INVARIANTS OF HYPERSURFACE SINGULARITIES IN POSITIVE CHARACTERISTIC

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Abstract. We study singularities $f \in \mathbb{K}[[x_1,\dots,x_n]]$ over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic with respect to right respectively contact equivalence, and we establish that the finiteness of the Milnor respectively the Tjurina number is equivalent to finite determinacy. We introduce the notion of *differential order* and use this to give improved bounds for the degree of determinacy in positive characteristic. Moreover, we consider different non-degeneracy conditions of Kouchnirenko, Wall and Beelen-Pellikaan in positive characteristic, and we show that Newton non-degenerate singularities satisfy Milnor's formula $\mu = 2\cdot\delta - r + 1$.

1. Introduction

Throughout this paper $\mathbb{K}$ shall be an algebraically closed field of arbitrary characteristic unless explicitly stated otherwise. By

$$\mathbb{K}[[\boldsymbol{x}]] = \mathbb{K}[[x_1,\dots,x_n]] = \left\{ \sum_{\alpha\in\mathbb{N}^n} a_\alpha \cdot \boldsymbol{x}^\alpha \;\middle|\; a_\alpha \in \mathbb{K} \right\}$$

we denote the formal power series ring over $\mathbb{K}$ in $n \geq 2$ indeterminates $x_1,\dots,x_n$ using the usual multiindex notation $\boldsymbol{x}^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$. Moreover, we denote by

$$\mathfrak{m} = \langle x_1,\dots,x_n\rangle \lhd \mathbb{K}[[\boldsymbol{x}]]$$

the unique maximal ideal of $\mathbb{K}[[\boldsymbol{x}]]$, so that the set of units in $\mathbb{K}[[\boldsymbol{x}]]$ is $\mathbb{K}[[\boldsymbol{x}]]^* = \mathbb{K}[[\boldsymbol{x}]] \setminus \mathfrak{m}$.

If we are interested in the geometry of a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ there are two natural equivalence relations with respect to which we may want to consider it.

We say that two power series $f, g \in \mathbb{K}[[\boldsymbol{x}]]$ are *right equivalent* to each other if and only if there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ such that $f = \varphi(g)$, and we denote this by $f \sim_r g$. If we replaced $\mathbb{K}$ by the complex numbers and formal power series by convergent ones then φ would induce an isomorphism of the zero fiber of

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f as well as of close by fibers. That is how we should interpret right equivalence also in this more general setting.

If we are only interested in the geometry of the zero fiber, then the second equivalence relation is the appropriate one. We call $f, g \in \mathbb{K}[[\boldsymbol{x}]]$ *contact equivalent* to each other if and only if there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ and a unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$ such that $f = u \cdot \varphi(g)$, and we denote this by $f \sim_c g$. The idea here is, that φ and u still induce an isomorphism of the zero fibers of f and g.

However, we have to replace the geometric notion of the zero fiber by the algebraic counterpart of its coordinate ring. That is, for a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ we call the analytic $\mathbb{K}$-algebra $R_f = \mathbb{K}[[\boldsymbol{x}]]/\langle f\rangle$ the induced *hypersurface singularity.* We obviously have

$$f \sim_c g \quad \Longleftrightarrow \quad R_f \cong R_g,$$

i.e. f and g are contact equivalent if and only if the induced hypersurface singularities are isomorphic as local $\mathbb{K}$-algebras.

Right equivalence as well as contact equivalence can be expressed via group actions. The group $\mathcal{R} = \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ operates on $\mathbb{K}[[\boldsymbol{x}]]$ and the equivalence classes of $\mathbb{K}[[\boldsymbol{x}]]$ with respect to right equivalence are the orbits of this group, which we therefore call the *right group.* Similarly, the semidirect product

$$\mathcal{K} = \mathbb{K}[[\boldsymbol{x}]]^* \ltimes \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$$

with multiplication

$$(u, \varphi) \cdot (v, \psi) = (u \cdot \varphi(v), \varphi \circ \psi)$$

operates on $\mathbb{K}[[\boldsymbol{x}]]$ and the orbits of this group operation are the equivalence classes with respect to contact equivalence. $\mathcal{K}$ is also known as the *contact group.*

Over the complex numbers we say that the origin is an isolated singular point of f if f is not singular at any point close-by, i.e. the origin is the only common zero of the partial derivatives of f. We reformulate this algebraically so that it works over any field. For a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ we thus denote by

$$\mathrm{j}(f) = \langle f_{x_1}, \ldots, f_{x_n}\rangle \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$$

the *Jacobian ideal* of f, i.e. the ideal generated by the partial derivatives of f, and we call the associated algebra

$$M_f = \mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{j}(f)$$

the *Milnor algebra* of f and its dimension

$$\mu(f) = \dim_{\mathbb{K}}(M_f)$$

the *Milnor number* of f. We then call the origin an isolated singular point of f, or we simply call f an *isolated singularity* if $\mu(f) < \infty$, which is equivalent to the existence of a positive integer k such that $\mathfrak{m}^k \subseteq \mathrm{j}(f)$.

Similarly, over $\mathbb{C}$ we would call the origin an isolated singular point of the hypersurface singularity defined by f if this hypersurface singularity has no other singular point close-by, i.e. the origin is the only common zero of f and its partial derivatives. Algebraically we thus consider the *Tjurina ideal*

$$\mathrm{tj}(f) = \langle f, f_{x_1}, \ldots, f_{x_n} \rangle = \langle f \rangle + \mathrm{j}(f) \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$$

of f, the associated *Tjurina algebra*

$$T_f = \mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{tj}(f)$$

of f and its dimension

$$\tau(f) = \dim_{\mathbb{K}}(T_f),$$

the *Tjurina number* of f. We then call the origin an isolated singular point of the hypersurface singularity R_f, or we will simply call R_f an *isolated hypersurface singularity* if $\tau(f) < \infty$ or, equivalently, if there is a positive integer such that $\mathfrak{m}^k \subseteq \mathrm{tj}(f)$.
It is straight forward to see ([Bou09, Lem. 1.2.7]) that for an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ and a unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$ we have

$$\mathrm{j}\left(\varphi(f)\right) = \varphi\bigl(\mathrm{j}(f)\bigr)$$

and

$$\mathrm{tj}\left(u\varphi(f)\right) = \varphi\bigl(\mathrm{tj}(f)\bigr).$$

In particular, the Milnor number is *invariant* under right equivalence and the Tjurina number is *invariant* under contact equivalence.
It is a non-trivial theorem, using methods from complex analysis, which cannot be extended to other fields that the Milnor number is indeed invariant under contact equivalence (see [Gre75, p. 262]), and it is even a topological invariant (see [TrR76]). Using the Lefschetz Principle the result for contact equivalence can be generalised to arbitrary algebraically closed fields of characteristic zero (see [Bou09, Prop. 5.2.1,Prop. 5.3.1] for a detailed proof of Theorem 1.1 and Theorem 1.2).

Theorem 1.1
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $f, g \in \mathbb{K}[[\boldsymbol{x}]]$.
If $f \sim_c g$, then $\mu(f) = \mu(g)$.

Proof: Since the Milnor number is invariant under right equivalence, it suffices to show that $\mu(f) = \mu(u \cdot f)$ for any unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$. If A denotes the subset of $\mathbb{K}$ containing the coefficients of u, f and all partial derivatives f_{x_i} of f, then A is at most countable infinite. Since $\mathrm{char}(\mathbb{K}) = 0$ and since $\mathbb{Q} \subset \mathbb{C}$ is a field extension of uncountable transcendence degree the field $\mathbb{Q}(A)$ is isomorphic to a subfield $\mathbb{L}$ of the field $\mathbb{C}$ of complex numbers, and we may suppose that f and $u \cdot f$ belong to

$\mathbb{L}[[\boldsymbol{x}]] \subseteq \mathbb{C}[[\boldsymbol{x}]]$. Now using the fact that over the complex numbers f and $u \cdot f$ have the same Milnor number we get

$$\begin{aligned}\mu(f) &= \dim_{\mathbb{L}}(\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{j}(f)) = \dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/\,\mathrm{j}(f)) \\ &= \dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/\,\mathrm{j}(u \cdot f)) = \dim_{\mathbb{L}}(\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{j}(u \cdot f)) = \mu(u \cdot f).\end{aligned}$$

□

In positive characteristic this result does not hold any more; e.g. if $\mathrm{char}(\mathbb{K}) = p > 0$ and $f = x^p + y^{p-1}$, then $\mu(f) = \infty$ while the contact equivalent series $g = (1+x) \cdot f$ has Milnor number $\mu(g) = p \cdot (p-2)$.
A well-known result in complex singularity theory states that the Milnor number of a power series is finite if and only if the Tjurina number is so, i.e. that the origin is an isolated singularity of f if and only if it is an isolated singular point of the corresponding hypersurface (see [GLS07, Lem. 2.3]). This fact can also be generalised to arbitrary fields of *characteristic zero* using the Lefschetz principle.

Theorem 1.2
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $f \in \mathbb{K}[[\boldsymbol{x}]]$.
Then $\mu(f) < \infty$ if and only if $\tau(f) < \infty$.

Proof: Let A be the set of coefficients of f and all its partial derivatives. As in the proof of Theorem 1.1 the field $\mathbb{Q}(A)$ is isomorphic to a subfield $\mathbb{L}$ of $\mathbb{C}$. We may therefore assume that $f \in \mathbb{L}[[\boldsymbol{x}]] \subset \mathbb{C}[[\boldsymbol{x}]]$, so that

$$\mu(f) = \dim_{\mathbb{L}}(\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{j}(f)) = \dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/\,\mathrm{j}(f)),$$

$$\tau(f) = \dim_{\mathbb{L}}(\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{tj}(f)) = \dim_{\mathbb{C}}(\mathbb{C}[[\boldsymbol{x}]]/\,\mathrm{tj}(f)).$$

Using the result for $\mathbb{C}$, $\tau(f)$ is finite if and only if $\mu(f)$ is finite. □

For fields of positive characteristic this is false. The same example as above shows that $\tau(f) = p \cdot (p-2)$ while $\mu(f) = \infty$.

