu=\mu}^{n} \frac{1}{n-\mu+1}$$

$$= \sum_{n=1}^{10} \sum_{\mu=1}^{n} \frac{1}{n-\mu+1} \underbrace{\sum_{\nu=\mu}^{n} 1}_{=n-\mu+1} = \sum_{n=1}^{10} \underbrace{\sum_{\mu=1}^{n} 1}_{=n}$$

$$\underset{\text{Weiner Gauß}}{=} \frac{10 \cdot 11}{2} = 55$$

c) $$\sum_{k=0}^{n} \sum_{\ell=0}^{n} \binom{\ell}{k} = \sum_{\ell=0}^{n} \sum_{k=0}^{n} \binom{\ell}{k} = \sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \binom{\ell}{k}$$

$\binom{\ell}{k} = 0$ für $k > \ell$

$$\underset{\text{Binomi}}{=} \sum_{\ell=0}^{n} 2^{\ell} \underset{\text{geom. Reihe}}{=} \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1$$

3

a) $$\lim_{n\to\infty} \frac{n^2 \sin(n^5)}{(n-1)^3} = \lim_{n\to\infty} \frac{n^2 \sin(n^5)}{n^3\left(1-\frac{1}{n}\right)^3}$$

$$= \lim_{n\to\infty} \frac{\sin(n^5)}{n\left(1-\frac{1}{n}\right)^3} = 0 \quad \text{da } |\sin(n^5)| \leq 1$$

b) $\lim_{n\to\infty} (-1)^n \left(2-\frac{1}{n}\right)$ existiert nicht, da

$$\lim_{n\to\infty} (-1)^{2n}\left(2-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(2-\frac{1}{2n}\right) = 2$$

$$\neq \lim_{n\to\infty} (-1)^{2n+1}\left(2-\frac{1}{2n+1}\right) = -\lim\left(2-\frac{1}{2n+1}\right) = -2$$

c) $$\lim_{n\to\infty} \left(n^2-\sqrt{n^4-n^2}\right) \cdot \frac{n^2+\sqrt{n^4-n^2}}{n^2+\sqrt{n^4-n^2}}$$

$$= \lim_{n\to\infty} \frac{n^4-(n^4-n^2)}{n^2+\sqrt{n^4-n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n^2\sqrt{1-\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n^2}}} = \frac{1}{1+1} = \frac{1}{2}$$

andere Möglichkeit mit Taylorreihe

$$(1+x)^{1/2} = 1 + \frac{x}{2} + o(x)$$

$$\lim_{n\to\infty} \left(n^2-\sqrt{n^4-n^2}\right) = \lim_{n\to\infty} \left(n^2 - n^2\left(1+\frac{1}{n^2}\right)^{1/2}\right)$$

$$= \lim_{n\to\infty} \left[n^2 - n^2\left(1+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{2}+o(1)\right] = \frac{1}{2}$$

d) $$\lim_{n\to\infty} \left(1-\frac{2}{n}\right)^{3n+1} = \lim_{n\to\infty} \left[\Bigg(\underbrace{\left(1-\frac{2}{n}\right)^n}_{\to e^{-2}}\Bigg)^3 \cdot \underbrace{\left(1-\frac{2}{n}\right)}_{\to 1}\right]$$

$$= \left(e^{-2}\right)^3 = e^{-6} = \frac{1}{e^6}$$

$\boxed{\text{zu } 3}$

e) $$\lim_{x\to 0} \frac{(\sin x - x)^2}{2x^9 - x^6} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + o(x^3) - x\right)^2}{2x^9 - x^6}$$

$$= \lim_{x\to 0} \frac{\frac{x^6}{36} + o(x^6)}{2x^9 - x^6} = \lim_{x\to 0} \frac{\frac{1}{36} + o(1)}{2x^3 - 1}$$

$$= -\frac{1}{36}$$

<u>oder</u> $6\times$ l'Hospital, aber das macht keinen Spaß...

