z. B.

$$f(x,y) = y^2 \sin x$$

$$\frac{\partial f}{\partial x}(x,y) = y^2 \cos x$$

$$\frac{\partial f}{\partial y}(x,y) = 2y \sin x$$

Calc-zilla: $f(x,y) = y^x = e^{\log(y^x)} = e^{x \cdot \log y}$

$$\frac{\partial f}{\partial x} = e^{x \cdot \log y} \cdot \log y = y^x \cdot \log y$$

$$f: \mathbb{R}^2 \to \mathbb{R}$$

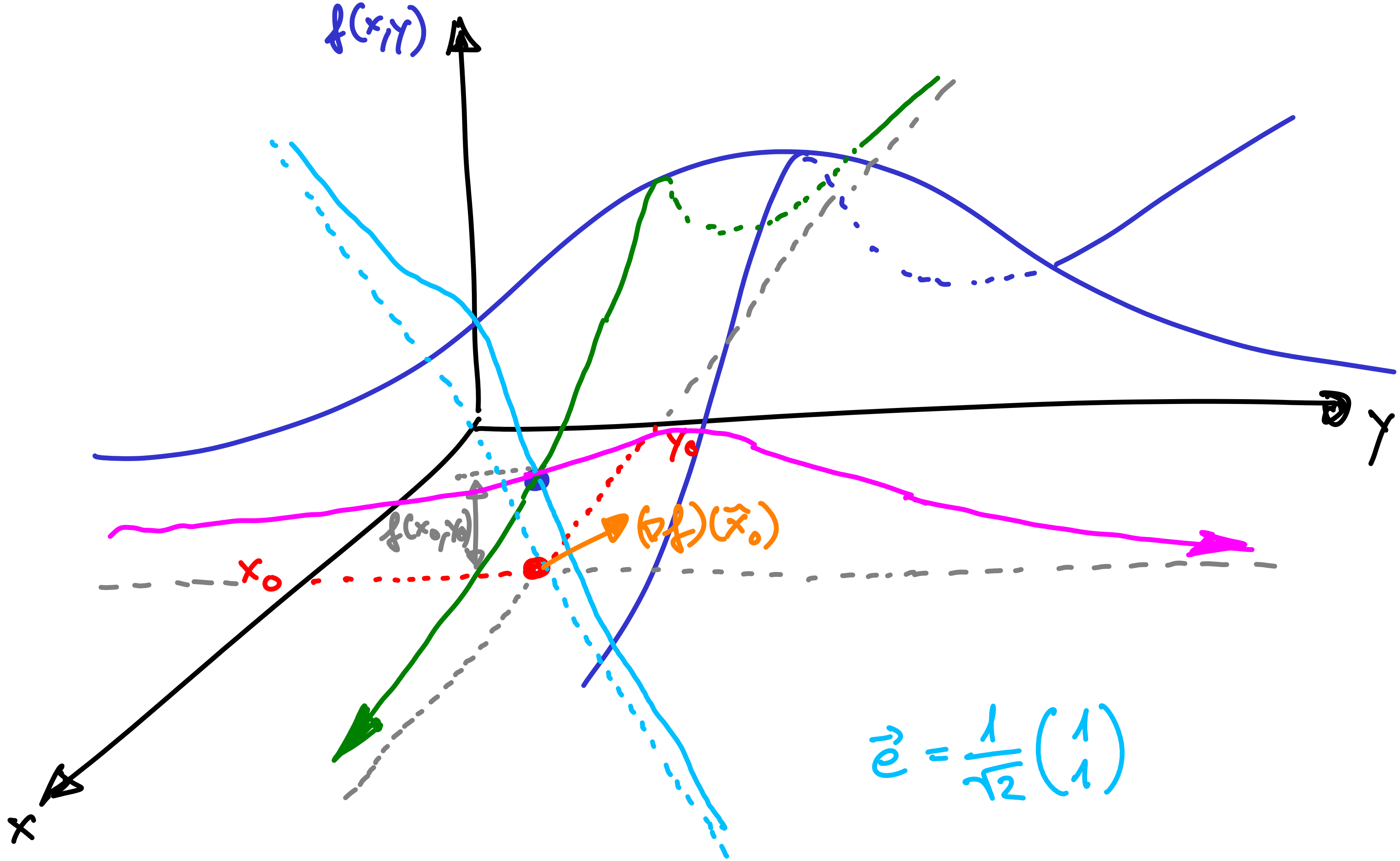
$$(x_1, x_2) \mapsto f(x_1, x_2) = f(\vec{x}) \quad , \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_1) - f(\vec{x})}{h}$$

$$\vec{x} + h\vec{e}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + h \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + h \\ x_2 \end{pmatrix}$$