Our principle interest is the classification of power series with respect to right respectively contact equivalence, where the latter is the same as to say that we are interested in classifying hypersurface singularities up to isomorphism (see also [BGM10]). In order to do this we need finiteness conditions and therefore we restrict to the isolated case, i.e. to the case that f is an isolated singularity for right equivalence and to the case that R_f is an isolated hypersurface singularity for contact equivalence, which are two distinct conditions in arbitrary characteristic.
A first important step in the attempt to classify singularities from a theoretical point of view as well as from a practical one is to know that the equivalence class is determined by a finite number of terms of the power series f and to find the (smallest) corresponding degree bound. We say that f is *right k-determined* if f is

right equivalent to every $g \in \mathbb{K}[[\boldsymbol{x}]]$ whose k-jet coincides with that of f, where the k-jet of f is

$$\mathrm{jet}_k(f) = \overline{f} \in \mathbb{K}[[\boldsymbol{x}]]/\mathfrak{m}^{k+1}$$

the residue class of f modulo the $k+1$-st power of the maximal ideal. I.e. f is right k-determined if it is right equivalent to every g which coincides with f up to order k. Similarly, we call f *contact k-determined* if f is contact equivalent to every g whose k-jet coincides with that of f. In both situations we say that f is *finitely determined* if it is k-determined for some positive integer k, and we call the least such k the *determinacy* of f.
Over the complex numbers it is well known that f is finitely determined w.r.t. right or contact equivalence if and only if f respectively R_f is an isolated singularity. It is straight forward to generalise this to any field of characteristic zero, using the infinitesimal characterisation of local triviality. Since the proof involves the solution of a differential equation, it does not work in positive characteristic. We will, however, prove the following theorem.

Theorem 2.8
Let $0 \neq f \in \mathfrak{m} \lhd \mathbb{K}[[\boldsymbol{x}]]$ be a power series.

(a) *f is an isolated singularity if and only if f is finitely right determined.*

(b) *R_f is an isolated hypersurface singularity if and only if f is finitely contact determined.*

In the complex case and thus for arbitrary fields of characteristic zero it is known that the right determinacy is at most $\mu(f) + 1$ and the contact determinacy is at most $\tau(f)+1$. For arbitrary characteristic it was shown in [GrK90] (among others) that $2 \cdot \mu(f)$ respectively $2 \cdot \tau(f)$ are bounds for the degree of right respectively contact determinacy. We will improve these bounds substantially, in particular for right equivalence by introducing the *differential order*, in Section 2. For the formulation of our result we introduce the *order* $\mathrm{ord}(f)$ of a non-zero power series as the largest integer k such that $f \in \mathfrak{m}^k$, i.e. the smallest integer k such that f has non-zero terms of degree k, and we set $\mathrm{ord}(0) = \infty$. Moreover, we introduce the *differential order* of f to be

$$d(f) := \min\{\mathrm{ord}(f_{x_1}), \dots, \mathrm{ord}(f_{x_n})\}.$$

Note that $d(f) = \mathrm{ord}(f) - 1$ if $\mathrm{char}(\mathbb{K}) \nmid \mathrm{ord}(f)$, e.g. if$\mathrm{char}(\mathbb{K}) = 0$, and that

$$d(f) = \sup\{l \mid \mathrm{j}(f) \subseteq \mathfrak{m}^l\}.$$

Hence $d(f)$ is invariant under right equivalence, while $\mathrm{ord}(f)$ is even invariant under contact equivalence.

Theorem 2.1
Let $0 \neq f \in \mathfrak{m}^2 \unlhd \mathbb{K}[[\boldsymbol{x}]]$ and $k \in \mathbb{N}$.

(a) *If* $\mathfrak{m}^{k+2} \subseteq \mathfrak{m}^2 \cdot \mathrm{j}(f)$*, then* f *is right* $(2k - d(f) + 1)$*-determined.*

(b) *If* $\mathfrak{m}^{k+2} \subseteq \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot \mathrm{j}(f)$*, then* f *is contact* $(2k - \mathrm{ord}(f) + 2)$*-determined.*

Corollary 2.4
Let $0 \neq f \in \mathfrak{m}^2 \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$.

(a) *If* $\mu(f) < \infty$*, then the right determinacy of* f *is at most* $2\mu(f) - d(f) + 1$.

(b) *If* $\tau(f) < \infty$*, then the contact determinacy of* f *is at most* $2\tau(f) - \mathrm{ord}(f) + 2$.

The Milnor number of a singularity is governed by the geometry of its Newton polytope. In [Kou76] Kouchnirenko introduced the *Newton number* of a singularity which only depends on the Newton polytope, and he showed that this number is a lower bound for the Milnor number. Moreover, these two numbers coincide for non-degenerate singularities with fixed Newton polytope – no matter what the characteristic of the base field is. Unfortunately, his non-degeneracy assumption (*Newton non-degeneracy*, NND) does not include all right semi-quasihomogeneous singularities (see page 20), which led Wall (see [Wal99]) to the modified notion of *strict Newton non-degeneracy* SNND in characteristic zero. Neither of these notions fully implies the other, but there are well-understood relations, and Wall's notion allows to determine the Milnor number of a singularity from the geometry of the Newton polytope in the same way. In Section 3 we introduce the different notions of non-degeneracy and the Newton number $\mu_N(f)$, recall Kouchnirenko's result and generalise Wall's result to positive characteristic.

Theorem 3.4 (Kouchnirenko, [Kou76])
For $f \in \mathbb{K}[[\boldsymbol{x}]]$ *we have* $\mu_N(f) \leq \mu(f)$*, and if* f *is NND and convenient then*

$$\mu(f) = \mu_N(f) < \infty.$$

Theorem 3.6 (Wall, [Wal99])
If $f \in \mathbb{K}[[\boldsymbol{x}]]$ *is SNND w.r.t. some* C*-polytope, then*

$$\mu(f) = \mu_N(f) = \mu_N\big(\Gamma_-(f)\big) < \infty.$$

Moreover, we will show that if the Newton polytope has only one facet then the right semi-quasihomogeneous singularities are precisely the strictly Newton non-degenerate ones (see Proposition 3.9). In the case of plane curve singularities we shall see that the condition of convenience can be dropped (in any characteristic) for the equality of the Milnor and the Newton number (see Proposition 4.5), and we show that Newton non-degeneracy implies strict Newton non-degeneracy (see Proposition 4.3).

Under a somewhat weaker assumption WNND than Newton non-degeneracy Beelen and Pellikaan showed in [BeP00] how the delta invariant of a plane curve singularity can be computed in terms of the Newton polygon. Combining this with

Kouchnirenko's formula we deduce that also in positive characteristic Newton non-degenerate singularities satisfy Milnor's well-known formula $\mu(f) = 2\cdot\delta(f) - r(f) + 1$, relating the Milnor number and the delta invariant.

Theorem 4.13
If $f \in \mathbb{K}[[\boldsymbol{x}]]$ is ND along each face of $\Gamma(f)$, then $\mu(f) = 2 \cdot \delta(f) - r(f) + 1$.

2. Finite determinacy

In this section we prove that finite determinacy is equivalent to the isolatedness of the singularity. We start by showing that an isolated singularity is finitely determined and improve previously known determinacy bounds.

Theorem 2.1
Let $0 \neq f \in \mathfrak{m}^2$ and $k \in \mathbb{N}$.

(a) *If $\mathfrak{m}^{k+2} \subseteq \mathfrak{m}^2 \cdot \mathrm{j}(f)$, then f is right $(2k - d(f) + 1)$-determined.*

(b) *If $\mathfrak{m}^{k+2} \subseteq \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot \mathrm{j}(f)$, then f is contact $(2k - \mathrm{ord}(f) + 2)$-determined.*

Proof: We first consider the case of contact determinacy and set $o = \mathrm{ord}(f)$. It follows that $\mathrm{ord}(f_{x_i}) \geq o - 1$ for all $i = 1, \ldots, n$ and by assumption we thus have

$$\mathfrak{m}^{k+2} \subseteq \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot \langle f_{x_1}, \ldots, f_{x_n} \rangle \subseteq \mathfrak{m}^{o+1}.$$

This implies $k \geq o - 1$. We set

$$N = 2k - o + 2 \geq k + 1,$$

and we consider $g \in \mathbb{K}[[\boldsymbol{x}]]$ such that $g - f \in \mathfrak{m}^{N+1}$, i.e. f and g have the same N-jet. We have to show that f and g are contact equivalent, i.e. that there exists a unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$ and an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ such that

$$g = u \cdot \varphi(f).$$

We want to construct u and φ inductively, i.e. we want to construct inductively sequences of units $(u_p)_{p\geq 1}$ and of automorphisms $(\varphi_p)_{p\geq 1}$ such that $u_p \cdot \varphi_p(f)$ converges in the $\mathfrak{m}$-adic topology to $u \cdot \varphi(f)$ for some unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$ and some automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ and at the same time

$$g - u_p \cdot \varphi_p(f) \in \mathfrak{m}^{N+1+p},$$

for all $p \geq 1$. The latter implies that the $u_p \cdot \varphi_p(f)$ converge to g as well, and thus

$$g = u \cdot \varphi(f).$$

Taking Lemma 2.2 into account and using its terminology with $M = N - k \geq 1$ it suffices to construct certain series $b_{p,0} \in \mathfrak{m}^{M+p-1}$ and $b_{p,i} \in \mathfrak{m}^{M+p}$ for $i = 1, \ldots, n$ and $p \geq 1$. For this we note that by assumption

$$g - f \in \mathfrak{m}^{N+1} = \mathfrak{m}^{M-1} \cdot \mathfrak{m}^{k+2} \subseteq \mathfrak{m}^{M} \cdot \langle f \rangle + \mathfrak{m}^{M+1} \cdot \mathrm{j}(f).$$

Thus there are series $b_{1,0} \in \mathfrak{m}^M$ and $b_{1,i} \in \mathfrak{m}^{M+1}$ for $i = 1, \ldots, n$ such that

$$g - f = b_{1,0} \cdot f + \sum_{i=1}^{n} b_{1,i} \cdot f_{x_i}. \tag{1}$$

As in Lemma 2.2 we define $v_1 = 1 + b_{1,0} \in \mathbb{K}[[\boldsymbol{x}]]^*$ and

$$\phi_1 : \mathbb{K}[[\boldsymbol{x}]] \longrightarrow \mathbb{K}[[\boldsymbol{x}]] : x_i \mapsto x_i + b_{1,i}.$$