$\boxed{4}$

a) $$f(x) = x^{\sqrt{x}} = e^{\log x^{\sqrt{x}}} = e^{\sqrt{x}\log x}$$

$$f'(x) = e^{\sqrt{x}\log x} \cdot \left(\frac{1}{2\sqrt{x}} \log x + \sqrt{x}\,\frac{1}{x} \right)$$

$$= x^{\sqrt{x}}\,\frac{1}{\sqrt{x}} \left(\frac{1}{2}\log x + 1 \right)$$

b) $$g(x) = \sin(\arccos x) = \sqrt{1 - \cos^2(\arccos x)}$$

positive Wurzel ist z.B. richtig für den Hauptzweig des $\arccos : [-1,1] \longrightarrow [0,\pi]$, denn

$\sin : [0,\pi] \longrightarrow [0,1]$ also positiv

<u>also:</u> $$g(x) = \sqrt{1 - x^2}$$

$$g'(x) = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

c) $$h(x) = \int_{-x^2}^{x^2} e^{t^2}\,dt$$

$$h'(x) = e^{x^4} \cdot 2x - e^{x^4}(-2x) = 4x\,e^{x^4}$$

$\boxed{5}$ i)

a) $$\frac{3}{8+x^3} = \frac{3}{8}\,\frac{1}{1+\frac{x^3}{8}} \underset{\substack{\text{geom.}\\ \text{Reihe}}}{=} \frac{3}{8}\sum_{\nu=0}^{\infty}\left(-\frac{x^3}{8}\right)^{\nu}$$

für $\left|\frac{x^3}{8}\right| < 1 \iff \underline{|x| < 2}$

$$= 3\sum_{\nu=0}^{\infty}(-1)^{\nu}\, 8^{-\nu-1}\, x^{3\nu}$$

b) $$\frac{1+x}{1-x} \underset{\substack{\text{geom.}\\ \text{Reihe}}}{=} (1+x)\sum_{\nu=0}^{\infty} x^{\nu} \quad \text{für } \underline{|x| < 1}$$

$$= \sum_{\nu=0}^{\infty} x^{\nu} + \sum_{\nu=0}^{\infty} x^{\nu+1} = \sum_{\nu=0}^{\infty} x^{\nu} + \sum_{\nu=1}^{\infty} x^{\nu}$$

$$= 1 + 2\sum_{\nu=1}^{\infty} x^{\nu}$$

c) $$\frac{\cos x - e^{-\sqrt{a}\,x^2}}{x^2}$$

$$= \frac{\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu)!}x^{2\nu} - \sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}a^{\nu/2}}{\nu!}x^{2\nu}}{x^2} \qquad \underline{\forall\, x \in \mathbb{R}}$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu}\underbrace{\left(\frac{1}{(2\nu)!} - \frac{a^{\nu/2}}{\nu!}\right)}_{=0 \text{ für } \nu=0} x^{2(\nu-1)}$$

$$= \sum_{\nu=1}^{\infty}(-1)^{\nu}\left(\frac{1}{(2\nu)!} - \frac{a^{\nu/2}}{\nu!}\right) x^{2(\nu-1)}$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu+1}\left(\frac{1}{(2\nu+2)!} - \frac{a^{\frac{\nu+1}{2}}}{(\nu+1)!}\right) x^{2\nu}$$

zu 5

ii) laut c gilt

$$\frac{\cos x - e^{-\sqrt{a}\,x^2}}{x^2} = -\left(\frac{1}{2} - \frac{\sqrt{a}}{1}\right) + \left(\frac{1}{4!} - \frac{a}{2}\right)x^2 - \left(\frac{1}{6!} - \frac{a^{3/2}}{3!}\right)x^4 + o(x^4)$$

d. h.