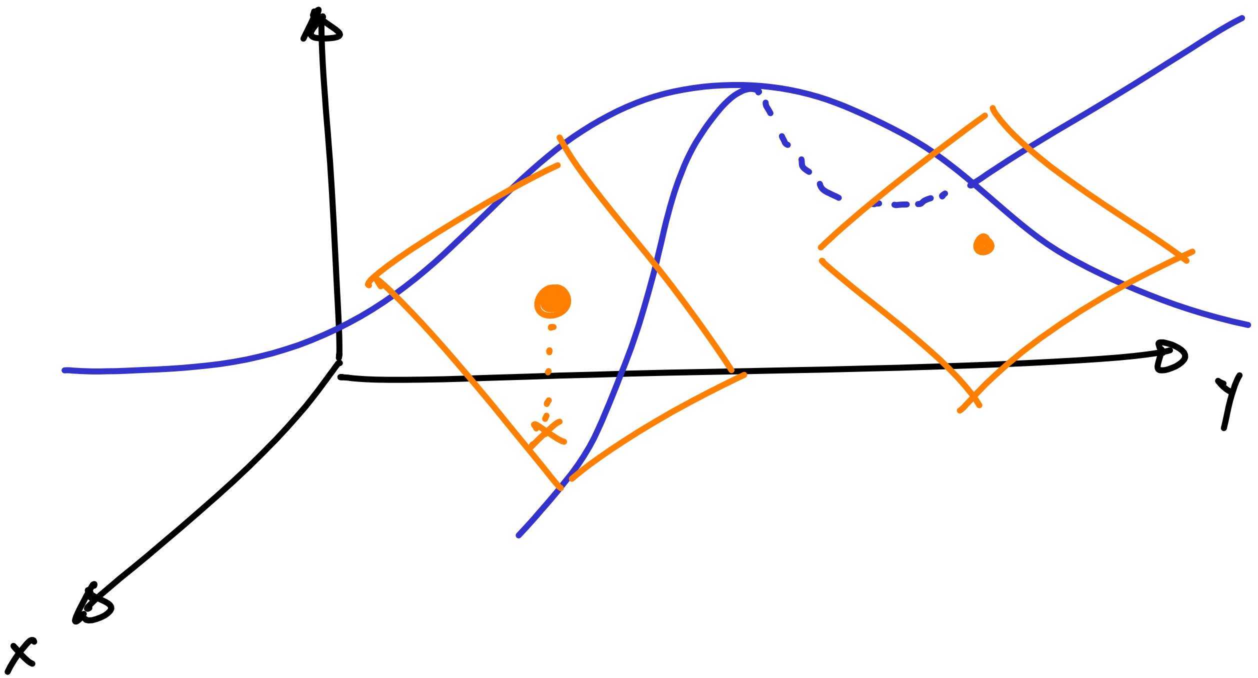
$$= \lim_{h \to 0} \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h} = \frac{\partial f}{\partial x_1}$$



$\frac{\partial f}{\partial x}(x_0, y_0)$ = Steigung des grünen Weges an der Stelle (x_0, y_0) < 0

$\frac{\partial f}{\partial y}(x_0, y_0)$ = Steigung des rosa Weges an der Stelle (x_0, y_0) > 0

$$\frac{\partial f}{\partial \vec{e}}(x_0, y_0) = \text{Steigung des hellblauen Weges an der Stelle } (x_0, y_0)$$



$$T_{\vec{x}_0}(\vec{x}) = f(\vec{x}_0) + \underbrace{\vec{a}\cdot(\vec{x}-\vec{x}_0)}_{= a_1(x_1 - x_{01}) + a_2(x_2 - x_{02}) + \dots}$$

$$\frac{\partial T_{\vec{x}_0}}{\partial x_1}(\vec{x}_0) = a_1 \stackrel{!}{=} \frac{\partial f}{\partial x_1}(\vec{x}_0)$$

$$\frac{\partial T}{\partial x_j}(\vec{x}_0) = a_j \stackrel{!}{=} \frac{\partial f}{\partial x_j}(\vec{x}_0)$$

d.h. $\vec{a} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}_0) \end{pmatrix}$ nennt man <u>Gradient</u>

und schreibt

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

"∇" = "Nabla"

$$T_{\vec{x}_0}(\vec{x}) = f(\vec{x}_0) + (\nabla f)(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) \qquad (*)$$