We now want to show that

$$g - v_1 \cdot \phi_1(f) \in \mathfrak{m}^{N+2},$$

since then we can replace f in the above argument by $v_1 \cdot \phi_1(f)$ and go on inductively. Let us consider the formal Taylor expansion of $f(\boldsymbol{z})$ at $\boldsymbol{x}$

$$f(\boldsymbol{z}) = f(\boldsymbol{x}) + \sum_{i=1}^{n} f_{x_i}(\boldsymbol{x}) \cdot (z_i - x_i) + \sum_{|\alpha| \geq 2} D^\alpha f(\boldsymbol{x}) \cdot (z_1 - x_1)^{\alpha_1} \cdots (z_n - x_n)^{\alpha_n},$$

where $D^\alpha f$ we denotes the corresponding formal partial derivative of f, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Note that

$$\operatorname{ord}(D^\alpha f) \geq o - |\alpha|.$$

Applying ϕ_1 to f amounts to substituting z_i by $x_i + b_{1,i}$ and we thus find

$$\phi_1(f) = f + \sum_{i=1}^{n} f_{x_i} \cdot b_{1,i} + h$$

where

$$h = \sum_{|\alpha| \geq 2} D^\alpha f(\boldsymbol{x}) \cdot b_{1,1}^{\alpha_1} \cdots b_{1,n}^{\alpha_n} \in \mathfrak{m}^{N+2},$$

since

$$\begin{aligned} \operatorname{ord}\left(D^\alpha f(\boldsymbol{x}) \cdot b_{1,1}^{\alpha_1} \cdots b_{1,n}^{\alpha_n}\right) \geq & \operatorname{ord}\left(D^\alpha f(\boldsymbol{x})\right) + \sum_{i=1}^{n} \operatorname{ord}(b_{1,i}) \cdot \alpha_i \\ & \geq o - |\alpha| + (M+1) \cdot |\alpha| \geq \ o + 2 \cdot M = N + 2. \end{aligned}$$

Multiplying $\phi_1(f)$ by $v_1 = 1 + b_{1,0}$ and using (1) we get

$$\begin{aligned} g - v_1 \cdot \phi_1(f) =& g - (1 + b_{1,0}) \cdot \left(f + \sum_{i=1}^{n} f_{x_i} \cdot b_{1,i} + h \right) \\ =& - \sum_{i=1}^{n} b_{1,0} \cdot b_{1,i} \cdot f_{x_i} - (1 + b_{1,0}) \cdot h \in \mathfrak{m}^{N+2}, \end{aligned} \tag{2}$$

since

$$\operatorname{ord}(b_{1,0} \cdot b_{1,i} \cdot f_{x_i}) \geq M + (M+1) + (o-1) = N + 2.$$

We thus can proceed inductively to construct sequences $(b_{p,i})_{p\geq 1}$ for $i = 0,\ldots,n$ with $b_{p,0} \in \mathfrak{m}^{M+p-1}$ and $b_{p,i} \in \mathfrak{m}^{M+p}$ for $i = 1,\ldots,n$. The generalisation of (2) holds by induction and with the notation of Lemma 2.2 it reads as

$$g - u_p \cdot \varphi_p(f) \in \mathfrak{m}^{N+1+p}$$

as required. This finishes the proof for the contact equivalence.
The proof for right equivalence works along the same lines. With the notation from above and $o := d(f) + 1$ the condition

$$\mathfrak{m}^{k+2} \subseteq \mathfrak{m}^2 \cdot \mathrm{j}(f) \subseteq \mathfrak{m}^{o+1}$$

implies that still $k \geq o - 1$ and that for any g with

$$g - f \in \mathfrak{m}^{N+1} = \mathfrak{m}^{M-1} \cdot \mathfrak{m}^{k+2} \subseteq \mathfrak{m}^{M+1} \cdot \mathrm{j}(f)$$

where $N = 2k - d(f) + 1 = 2k - o + 2 \geq k + 1$ and $M = N - k \geq 1$, there are $b_{1,i} \in \mathfrak{m}^{M+1}$ with

$$g - f = b_{1,1} \cdot f_{x_1} + \ldots + b_{1,n} \cdot f_{x_n}.$$

We can then define ϕ_1 as above and see that

$$g - \phi_1(f) = h \in \mathfrak{m}^{N+2}.$$

Going on by induction and applying Lemma 2.2 we get an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ such that $g = \varphi(f)$. □

Lemma 2.2
Let $M \geq 1$ be an integer and let $b_{p,0} \in \mathfrak{m}^{M+p-1}$ and $b_{p,i} \in \mathfrak{m}^{M+p}$ for $i = 1,\ldots,n$ and $p \geq 1$. Consider the units $v_p = 1 + b_{p,0} \in \mathbb{K}[[\boldsymbol{x}]]^*$ and the automorphisms $\phi_p \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ given by

$$\phi_p : x_i \mapsto x_i + b_{p,i} \quad \text{for } \ i = 1,\ldots,n.$$

We denote by

$$\varphi_p = \phi_p \circ \phi_{p-1} \circ \ldots \circ \phi_1 \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$$

the composition of the first p automorphisms, and we define inductively

$$u_p = v_p \cdot \phi_p(u_{p-1}),$$

where $u_0 = 1$.
Then the following hold true:

(a) *The sequences $\big(\varphi_p(x_i)\big)_{p\geq 1}$ converge in the $\mathfrak{m}$-adic topology of $\mathbb{K}[[\boldsymbol{x}]]$ to power series $x_i + b_i$ with $b_i \in \mathfrak{m}^{M+1}$ for $i = 1,\ldots,n$. In particular, the map*

$$\varphi : \mathbb{K}[[\boldsymbol{x}]] \longrightarrow \mathbb{K}[[\boldsymbol{x}]] : x_i \mapsto x_i + b_i$$

is a local $\mathbb{K}$-algebra automorphism of $\mathbb{K}[[\boldsymbol{x}]]$.

(b) *The sequence $(u_p)_{p\geq 1}$ converges in the $\mathfrak{m}$-adic topology to a unit $u = 1 + b_0 \in \mathbb{K}[[\boldsymbol{x}]]^*$ with $b_0 \in \mathfrak{m}^M$.*

(c) *For any power series* $f_0 \in \mathbb{K}[[\boldsymbol{x}]]$ *the sequence* $\big(\varphi_p(f_0)\big)_{p\geq 1}$ *converges in the* $\mathfrak{m}$*-adic topology to* $\varphi(f_0)$.

(d) *For any power series* $f_0 \in \mathbb{K}[[\boldsymbol{x}]]$ *the sequence* $\big(u_p \cdot \varphi_p(f_0)\big)_{p\geq 1}$ *converges in the* $\mathfrak{m}$*-adic topology to* $u \cdot \varphi(f_0)$.

Proof: Since $b_{p,i} \in \mathfrak{m}^{M+p}$ for $i = 1, \ldots, n$ we have by construction that

$$\varphi_p(x_i) - \varphi_{p-1}(x_i) = \phi_p\big(\varphi_{p-1}(x_i)\big) - \varphi_{p-1}(x_i) \in \mathfrak{m}^{M+p},$$

and thus for any $N \geq 1$ there is a $P = \max\{N - M, 1\} \geq 1$ such that for all $p > q > P$

$$\varphi_p(x_i) - \varphi_q(x_i) = \sum_{j=q+1}^{p} \varphi_j(x_i) - \varphi_{j-1}(x_i) \in \mathfrak{m}^{M+P} \subseteq \mathfrak{m}^N.$$

This shows that the $\varphi_p(x_i)$ converge to a power series of the form $x_i + b_i$ with $b_i \in \mathfrak{m}^{M+1}$. Note that $b_i \equiv b_{i,1} \pmod{\mathfrak{m}^{M+1}}$. Similarly we have that

$$\phi_p(u_{p-1}) - u_{p-1} \in \mathfrak{m}^{M+p},$$

and since $b_{p,0} \in \mathfrak{m}^{M+p-1}$ thus also

$$u_p - u_{p-1} = (1 + b_{p,0}) \cdot \phi_p(u_{p-1}) - u_{p-1} \in \mathfrak{m}^{M+p-1}.$$

With basically the same argument as above we see that u_p converges in the $\mathfrak{m}$-adic topology to a power series of the form $1 + b_0$ with $b_0 \in \mathfrak{m}^M$. Note that $b_0 \equiv b_{0,1} \pmod{\mathfrak{m}^M}$.
Let now $f_0 \in \mathbb{K}[[\boldsymbol{x}]]$ be any power series and let $N \in \mathbb{N}$ be given. Since the $\varphi_p(x_i)$ converge to $\varphi(x_i)$ and the u_p converge to u, there is a $P \geq 1$ such that for all $p \geq P$ and $i = 1, \ldots, n$

$$\varphi(x_i) - \varphi_p(x_i) \in \mathfrak{m}^N \quad \text{as well as} \quad u - u_p \in \mathfrak{m}^N.$$

It follows that also

$$\varphi(f_0) - \varphi_p(f_0) \in \mathfrak{m}^N,$$

and

$$u \cdot \varphi(f_0) - u_p \cdot \varphi_p(f_0) = u \cdot \big(\varphi(f_0) - \varphi_p(f_0)\big) + (u - u_p) \cdot \varphi_p(f_0) \in \mathfrak{m}^N$$

for all $p \geq P$. Thus the $\varphi_p(f_0)$ converge to $\varphi(f_0)$ and the $u_p \cdot \varphi_p(f_0)$ converge to $u \cdot \varphi(f_0)$. □

Remark 2.3 (a) If the base field $\mathbb{K}$ has characteristic zero it is known that

$$\mathfrak{m}^{k+2} \subseteq \mathfrak{m}^2 \cdot \mathrm{j}(f)$$

implies right-$(k+1)$-determinacy of f, and that

$$\mathfrak{m}^{k+2} \subseteq \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot \mathrm{j}(f)$$

implies contact-$(k+1)$-determinacy of f – see e.g. [GLS07, Thm. 2.23] for $\mathbb{K} = \mathbb{C}$ and [Bou09, Thm. 3.1.13] for the general case. In particular, the assumptions of Theorem 2.1 would in each case imply $k+1$ as determinacy bound, which in general is strictly better than $2k - d(f) + 1 \geq k+1$ resp. $2k - \operatorname{ord}(f) + 2 \geq k+1$.