Minimum, falls $\frac{1}{4!} - \frac{a}{2} > 0 \iff a < \frac{1}{12}$

Maximum, falls $\frac{1}{4!} - \frac{a}{2} < 0 \iff a > \frac{1}{12}$

für $a = \frac{1}{12}$ bestimmt der x^4-Term das Verhalten:

$$-\left(\frac{1}{6!} - \frac{a^{3/2}}{3!}\right) = \frac{1}{3!}\left(\left(\frac{1}{12}\right)^{3/2} - \frac{1}{120}\right)$$

$$= \frac{1}{3!}\left(\frac{1}{\sqrt{12}^3} - \frac{1}{120}\right)$$

$$= \frac{1}{3!}\left(\frac{1}{12\cdot\sqrt{12}} - \frac{1}{120}\right)$$

$$= \frac{1}{3!\cdot 12}\underbrace{\left(\frac{1}{\sqrt{12}} - \frac{1}{10}\right)}_{>0 \text{ da } \sqrt{12} < 10}$$

d. h. dann liegt ein Minimum vor.

$\boxed{6}$ $f(x) = \dfrac{1-x^3}{x^2-1}$

a) Definitionsbereich: $\mathbb{R} \setminus \{-1, 1\}$ Nennernullstellen

$x=1$ ist auch Zählernullstelle, f dort stetig fortsetzbar durch

$$\lim_{x\to 1} \frac{1-x^3}{x^2-1} \underset{\text{l'H.}}{=} \lim_{x\to 1} \frac{-3x^2}{2x} = -\frac{3}{2}$$

d.h. $f(x) := \begin{cases} \frac{1-x^3}{x^2-1}, & x \neq 1 \\ -\frac{3}{2}, & x = 1 \end{cases}$

ist definiert für $x \in \mathbb{R} \setminus \{-1\}$.

b) Senkrechte Asymptote: $x = -1$ Pol, Nenner dort Null, Zähler nicht

keine waagerechten Asymptoten

Schiefe Asymptote durch Polynomdivision:

$$\underbrace{(-x^3+1):(x^2-1) = -x + \frac{1-x}{x^2-1} = -x - \frac{1}{x+1}}_{=f(x)}$$

$$\begin{array}{l} -x^3 + x \\ \hline -x+1 \end{array}$$

Schiefe Asymptote: $y = -x$

c) Nullstellen:

$f(x) = 0 \iff -x - \frac{1}{x+1} = 0 \iff -x^2 - x - 1 = 0$

$\iff x = \frac{1 \pm \sqrt{1-4}}{-2} \notin \mathbb{R}$ also keine reellen Nullstellen

Extrema:

$f'(x) = -1 + \frac{1}{(x+1)^2}$

$f'(x) = 0 \underset{x \neq 1}{\iff} (x+1)^2 = 1 \iff x=0$ oder $x=-2$

$f''(x) = -\frac{2}{(x+1)^3}$

$\boxed{\text{zu } 6}$ $f''(x) = -\frac{2}{(x+1)^3}$

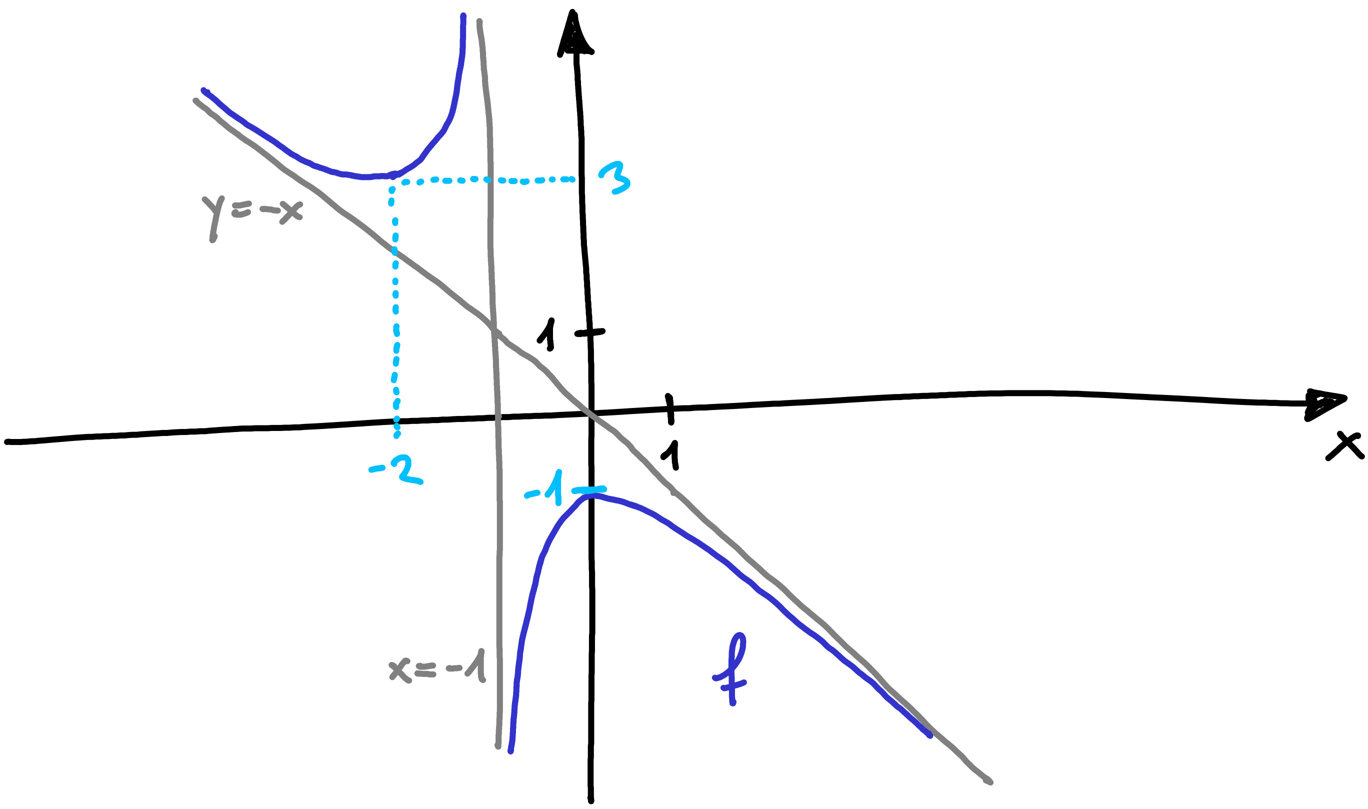
zu c) $f''(0) = -2$ also Maximum bei $x=0$

$f''(-2) = 2$ also Minimum bei $x=-2$

Funktionswerte: $f(0) = -1$, $f(-2) = \frac{9}{3} = 3$

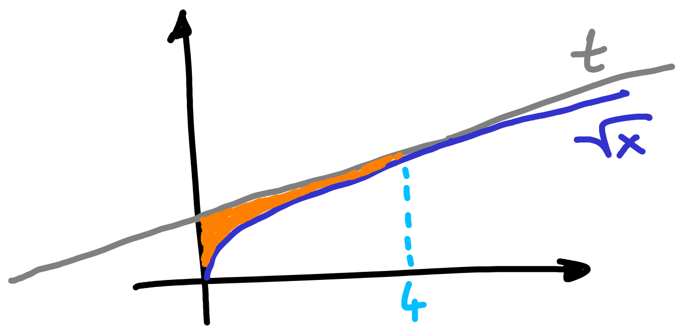
also Hochpunkt $(0,-1)$ und Tiefpunkt $(-2,3)$

d)



$\boxed{7}$ $f(x) = \sqrt{x}$, $f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(4) = \frac{1}{4}$



Tangente:

$$t(x) = f(4) + f'(4)(x-4)$$
$$= 2 + \frac{x-4}{4}$$
$$= 1 + \frac{x}{4}$$

gesuchte Fläche

$$A = \int_0^4 [t(x) - f(x)]\,dx$$
$$= \left[x + \frac{x^2}{8} - \frac{2}{3}x^{3/2}\right]_0^4$$
$$= 4 + 2 - \frac{16}{3} = \frac{2}{3}$$