Richtungsableitung

$$\frac{\partial f}{\partial \vec{e}}(\vec{x}_0) = \frac{\partial T_{\vec{x}_0}}{\partial \vec{e}}(\vec{x}_0) = \lim_{h \to 0} \frac{T_{\vec{x}_0}(\vec{x}_0 + h\vec{e}) - T_{\vec{x}_0}(\vec{x}_0)}{h}$$

$$\overset{(*)}{=} \lim_{h \to 0} \frac{\cancel{f(\vec{x}_0)} + (\nabla f)(\vec{x}_0) \cdot (\cancel{\vec{x}_0} + h\vec{e} - \cancel{\vec{x}_0}) - \cancel{f(\vec{x}_0)}}{h}$$

$$= \lim_{h \to 0} \frac{(\nabla f)(\vec{x}_0) \cdot \cancel{h}\vec{e}}{\cancel{h}} = (\nabla f)(\vec{x}_0) \cdot \vec{e}$$

Bei Bewegung entlang einer Höhenlinie ändert sich der Fkt.-Wert nicht; starte an Stelle $\vec{x}_0$

$$\underbrace{f(\vec{x})}_{\vec{x} \text{ nahe } \vec{x}_0} \approx T_{\vec{x}_0}(\vec{x}) = f(\vec{x}_0) + \underbrace{(\nabla f)(\vec{x}_0) \cdot \underbrace{(\vec{x} - \vec{x}_0)}_{\text{daher laufen wir}}}_{= 0 \text{ auf Höhenlinie}}$$

$$\iff (\vec{x} - \vec{x}_0) \perp (\nabla f)(\vec{x}_0)$$

Umgekehrt wird Diff. $f(\vec{x}) - f(\vec{x}_0)$ maximal,
wenn $(\vec{x} - \vec{x}_0) \parallel (\nabla f)(\vec{x}_0)$

d.h. ∇f zeigt in Richtung des stärksten Anstiegs zweite partielle Ableitungen, Bsp

$$f(x,y) = x^2 + x \sin y$$

$$\frac{\partial f}{\partial x} = 2x + \sin y \quad , \quad \frac{\partial f}{\partial y} = x \cos y$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \cos y$$

Zufall?
Immer gleich?
Ja, falls 2. part. Ableitungen existieren & stetig sind

$$\frac{\partial^2 f}{\partial y^2} = - x \sin y$$

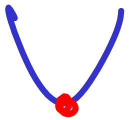
damit Hesse-Matrix: $H = \begin{pmatrix} 2 & \cos y \\ \cos y & -x \sin y \end{pmatrix} = f''(x,y)$

lokale Extrema ....

$\nabla f = 0$ heißt für $f: \mathbb{R}^2 \to \mathbb{R}$

horizontale Tangentialebene

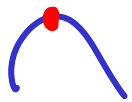
1D



Minimum

$f'' > 0$

z.B. $f(x) = x^2$

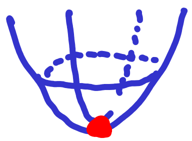


Maximum

$f'' < 0$

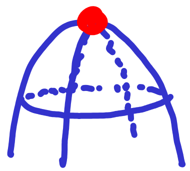
$f(x) = -x^2$

2D



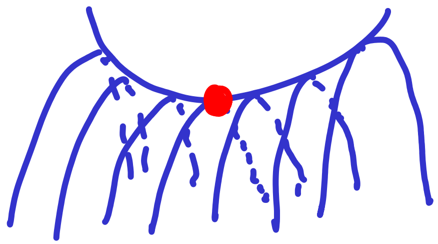
Minimum

z.B. $f(x,y) = x^2 + y^2$



Maximum

$f(x,y) = -x^2 - y^2$



Sattel

$f(x,y) = x^2 - y^2$

$\vec{x} \cdot (A\vec{x})$ für $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = ax^2 + bxy + cxy + dy^2$$

A symmetrisch: $c = b$

$$= ax^2 + 2bxy + dy^2$$