(b) In positive characteristic the bounds in (a) do not hold any longer. Consider the power series $f = y^2 + x^3y \in \mathbb{K}[x,y]$, $\operatorname{char}(\mathbb{K}) = 2$. Then $\operatorname{tj}(f) = \langle y^2, x^2y, x^3\rangle$ and thus $\tau(f) = 5$. In particular, f defines an isolated hypersurface singularity R_f. Moreover, we have

$$\langle f\rangle + \mathfrak{m}\cdot \mathrm{j}(f) = \langle y^2, x^3y, x^4\rangle \supset \mathfrak{m}^4,$$

and if (a) would hold then f would be 5-determined. However, f is reducible while $f + x^5$ is irreducible as can be checked by the procedure `is_irred` in SINGULAR [DGPS10]. Therefore, R_f and R_{f+x^5} cannot be isomorphic, i.e. $f \not\sim_c f + x^5$, and f is not 5-determined. Theorem 2.1 asserts that f is actually 6-determined, i.e. our result is sharp in this example.

(c) The determinacy bounds given in Theorem 2.1 are always at least as good as the previously known bounds $2\cdot\mu(f)$ respectively $2\cdot\tau(f)$ for arbitrary characteristic, and they are in general much better (see e.g. the example in Part (b)). This follows from the following two facts, which are easy consequences of Nakayama's Lemma:
- If $\mu(f) < \infty$, then $\mathfrak{m}^{\mu(f)} \subseteq \mathrm{j}(f)$.
- If $\tau(f) < \infty$, then $\mathfrak{m}^{\tau(f)} \subseteq \operatorname{tj}(f)$.

To see this it suffices to show that for any ideal $I \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ with $d = \dim_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]]/I) < \infty$ we have $\mathfrak{m}^d \subseteq I$. In the ring $\mathbb{K}[[\boldsymbol{x}]]/I$ by Nakayama's Lemma the descending sequence of powers $\overline{\mathfrak{m}}^k$ of the maximal ideal has to be strictly descending until it finally becomes zero. Thus in each step the codimension of $\overline{\mathfrak{m}}^k$ grows at least by one, and it can do so at most d times. Thus $\overline{m}^d = 0$ or equivalently $\mathfrak{m}^d \subseteq I$.

(d) In concrete examples the integers k in Theorem 2.1 can be computed in SINGULAR with the aid of the procedure `highcorner`.

Corollary 2.4
Let $0 \neq f \in \mathfrak{m}^2 \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$.

(a) *If* $\mu(f) < \infty$*, then the right determinacy of* f *is at most* $2\mu(f) - d(f) + 1$.

(b) *If* $\tau(f) < \infty$*, then the contact determinacy of* f *is at most* $2\tau(f) - \operatorname{ord}(f) + 2$.

Proof: This follows from Remark 2.3 (c) and Theorem 2.1. □

The assumptions in Theorem 2.1 are fulfilled if f respectively R_f is an isolated singularity, and thus these are finitely determined. In the complex setting it is well known that the converse holds as well (see e.g. [GLS07, Cor. 2.39]), and the same is true in arbitrary characteristic.

Theorem 2.5
Let $0 \neq f \in \mathfrak{m} \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ be a power series.

(a) *If f is right k-determined, then $\mathfrak{m}^{k+1} \subseteq \mathfrak{m} \cdot \mathrm{j}(f)$. In particular, f is an isolated singularity.*

(b) *If f is contact k-determined, then $\mathfrak{m}^{k+1} \subseteq \langle f \rangle + \mathfrak{m} \cdot \mathrm{j}(f)$. In particular, R_f is an isolated hypersurface singularity.*

In the proof of Theorem 2.5 we will restrict ourselves to the case of contact equivalence, since the case of right equivalence can be treated analogously. But before we come to the proof we would like to fix some notation. We have already seen that contact equivalence can be phrased via the action of the contact group $\mathcal{K}$ on $\mathbb{K}[[\boldsymbol{x}]]$. If a power series is finitely determined then only terms up to some finite order are relevant. We thus consider $\mathbb{K}[[\boldsymbol{x}]]$ as well as the contact group modulo some power of the maximal ideal, and we will now introduce the necessary notation.
We denote by

$$J_l = \mathbb{K}[[\boldsymbol{x}]]/\mathfrak{m}^{l+1}$$

the space of l-jets of power series in $\mathbb{K}[[\boldsymbol{x}]]$. Recall that each local $\mathbb{K}$-algebra automorphism φ of $\mathbb{K}[[\boldsymbol{x}]]$ is uniquely represented by a tuple $(\varphi_1, \ldots, \varphi_n) \in \mathbb{K}[[\boldsymbol{x}]]^n$ of power series such that

$$\varphi_i(0) = 0 \quad \text{for all } i = 1, \ldots, n$$

and

$$\det \left(\frac{\partial \varphi_i}{\partial x_j}(0) \right)_{i,j=1,\ldots,n} \neq 0.$$

We define the l-jet of the automorphism φ as

$$\mathrm{jet}_l(\varphi) := (\mathrm{jet}_l(\varphi_1), \ldots, \mathrm{jet}_l(\varphi_n)),$$

and the l-jet of the contact group

$$\mathcal{K}_l := \mathrm{jet}_l(\mathbb{K}[[\boldsymbol{x}]]^*) \ltimes \mathrm{jet}_l \big(\mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]]) \big)$$

via the multiplication

$$\big(\mathrm{jet}_l(u), \mathrm{jet}_l(\varphi)\big) \cdot \big(\mathrm{jet}_l(v), \mathrm{jet}_l(\psi)\big) := \big(\mathrm{jet}_l(u \cdot \varphi(v)), \mathrm{jet}_l(\varphi \circ \psi)\big)$$

which is independent of the chosen representatives. The l-jet of the contact group then operates on the l-jet of $\mathbb{K}[[\boldsymbol{x}]]$ via

$$\Phi_l : \mathcal{K}_l \times J_l \longrightarrow J_l : \big((\mathrm{jet}_l(u), \mathrm{jet}_l(\varphi)), \mathrm{jet}_l(f)\big) \mapsto \mathrm{jet}_l \big(u \cdot \varphi(f)\big),$$

i.e. by taking representatives, let them act and taking the l-jet.
Analogously, we define the l-jet $\mathcal{R}_l$ of the right group $\mathcal{R} = \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ and it operates on J_l.

Remark 2.6
Note also that J_l is an affine space and $\mathcal{K}_l$ and $\mathcal{R}_l$ are affine algebraic groups acting on J_l via a regular *separable* algebraic action.

Proof that the actions are separable: We restrict to the case of the action of $\mathcal{K}_l$ and we choose coordinates on J_l and of $\mathcal{K}_l$. Writing

$$\operatorname{jet}_l(f) = \sum_{|\alpha|=0}^{l} a_\alpha \overline{\boldsymbol{x}}^\alpha,$$

and

$$\operatorname{jet}_l(\varphi_i) = \sum_{|\beta|=1}^{l} b_{i,\beta} \overline{\boldsymbol{x}}^\beta,$$

and

$$\operatorname{jet}_l(u) = \sum_{|\gamma|=0}^{l} c_\gamma \overline{\boldsymbol{x}}^\gamma$$

we have the coordinates

$$(a_\alpha, b_{i,\beta}, c_\gamma)_{\alpha,i,\beta,\gamma}$$

on $\mathcal{K}_l \times J_l$ with $c_0 \neq 0$ and $\det(B) \neq 0$ where $B = (B_{ij})$ with $B_{ij} = \frac{\partial \varphi_i}{\partial x_j}(0) = b_{i,e_j}$ and e_j is the j-th canonical basis vector in $\mathbb{Z}^n$. Using in the same manner the coordinates

$$(a'_\delta)_{|\delta|=0,\ldots,l}$$

on the target space the action is given by polynomial maps

$$a'_\delta = F_\delta(a_\alpha, b_{i,\beta}, c_\gamma),$$

and it is important to note that the inverse of the action is given by rational maps

$$a_\alpha = \frac{G_\alpha(a'_\delta, b_{i,\beta}, c_\gamma)}{H_\alpha(a'_\delta, b_{i,\beta}, c_\gamma)}.$$

The reason for this is that we can solve for the a_α degree by degree starting basically with Cramer's rule, and this property ensures that for the extension of the fields of rational functions induced by the operation Φ_l we have

$$\mathbb{K}(J_l) = \mathbb{K}(a'_\delta) \subset \mathbb{K}(\mathcal{K}_l \times J_l) = \mathbb{K}(a_\alpha, b_{i,\beta}, c_\gamma) = \mathbb{K}(a'_\delta, b_{i,\beta}, c_\gamma) = \mathbb{K}(J_l)(b_{i,\beta}, c_\gamma).$$

The $b_{i,\beta}$ and c_γ are algebraically independent over $\mathbb{K}(a_\alpha)$ and comparing transcendence degrees they must be so over $\mathbb{K}(J_l)$. Thus $\mathbb{K}(\mathcal{K}_l \times J_l)$ is a purely transcendental extension of $\mathbb{K}(J_l)$, and it is thus by default a separably generated extension in the sense of [Har77, p. 27]. Thus $\mathcal{K}_l$ operates separably on J_l. $\square$

This allows us to describe the tangent space to the orbits also in positive characteristic (for $\mathbb{K} = \mathbb{C}$ see [GLS07, Prop. 2.38]).