$\boxed{8}$

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 3 & 4 & 0 & 2 & 6 \\ 0 & 6 & 4 & 0 & 3 & 7 \\ 7 & 0 & 0 & 8 & 4 & 8 \end{array}\right) \begin{array}{l} -7 \\ -2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 3 & 4 & 0 & 2 & 6 \\ 0 & 0 & -4 & 0 & -1 & -5 \\ 0 & 0 & 0 & -6 & -3 & -27 \end{array}\right) \begin{array}{l} \\ \\ \\ 1/3 \end{array}$$

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & 0 & -1 & -5 \\ 0 & 0 & 0 & -6 & -3 & -27 \end{array}\right) \begin{array}{l} \\ |\cdot 1/3 \\ |\cdot(-1/4) \\ |\cdot(-1/6) \end{array}$$

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/4 & 5/4 \\ 0 & 0 & 0 & 1 & 1/2 & 9/2 \end{array}\right)$$

also

$$X = \begin{pmatrix} 0 & -4 \\ 1/3 & 1/3 \\ 1/4 & 5/4 \\ 1/2 & 9/2 \end{pmatrix}$$

$\boxed{9}$

a) $$\det A = \det\begin{pmatrix} 1 & iz & 0 \\ 0 & i+1 & z \\ i & z & i-1 \end{pmatrix}$$

$$= -1-1-z+i(iz^2) = -2-2z^2$$

$$= -2(1+z^2)$$

b) A invertierbar

$\Leftrightarrow \det A \neq 0$

$\Leftrightarrow z \neq \pm i$

c) $$\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} i \\ i+1 \\ 1 \end{pmatrix}, \quad \vec{a}_3 = \begin{pmatrix} 0 \\ z \\ i-1 \end{pmatrix}$$

$$\vec{c}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \qquad \text{da } \|\vec{a}_1\|^2 = 2$$

$$\vec{b}_2 = \vec{a}_2 - \Big[\underbrace{\tfrac{1}{\sqrt{2}}(1,0,-i)\begin{pmatrix} i \\ i+1 \\ 1 \end{pmatrix}}_{=0}\Big]\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \vec{a}_2$$

$$\vec{c}_2 = \frac{1}{2}\begin{pmatrix} i \\ i+1 \\ 1 \end{pmatrix} \qquad \text{da } \|\vec{a}_2\|^2 = 4$$

$$\vec{b}_3 = \vec{a}_3 - \Big[\underbrace{\tfrac{1}{\sqrt{2}}(1,0,-i)\begin{pmatrix} 0 \\ 1 \\ i-1 \end{pmatrix}}_{\frac{1+i}{\sqrt{2}}}\Big]\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} - \Big[\underbrace{\tfrac{1}{2}(-i,-i+1,1)\begin{pmatrix} 0 \\ 1 \\ i-1 \end{pmatrix}}_{=0}\Big]\frac{1}{2}\begin{pmatrix} i \\ i+1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ i-1 \end{pmatrix} - \frac{1+i}{2}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1-i \\ 2 \\ i-1 \end{pmatrix}$$

$$\vec{c}_3 = \frac{\vec{b}_3}{\|\vec{b}_3\|} = \frac{1}{2\sqrt{2}}\begin{pmatrix} -1-i \\ 2 \\ i-1 \end{pmatrix}$$

Die gesuchte ON-Basis ist $\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$

$\boxed{10}$

a) $A, B \in M$ d.h. $\det A = \det B = 1$

$\det(AB) = \det A \cdot \det B = 1$ also $AB \in M$

b) neutrales Element: Einheitsmatrix

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\det I = 1$ also $I \in M$

c) $A \in M$

$\Rightarrow \det A = 1 \neq 0$

$\Rightarrow A$ invertierbar

weiter gilt $\det A^{-1} = 1$ wegen

$$\det(A \cdot A^{-1}) = \det I = 1$$
$$= \underbrace{\det A}_{=1} \cdot \det A^{-1} = \det A^{-1}$$

d.h. $A^{-1} \in M$

d) es genügt ein Beispiel...

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $BA = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \neq AB$