Proposition 2.7
Let $f \in \mathbb{K}[[\boldsymbol{x}]]$. Then the tangent space to the orbit of $\operatorname{jet}_l(f)$ *under the action of* $\mathcal{R}_l$ *respectively* $\mathcal{K}_l$ *considered as a subspace of* J_l *is*

$$T_{\operatorname{jet}_l(f)}\big(\mathcal{R}_l \cdot \operatorname{jet}_l(f)\big) = \big(\mathfrak{m} \cdot \mathrm{j}(f) + \mathfrak{m}^{l+1}\big)/\mathfrak{m}^{l+1}$$

respectively

$$T_{\operatorname{jet}_l(f)}\big(\mathcal{K}_l \cdot \operatorname{jet}_l(f)\big) = \big(\langle f\rangle + \mathfrak{m} \cdot \mathrm{j}(f) + \mathfrak{m}^{l+1}\big)/\mathfrak{m}^{l+1}.$$

Proof: If G denotes one of the two above groups then the action of G on J_l induces a surjective separable morphism $G \longrightarrow G \cdot \operatorname{jet}_l(f)$ of smooth varieties. Thus the induced differential map on the tangent spaces is generically surjective (see e.g. the proof of [Har77, Lem. III.10.5.1]). However, since we can translate each point in G to a point where the tangent map is surjective and the translation is an isomorphism the differential map is actually always surjective. It thus suffices to understand the image of the tangent space to G at the neutral element of the group and its image under the differential map. We restrict here to the case $G = \mathcal{K}_l$ since the proof for $\mathcal{R}_l$ works analogously.
The tangent space to $\mathcal{K}_l$ at $(1, \mathrm{id})$ can be described via the local $\mathbb{K}$-algebra homomorphisms from the local ring of $\mathcal{K}_l$ to $\mathbb{K}[\varepsilon]$ where $\varepsilon^2 = 0$. In this sense, a tangent vector is represented by the residue class modulo $\mathfrak{m}^{l+1}$ of a tuple

$$\big(1 + \varepsilon \cdot a, \mathrm{id} + \varepsilon \cdot \phi\big)$$

with $a \in \mathbb{K}[[\boldsymbol{x}]]$ and $\phi = (\phi_1, \ldots, \phi_n)$ where $\phi_i \in \mathfrak{m}$. We now apply the differential map by acting with the above tuple on f modulo $\mathfrak{m}^{l+1}$. Taking $\varepsilon^2 = 0$ and Taylor expansion into account we get

$$(1 + \varepsilon \cdot a) \cdot f\big(\boldsymbol{x} + \varepsilon \cdot \phi\big) = f + \varepsilon \cdot \left(a \cdot f + \sum_{i=1}^{n} f_{x_i} \cdot \phi_i\right).$$

Interpreted in J^l this tangent vector is just the l-jet of

$$a \cdot f + \sum_{i=1}^{n} f_{x_i} \cdot \phi_i,$$

which proves the claim. $\square$

Proof of Theorem 2.5: We only do the proof for contact determinacy since the other case works analogously. If f is contact k determined and $g \in \mathfrak{m}^{k+1}$ then for any $t \in \mathbb{K}$ the $k+1$-jet $\operatorname{jet}_{k+1}(f) + t \cdot \operatorname{jet}_{k+1}(g)$ is in the orbit of $\operatorname{jet}_{k+1}(f)$ under $\mathcal{K}^{k+1}$. But then

$$\operatorname{jet}_{k+1}(g) \in T_{\operatorname{jet}_{k+1}(f)}\big(\mathcal{K}^{k+1} \cdot \operatorname{jet}_{k+1}(f)\big) = \langle f\rangle + \mathfrak{m} \cdot \mathrm{j}(f) + \mathfrak{m}^{k+2}/\mathfrak{m}^{k+2}.$$

This implies

$$g \in \langle f\rangle + \mathfrak{m} \cdot \mathrm{j}(f) + \mathfrak{m}^{k+2},$$

and hence

$$\mathfrak{m}^{k+1} \subseteq \langle f \rangle + \mathfrak{m} \cdot \mathrm{j}(f) + \mathfrak{m}^{k+2}.$$

By Nakayama's Lemma we finally get

$$\mathfrak{m}^{k+1} \subseteq \langle f \rangle + \mathfrak{m} \cdot \mathrm{j}(f),$$

as claimed. □

Combining the results of Corollary 2.4 and of Theorem 2.5 we get the following result.

Theorem 2.8
Let $0 \neq f \in \mathfrak{m} \lhd \mathbb{K}[[\boldsymbol{x}]]$ be a power series.

(a) *f is an isolated singularity if and only if f is finitely right determined.*

(b) *R_f is an isolated hypersurface singularity if and only if f is finitely contact determined.*

3. Non-degenerate singularities

For the convenience of the reader let us recall the definition of the Newton polytope and Wall's notion of a C-polytope (see [Wal99]). To each power series $f = \sum_\alpha a_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[[\boldsymbol{x}]]$ we can associate its *Newton diagram* $\Gamma_+(f)$ as the convex hull of the set

$$\bigcup_{\alpha \in \mathrm{supp}(f)} (\alpha + \mathbb{R}^n_{\geq 0})$$

where $\mathrm{supp}(f) = \{\alpha \mid a_\alpha \neq 0\}$ denotes the support of f. This is an unbounded polytope in $\mathbb{R}^n$. We call the union $\Gamma(f)$ of its compact faces the *Newton polytope* of f. By $\Gamma_-(f)$ we denote the union of all line segments joining the origin to a point on $\Gamma(f)$. (See Figure 1 for an example.)

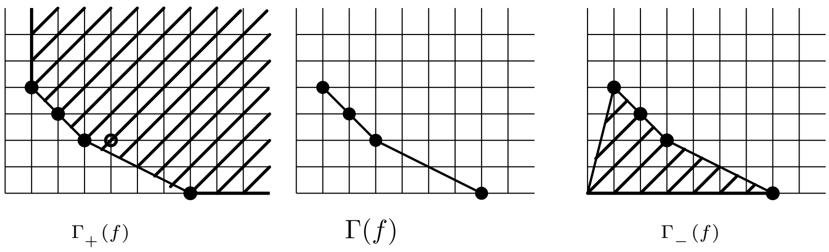

Figure 1. The Newton polytope of $x \cdot (y^4 + xy^3 + x^2y^2 - x^3y^2 + x^6)$.

If the Newton polytope of a singularity f meets all coordinate axes we call f *convenient*. In this case the Newton polytope of f can be used to define a filtration on $\mathbb{K}[[\boldsymbol{x}]]$ by finite dimensional vector spaces. However, not every isolated singularity is convenient, and one then has to enlarge the Newton polytope. A compact rational polytope P of dimension $n-1$ in the positive orthant $\mathbb{R}^n_{\geq 0}$ is called a C-polytope if the region above P is convex and if every ray in the positive orthant emanating from the origin meets P in exactly one point. Typical examples are the Newton polytopes of convenient series.

We will now first introduce the different notions of non-degeneracy. For this let $f = \sum_\alpha a_\alpha \cdot \boldsymbol{x}^\alpha \in \mathfrak{m}$ be a power series, let P be a C-polytope such that $\operatorname{supp}(f)$ has *no* point below P, and let Δ be a face of P. By $\operatorname{in}_\Delta(f) = \sum_{\alpha\in\Delta} a_\alpha \cdot \boldsymbol{x}^\alpha$ we denote the *initial form* or *principal part* of f along Δ.
Following Wall we call f *non-degenerate ND along* Δ if the Jacobian ideal $\operatorname{j}(\operatorname{in}_\Delta(f))$ has no zero in the torus $(\mathbb{K}^*)^n$. f is then said to be *Newton non-degenerate* NND if f is non-degenerate along each *face* of the Newton polytope $\Gamma(f)$. Note that unlike Wall we do *not* require f to be *convenient*.
To define strict non-degeneracy we need to fix two more notions. The face Δ is an *inner face* of P if it is not contained in any coordinate hyperplane. And each point $q \in \mathbb{K}^n$ determines a coordinate hyperspace $H_q = \bigcap_{q_i=0}\{x_i = 0\} \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$. We call f *strictly non-degenerate SND along* Δ if for each zero q of the Jacobian ideal $\operatorname{j}(\operatorname{in}_\Delta(f))$ the polytope Δ contains no point on H_q. And we finally call f *strictly Newton non-degenerate* SNND w.r.t. a C-polytope P if f is strictly non-degenerate along each *inner* face of P.
Finally, we call f *weakly non-degenerate WND along* Δ if the Tjurina ideal $\operatorname{tj}(\operatorname{in}_\Delta(f))$ has no zero in the torus $(\mathbb{K}^*)^n$, and f is called *weakly Newton non-degenerate* WNND if f is weakly non-degenerate along each *facet* of $\Gamma(f)$. Recall that a facet is a top-dimensional face.

Remark 3.1
We collect some easy facts on and relations between the different types of non-degeneracy. For any occuring C-polytope and power series f we assume that no point in $\operatorname{supp}(f)$ lies below P.

(a) Each of the non-degeneracy conditions introduced above only depends the principal part $\operatorname{in}_P(f) = \sum_{\alpha\in P} a_\alpha \cdot \boldsymbol{x}^\alpha$ of f w.r.t. P.

(b) Obviously ND along Δ implies WND along Δ and both are equivalent in characteristic zero, or, more generally, if $\operatorname{char}(\mathbb{K})$ does not divide the weighted degree of $\operatorname{in}_\Delta(f)$.

(c) WNND is strictly weaker than NND.
E.g. $f = x^3 + y^2$ with $\operatorname{char}(\mathbb{K}) = 3$, then f is WNND but not NND, since f is not ND along $\Delta = \{(3, 0)\}$.

(d) If f is SND along Δ, then f is ND along Δ.

(e) If Δ does not meet any coordinate hyperplane, then f is ND along Δ if and only if f is SND along Δ.

(f) ND does not in general imply SND.
E.g. $f = x^2y^2 + y^4$ and Δ the line segment from $(4,0)$ to $(0,4)$, then f satisfies ND along Δ, but not SND.

(g) f can be convenient and SNND without satisfying NND.
E.g. $f = (x+y)^2 + xz + z^2$ with $\mathrm{char}(\mathbb{K}) \neq 2$, then $\Gamma(f)$ has a unique facet Δ with $f = \mathrm{in}_\Delta(f)$ and $\mathrm{Sing}(f) = \{0\}$. Thus f is SNND, but f is not ND along the line segment from $(2,0,0)$ to $(0,2,0)$ which is a face of $\Gamma(f)$ (see also [Kou76]).

(h) If f is NND and $k \cdot e_i \in \Gamma(f)$, where e_i is the i-th standard basis vector of $\mathbb{R}^n$, then $\mathrm{char}(\mathbb{K})$ does not divide k.

(i) f satisfies SND at an inner vertex $\alpha = (\alpha_1, \ldots, \alpha_n)$ of P if and only if α is a vertex of $\Gamma(f)$ and some α_i is not divisible by $\mathrm{char}(\mathbb{K})$.

(j) In characteristic zero each of the above non-degeneracy conditions is a generality condition in the sense that fixing a C-polytope P then, among all polynomials f with $\mathrm{supp}(f) \subseteq P$, there is a Zariski open dense subset which satisfies the non-degeneracy condition. In positive characteristic some additional assumptions on the C-polytope P are necessary, like that not all coordinates of a vertex should be divisible by the characteristic.

$\square$

The following two remarks shed some light on the definition of NND and SNND.

Remark 3.2 ([Kou76])
Kouchnirenko defines ND and NND actually by considering common zeros of $x_i \cdot \mathrm{in}_\Delta(f)_{x_i}$ for $i = 1, \ldots, n$, since these polynomials are better suited with respect to the piecewise filtration induced by the C-polytope $P = \Gamma(f)$ (see [Kou76] or [BGM10, Sec. 3] for details on this filtration). However, they have no common zero in the torus if and only if $\mathrm{j}(\mathrm{in}_\Delta(f))$ has no zero in the torus, so that the two definitions coincide.
Each face Δ of the Newton polytope of a convenient power series f determines a finitely generated semigroup C_Δ in $\mathbb{Z}^n$ by considering those lattice points which lie in the cone over Δ with the origin as base. This semigroup then determines a finitely generated $\mathbb{K}$-algebra $\mathbb{K}[C_\Delta] = \mathbb{K}[\boldsymbol{x}^\alpha | \alpha \in C_\Delta]$, and the polynomials $x_i \cdot \mathrm{in}_\Delta(f)_{x_i}$, $i = 1, \ldots, n$ generate an ideal, say I_Δ, in $\mathbb{K}[C_\Delta]$.
It then turns out that (see [Kou76, Thm. 6.2] or [Wal99, Prop. 2.2])

$$\dim_{\mathbb{K}}(\mathbb{K}[C_\Delta]/I_\Delta) < \infty \quad \Longleftrightarrow \quad f \text{ is ND along all faces of } \Delta.$$

We should like to point out that f ND along Δ is not sufficient for the finiteness of $\dim_{\mathbb{K}}(\mathbb{K}[C_\Delta]/I_\Delta)$. Consider e.g. $f = x^3 + y^2$ with $\mathrm{char}(\mathbb{K}) = 3$ and $\Delta = \Gamma(f)$, then $f = \mathrm{in}_\Delta(f)$ and $\mathrm{j}(f)$ has no zero in the torus, yet $\mathbb{K}[C_\Delta]/I_\Delta = \mathbb{K}[x,y]/\langle y \rangle$ has infinite dimension.

The piecewise filtration induced by P determines a graded algebra $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I_P)$ for the piecewise homogeneous ideal $I_P = \langle x_i \cdot \mathrm{in}_P(f)_{x_i} \mid i = 1, \ldots, n\rangle$. We can view $\mathbb{K}[C_\Delta]/I_\Delta$ in a natural way as a quotient of $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I_P)$, and we then get an injective map (see [Kou76, Prop. 2.6])

$$\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I_P) \longrightarrow \bigoplus_{\Delta \text{ face of } \Gamma(f)} \mathbb{K}[C_\Delta]/I_\Delta.$$

This shows right away that $\dim_{\mathbb{K}}(\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I_P)$ is finite, if f is NND. From this it is not hard to see that a monomial K-vector space basis of $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I_P)$ actually generates M_f (see [BGM10, Sec. 3]), and it thus follows:

$$f \text{ is NND and convenient} \quad \Longrightarrow \quad \mu(f) < \infty.$$

□

Remark 3.3 ([Wal99])
Wall adapts Kouchnirenko's arguments and replaces the ideal I_Δ by a possibly somewhat larger ideal. The ideal I_Δ was achieved by applying the $\mathbb{K}[C_\Delta]$-module D^0_Δ generated by the derivations $x_i\partial_{x_i}$ to $\mathrm{in}_\Delta(f)$. Wall replaces D^0_Δ by the $\mathbb{K}[C_\Delta]$-module

$$D_\Delta = \langle \boldsymbol{x}^\alpha \cdot \partial_{x_i} \mid \boldsymbol{x}^\alpha \cdot \partial_{x_i}(f) \in \mathbb{K}[C_\Delta] \ \ \forall\, f \in \mathbb{K}[C_\Delta]\rangle$$

generated by all monomial derivations which leave $\mathbb{K}[C_\Delta]$ invariant and considers the ideal

$$J_\Delta = \{\xi(\mathrm{in}_\Delta(f)) \mid \xi \in D_\Delta\}$$

which results by applying D_Δ to $\mathrm{in}_\Delta(f)$. He then shows that (see [Wal99, Prop. 2.2])

$$\dim_{\mathbb{K}}(\mathbb{K}[C_\Delta]/J_\Delta) < \infty \quad \Longleftrightarrow \quad f \text{ is SND along all } \textit{inner} \text{ faces of } \Delta.$$

The rings $\mathbb{K}[C_\Delta]/J_\Delta$ can be stacked neatly in an exact sequence of complexes whose homology was used by Wall to show (see [Wal99, Prop. 1.2, Prop. 2.3] and [BGM10, Thm. 4.13]):

$$f \text{ is SNND} \quad \Longrightarrow \quad \mu(f) < \infty.$$

Wall's arguments use only standard facts from toric geometry and homological algebra and do not depend on the characteristic of the base field. □

In [Kou76] Kouchnirenko not only showed that a Newton non-degenerate singularity f is isolated, but he gives a formula for the Milnor number in terms of certain volumes of the faces of $\Gamma_-(f)$.
For any compact polytope Q in $\mathbb{R}^n_{\geq 0}$ we denote by $V_k(Q)$ the sum of the k-dimensional Euclidean volumes of the intersections of Q with the k-dimensional coordinate subspaces of $\mathbb{R}^n$, and following Kouchnirenko we then call

$$\mu_N(Q) = \sum_{k=0}^{n} (-1)^{n-k} \cdot k! \cdot V_k(Q)$$

the *Newton number* of Q. For a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ we define the *Newton number* of f to be

$$\mu_N(f) = \sup\Big\{\mu_N(\Gamma_-(f_m)) \;\Big|\; f_m = f + x_1^m + \ldots + x_n^m,\ m \geq 1\Big\} \in \mathbb{Z} \cup \{\infty\}.$$

If f is convenient, then

$$\mu_N(f) = \mu_N(\Gamma_-(f)).$$

The following theorem was proved by Kouchnirenko in arbitrary characteristic.

Theorem 3.4 (Kouchnirenko, [Kou76])
For $f \in \mathbb{K}[[\boldsymbol{x}]]$ we have $\mu_N(f) \leq \mu(f)$, and if f is NND and convenient then

$$\mu(f) = \mu_N(f) < \infty.$$

Actually, Kouchnirenko shows that in *characteristic zero* the result still holds if f is not convenient. We will show in Proposition 4.5 that at least in the planar case this also holds in arbitrary characteristic.

Example 3.5
Newton non-degeneracy is sufficient but not necessary to ensure that the Milnor number coincides with the Newton number and both are finite.
If $\mathrm{char}(\mathbb{K}) \neq 2$ then $f = (x+y)^2 + xz + z^2$ is not NND (see Remark 3.1), but

$$\mu(f) = \mu_N(f) = 1.$$

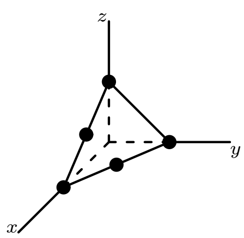

FIGURE 2. The Newton polytope of $f = (x+y)^2 + xz + z^2$

Wall proved in [Wal99] the analogous result for strictly Newton non-degenerate singularities in characteristic zero, and his proof generalises to arbitrary characteristic.

Theorem 3.6 (Wall, [Wal99])
If $f \in \mathbb{K}[[\boldsymbol{x}]]$ is SNND w.r.t. some C-polytope, then

$$\mu(f) = \mu_N(f) = \mu_N\big(\Gamma_-(f)\big) < \infty.$$

Proof: By [Wal99, Prop. 1.2] respectively [BGM10, Thm. 4.13] we know that $\mu(f) < \infty$. It thus remains to show that $\mu(f) = \mu_N(f) = \mu_N\big(\Gamma_-(f)\big)$, but the proof for this is the same as in [Wal99, Thm. 1.6] if we take into account that $\mu(f) < \infty$ implies that f is finitely determined (see Theorem 2.8). $\square$

Example 3.7 ($W_{1,1}$-Singularities)
A series f with principal part $\mathrm{in}_P(f) = x^7 + x^3y^2 + y^4$ for $P = \Gamma(f)$ is SNND if $\mathrm{char}(\mathbb{K}) \notin \{2, 3, 7\}$, and thus f is an isolated singularity.

At the beginning of this section we mentioned that strict Newton non-degeneracy has the advantage over Newton non-degeneracy that all right semi-quasihomogeneous singularities satisfy this condition, even if they are not convenient. This is an easy consequence of the observation in Lemma 3.8 as we will see in Proposition 3.9.
A polynomial $f \in \mathbb{K}[\boldsymbol{x}]$ is said to be *quasihomogeneous* QH w.r.t. a weight vector $w \in \mathbb{Z}^n_{>0}$ if all monomials $\boldsymbol{x}^\alpha$, $\alpha \in \mathrm{supp}(f)$, have the same *weighted degree* $\deg_w(\boldsymbol{x}^\alpha) = w \cdot \alpha = w_1 \cdot \alpha_1 + \ldots + w_n \cdot \alpha_n$. We call a power series $f = \sum_\alpha a_\alpha \cdot \boldsymbol{x}^\alpha \in \mathbb{K}[[\boldsymbol{x}]]$ *right semi-quasihomogeneous* rSQH if there is a weight vector $w \in \mathbb{Z}^n_{>0}$ such that the principal part $\mathrm{in}_w(f) = \sum_{w\cdot\alpha \text{ minimal}} a_\alpha \cdot \boldsymbol{x}^\alpha$ has a finite Milnor number. Note that since in positive characteristic the finiteness of the Milnor number and the Tjurina number are no longer equivalent we have to distinguish between semi-quasihomogeneity for right and contact equivalence (see also [BGM10, Sec. 2]).

Lemma 3.8
Let $f \in \mathbb{K}[\boldsymbol{x}]$ be QH w.r.t. $w \in \mathbb{Z}^n_{>0}$, then $\mu(f) < \infty$ if and only if 0 is the only zero of $\mathrm{j}(f)$.

Proof: If a monomial $\boldsymbol{x}^\alpha$ is a linear combination of the partial derivatives of f in $\mathbb{K}[[\boldsymbol{x}]]$ then we only have to consider the suitable weighted homogeneous part, and it actually is a linear combination in $\mathbb{K}[\boldsymbol{x}]$. Thus $\mu(f)$ is finite if and only if $\dim_{\mathbb{K}}(\mathbb{K}[x, y]/\mathrm{j}(f)) < \infty$. By Hilbert's Nullstellensatz the latter is equivalent to the fact that $\mathrm{j}(f)$ has only finitely many zeros in $\mathbb{K}^n$. But since f is weighted homogeneous for each zero $q = (q_1, \ldots, q_n)$ of $\mathrm{j}(f)$ also $(t^{w_1}q_1, \ldots, t^{w_n}q_n)$ is a zero of $\mathrm{j}(f)$ for all $t \in \mathbb{K}$. Thus $\mathrm{j}(f)$ has only finitely many zeros if and only if 0 is the only zero of $\mathrm{j}(f)$. □

Proposition 3.9
Let P be a C-polytope with a single facet Δ with weight vector w and suppose that $f \in \mathbb{K}[[\boldsymbol{x}]]$ has principal part $\mathrm{in}_w(f) = \mathrm{in}_\Delta(f)$ w.r.t. P. Then f is SNND w.r.t. P if and only f is rSQH w.r.t. w.
In particular, if f is rSQH w.r.t. w of weighted degree d, then

$$\mu(f) = \left(\frac{d}{w_1} - 1\right) \cdot \ldots \cdot \left(\frac{d}{w_n} - 1\right).$$

Proof: Since P is a C-polytope the unique facet Δ meets all coordinate subspaces except possibly $\{0\}$. Thus f is SND along Δ if and only if $\mathrm{Sing}(\mathrm{in}_\Delta(f)) = \{0\}$. By Lemma 3.8 this is equivalent to $\mu(\mathrm{in}_w(f)) = \mu(\mathrm{in}_\Delta(f)) < \infty$, i.e. that f is rSQH w.r.t. w.

The formula for $\mu(f)$, first proved by Milnor and Orlik [MiO70] for isolated QH singularities in characteristic zero, follows from Theorem 3.6 since $\mu_N(f)$ is easily seen to be the product of the $\frac{d}{w_i} - 1$. □

Example 3.10
Generalising Example 3.5 we consider $f = (x+y)^k + xz^{k-1} + z^k$ for some $k \geq 2$ such that $\mathrm{char}(\mathbb{K})$ neither divides k nor $k-1$. Then f is QH w.r.t. $w = (1,1,1)$ and $\mathrm{Sing}(f) = \{0\}$. Thus by Lemma 3.8 and Proposition 3.9 f is an isolated singularity and SNND with $\mu(f) = \mu_N(f) = (k-1)^3$. Note that f is not NND.

4. Invariants of plane curve singularities

For the convenience of the reader we start this section by gathering numerical invariants of a singularity f respectively numbers associated to the geometry of its Newton polygon that will be introduced and compared throughout. We will comment on these and their relations further down.

Remark 4.1
Let $f \in \mathbb{K}[[x,y]]$ be a power series and suppose that the Newton polygon of f has k facets $\Delta_1, \ldots, \Delta_k$. By $l(\Delta_i)$ we denote the *lattice length* of Δ_i, i.e. the number of lattice points on Δ_i minus one.
We fix a minimal resolution of the singularity computed via successively blowing up points, denote by $Q \to 0$ that Q is an infinitely near point of the origin and by m_Q the multiplicity of the strict transform of f at Q. Finally, for $m \in \mathbb{N}$ we set $f_m := f + x^m + y^m$.

(a) $\mu(f) = \dim_{\mathbb{K}}(\mathbb{K}[[x,y]]/\langle f_x, f_y\rangle)$ is the *Milnor number* of f.
(b) $\delta(f) = \sum_{Q\to 0} \frac{m_Q\cdot(m_Q-1)}{2}$ is the *delta invariant* of f.
(c) $\nu(f) = \sum_{Q \text{ special}} \frac{m_Q\cdot(m_Q-1)}{2}$, where an infinitely near point Q is special if it is zero or the origin of the corresponding chart of the blowing up.
(d) $r(f)$ is the *number of branches* of f counted with multiplicity.
(e) If f is convenient, then the *Newton number* of f is

$$\mu_N(f) = 2 \cdot V_2\big(\Gamma_-(f)\big) - V_1\big(\Gamma_-(f)\big) + 1,$$

and otherwise it is $\mu_N(f) = \sup\{\mu_N(f_m) \mid m \in \mathbb{N}\}$.
(f) If f is convenient, we define

$$\nu_N(f) = V_2(\Gamma_-(f)) - \frac{V_1(\Gamma_-(f))}{2} + \frac{\sum_{i=1}^k l(\Delta_i)}{2},$$

and otherwise we set $\nu_N(f) = \sup\{\nu_N(f_m) \mid m \in \mathbb{N}\}$.
(g) $r_N(f) = \sum_{i=1}^k l(\Delta_i) + \max\{j \mid x^j \text{ divides } f\} + \max\{l \mid y^l \text{ divides } f\}$.

Coming back to the different notions of non-degeneracy, a particularly interesting situation is that of plane curve singularities. We will end this paper by investigating this case more closely. One of the aims is to show that for non-degeneracy the condition of *convenience* is often not necessary, even in positive characteristic. We now elaborate on the conditions SND and SNND in the planar case.

Remark 4.2
Let $f \in \mathbb{K}[[x,y]$ and P be a C-polytope such that no point in $\operatorname{supp}(f)$ lies below P.

(a) Then f is SND along an edge Δ of P if and only if
- all zeros of $\mathrm{j}(\mathrm{in}_\Delta(f))$ have at least one coordinate zero if Δ does not meet any coordinate axis;
- all zeros of $\mathrm{j}(\mathrm{in}_\Delta(f))$ have x-coordinate zero if Δ only meets the x-axis;
- all zeros of $\mathrm{j}(\mathrm{in}_\Delta(f))$ have y-coordinate zero if Δ only meets the y-axis;
- the only zero of $\mathrm{j}(\mathrm{in}_\Delta(f))$ is $(0,0)$ if Δ meets both axes.

(b) In [Wal99] Wall describes how much the C-polytope P may differ from $\Gamma(f)$ if f is SNND w.r.t. P:
- each inner vertex of P is a vertex of $\Gamma(f)$;
- an edge of P which does not meet a coordinate axis is an edge of $\Gamma(f)$;
- an edge of P which meets exactly one of the coordinate axes is either itself an edge of $\Gamma(f)$ or, replacing the point on the coordinate axis by the point on the edge with distance one from the coordinate axis, leads to an edge or a vertex of $\Gamma(f)$;
- if P consists of a single edge meeting both coordinate axes, then f is rSQH w.r.t. any weight vector defining this edge (see Prop. 3.9); in particular, the principal part of f is reduced and unless it is xy the edge P contains an edge of the Newton polygon whose end points have distance at most one from the corresponding axes.

Wall gives this characterisation over the complex numbers, but it actually holds in the same way in any characteristic.

It turns out that in the planar situation NND implies SNND.

Proposition 4.3
If $f \in \mathbb{K}[[x,y]]$ is NND, then f is SNND w.r.t. $\Gamma(f)$.

Proof: Let Δ be any inner face of $\Gamma(f)$. If Δ intersects none of the two coordinate axes, then f is SND along Δ since it is ND along Δ by Remark 3.1.
If Δ meets the y-axis we have to show that there is no zero of $\mathrm{j}(\mathrm{in}_\Delta(f))$ with non-zero y-coordinate. In this situation Δ is an edge of the Newton polygon whose one end point lies on the y-axis, i.e.

$$\mathrm{in}_\Delta(f) = a \cdot y^k + x \cdot g$$

for some $a \in \mathbb{K}^*$, $k \geq 1$ and $g \in \mathbb{K}[x, y]$.
By assumption $\mathrm{j}(\mathrm{in}_\Delta(f))$ has no zero in $(\mathbb{K}^*)^2$ and we have to exclude the possibility that it has a zero $q = (0, z) \in \{0\} \times \mathbb{K}^*$. This is the case since

$$\mathrm{in}_\Delta(f)_y = a \cdot k \cdot y^{k-1} + x \cdot g_y$$

and

$$\mathrm{in}_\Delta(f)_y(q) = a \cdot k \cdot z^{k-1} \neq 0,$$

where, for the second statement, we note that by Remark 3.1 $\mathrm{char}(\mathbb{K})$ does not divide k.
Similarly, if Δ meets the x-axis there is no zero of $\mathrm{j}(\mathrm{in}_\Delta(f))$ with non-zero x-coordinate.
Thus f is also SND along any inner face which meets any of the two coordinate axes by Remark 4.2, and altogether we have that f is SND w.r.t. $\Gamma(f)$. □

Since on each face SND implies ND, the previous result can be rephrased as follows.

Corollary 4.4
If $f \in \mathbb{K}[[x, y]]$, then the following are equivalent:

(a) *f is NND.*

(b) *f is SNND w.r.t. $\Gamma(f)$, and in case $\Gamma(f)$ meets the x-axis or the y-axis then the corrsponding coordinate is not divisible by* $\mathrm{char}(\mathbb{K})$.

We show now that in the planar case Kouchnirenko's result holds in arbitrary characteristic without the assumption that f is convenient.

Proposition 4.5
Suppose that $f \in \mathbb{K}[[x, y]]$ is NND, then $\mu(f) = \mu_N(f)$.

Proof: We may assume that $f \in \mathfrak{m}^2$. Moreover, if $\Gamma(f)$ consists of a single point α then either $\alpha = (1, 1)$ with $\mu(f) = \mu_N(f) = 1$ or $\mu(f) = \mu_N(f) = \infty$. We thus also may assume that $\Gamma(f)$ has at least one edge.
Let $\alpha = (k, l)$ be the end point of $\Gamma(f)$ closest to the y-axis, and suppose that $k \geq 2$, then $\mu_N(f) = \infty$ and by Theorem 3.4 also $\mu(f) = \infty$. Thus we may assume that either α is on the y-axis or its distance k to the y-axis is one. Similarly, we may assume that the end point of $\Gamma(f)$ closest to the x-axis has distance at most one from the x-axis.
Note that there is a unique C-polytope P which contains $\Gamma(f)$ and which has the same number of edges. It is derived from $\Gamma(f)$ by prolonging the obvious edges to the coordinate axes. We want to show that f is SNND w.r.t. P.
Let Δ be an inner face of P. If Δ does not meet any of the coordinate axes, then Δ is a face of $\Gamma(f)$ and condition ND implies that f is also SND along Δ.
If Δ meets the y-axis, then it prolongs the edge Δ' of $\Gamma(f)$ whose end point closest to the y-axis is $\alpha = (k, l)$ with $k \leq 1$. If $k = 0$ then $\Delta = \Delta'$ is an edge of $\Gamma(f)$

and we can see as in the proof of Proposition 4.3 that $\mathrm{j}(\mathrm{in}_\Delta(f))$ has no zero with non-zero y-coordinate. If $k = 1$ then

$$\mathrm{in}_\Delta(f) = \mathrm{in}_{\Delta'}(f) = a \cdot x \cdot y^l + x^2 \cdot g$$

for some $a \in \mathbb{K}^*$ and some $g \in \mathbb{K}[x, y]$. Since f satisfies ND along Δ' there is no point $q \in \mathrm{Sing}(\mathrm{in}_\Delta(f))$ with both coordinates non-zero, and since

$$\mathrm{in}_\Delta(f)_x = a \cdot y^l + x \cdot (2g + x \cdot g_x)$$

there can also be no point $q \in \mathrm{Sing}(\mathrm{in}_\Delta(f))$ with only the y-coordinate non-zero. Thus, in any case we see that $\mathrm{j}(\mathrm{in}_\Delta(f))$ has no zero with a non-zero y-coordinate. Similarly, if Δ meets the x-axis there is no zero of $\mathrm{j}(\mathrm{in}_\Delta(f))$ with a non-zero x-coordinate.

Again by Remark 4.2 f is SND along each inner face of P which meets any of the coordinate axes, and thus altogether f is SNND. Theorem 3.6 implies that $\mu(f) = \mu_N(f)$. □

Example 4.6

We now give an example for a reduced power series which is not SNND w.r.t. any C-polytope. Let $f = x^6 + y^3 + x^5y \in \mathbb{K}[[x, y]]$ with $\mathrm{char}(\mathbb{K}) = 2$ and suppose that f is SNND w.r.t. some C-polytope P. By Remark 4.2 P must be the Newton polygon of f and by Proposition 3.9 f is then rSQH in contradiction to $\mu(\mathrm{in}_P(f)) = \mu(x^6 + y^3) = \infty$. Note that $\mu(f) = 13 > 10 = \mu_N(f)$.

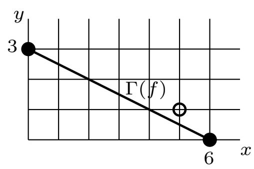

FIGURE 3. The Newton polygon of $f = x^6 + y^3 + x^5y$.

Beelen and Pellikaan investigate in [BeP00] plane curve singularities in arbitrary characteristic, and under the assumption of convenience and weak Newton non-degeneracy they give a formula for the delta invariant of f in terms of the Newton polygon. We generalise this by dropping the condition of convenience. Moreover, their proof shows that if f is WND along an edge Δ of $\Gamma(f)$ of lattice length k, then there are exactly k branches of f corresponding to Δ (see [BeP00, Rem. 3.18]). Combining these results Milnor's formula with the Newton number instead of the Milnor number follows in arbitrary characteristic.

If $f \in \mathbb{K}[[x, y]]$ is convenient and $\Delta_1, \ldots, \Delta_k$ are the facets of the Newton polygon $\Gamma(f)$, then we define

$$\nu_N(f) := V_2(\Gamma_-(f)) - \frac{V_1(\Gamma_-(f))}{2} + \frac{\sum_{i=1}^k l(\Delta_i)}{2},$$

where $l(\Delta_i)$ is the lattice length of Δ_i, i.e. one less than the number of lattice points on Δ_i. If f is not convenient, we generalise this definition to

$$\nu_N(f) := \sup\{\nu_N(f_m) \mid f_m = f + x^m + y^m, m \in \mathbb{N}\}.$$

Example 4.7
If $f = x^4y + x^2y^2 + y^5$ and $m \geq 6$, then the Newton polygon of f_m has three facets $\Delta_1, \Delta_2, \Delta_3$ of lattice length one (see Figure 4). We thus get

$$\begin{aligned}\nu_N(f_m) &= V_2(\Gamma_-(f_m)) - \frac{V_1(\Gamma_-(f_m))}{2} + \frac{l(\Delta_1) + l(\Delta_2) + l(\Delta_3)}{2} \\ &= \left(10 + \frac{m-4}{2}\right) - \frac{5 + 4 + (m-4)}{2} + \frac{1+1+1}{2} = 7,\end{aligned}$$

where the $\frac{m-4}{2}$ corresponds to both, the area of the gray triangle in Figure 4 and half the length of the intersection of this triangle with the x-axis. We thus have

$$\nu_N(f) = \nu_N(f_6) = 7.$$

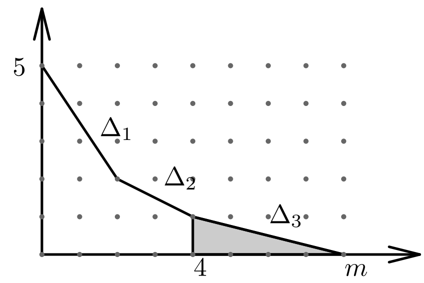

FIGURE 4. The Newton polygon of $x^6 + x^2y^2 + y^5$.

The number $\nu_N(f)$ is related to the delta invariant of f. If we consider a minimal resolution of the singularity computed via successive blowing up and denote by $Q \to 0$ that Q is an infinitely near point of the origin, then we know that (see [GLS07, Prop. 3.34])

$$\delta(f) = \sum_{Q \to 0} \frac{m_Q \cdot (m_Q - 1)}{2},$$

where m_Q denotes the multiplicity of the strict transform of f at Q. Beelen and Pellikaan introduced the number

$$\nu(f) := \sum_{Q \text{ special}} \frac{m_Q \cdot (m_Q - 1)}{2} \leq \delta(f),$$

where an infinitely near point is *special* if it is 0 or the origin in the corresponding chart of the blowing up procedure. Clearly, $\nu(f)$ depends on the coordinates of f, while $\delta(f)$ does not. Beelen and Pellikaan then show that $\nu(f)$ and $\nu_N(f)$ coincide if f is convenient. Using our generalisation of ν_N, the condition of convenience can be dropped.

Lemma 4.8
If $f \in \mathbb{K}[[x,y]]$, then $\nu(f) = \nu_N(f)$.

Proof: If x^2 or y^2 divides f then both numbers are infinite, so we may assume that this is not the case.
If y divides f then passing from f to $f + x^m$ for some large m replaces the smooth branch y by some other smooth branch with the same tangent direction, and the analogous argument holds if x divides f. Therefore, $\nu(f) = \nu(f_m)$ for sufficiently large m. Moreover, as in Example 4.7 the values of $\nu_N(f_m)$ stabilise for sufficiently large m, since the area that is added in the computation of $V_2(\Gamma_-(f_m))$ coincides with the length that is subtracted in the computation of $V_1(\Gamma_-(f_m))$. Using the result of Beelen and Pellikaan [BeP00, Thm. 3.11] we can summarise that for a sufficiently large m

$$\nu_N(f) = \nu_N(f_m) = \nu(f_m) = \nu(f).$$

□

One would like to know under which conditions $\nu(f)$ actually coincides with $\delta(f)$, and Beelen and Pellikaan show in [BeP00, Prop. 3.17] that for a convenient f weak Newton non-degeneracy is a sufficient condition to assure this. Again we can drop the condition of convenience.

Proposition 4.9
If $f \in \mathbb{K}[[x,y]]$ is WNND, then $\nu_N(f) = \nu(f) = \delta(f)$.

Proof: If f is divisible by x^2 or y^2, then all of these numbers are infinite, so we may exclude this case. Moreover, we may restrict to the case that y divides f but x does not, as the remaining cases work analogously. As above, passing from f to $f + x^m$ for a large m replaces the smooth branch y by some smooth branch with the same tangent direction, so the delta invariant does not change. Moreover, if m is sufficiently large then $\Gamma(f)$ differs from $\Gamma(f_m)$ by one additional facet Δ, a line segment with end points $(m, 0)$ and $(k, 1)$. The initial form along Δ is $\mathrm{in}_\Delta(f_m) = x^m + c\cdot x^k y$ and $\mathrm{j}(x^m + c\cdot x^k y)$ has no zero in the torus $(\mathbb{K}^*)^2$. Therefore, f_m is convenient and WNND, so that [BeP00, Prop. 3.17] and Lemma 4.8 imply that for sufficiently large m

$$\delta(f) = \delta(f_m) = \nu(f_m) = \nu_N(f_m) = \nu_N(f) = \nu(f).$$

□

We denote by $r(f)$ the number of branches of f counted with multiplicity, i.e. the number of irreducible factors of f. Moreover, we introduce the combinatorial counterpart of r as

$$r_N(f) = \sum_{i=1}^{k} l(\Delta_i) + \max\{j \mid x^j \text{ divides } f\} + \max\{l \mid y^l \text{ divides } f\}.$$

Beelen and Pellikan ([BeP00]) realised that weak Newton non-degeneracy is a sufficient condition for these two numbers to coincide.

Lemma 4.10
$r(f) \leq r_N(f)$, and if f is WNND then $r_N(f) = r(f)$.

Proof: If j and l are the maximal such that x^j and y^l divide f, then

$$r_N(f) = \sum_{i=1}^{k} l(\Delta_i) + j + l.$$

It is well known that the lattice length of a facet of the Newton polygon of f is an upper bound for the number of branches of f corresponding to this facet. This implies the inequality $r(f) \leq r_N(f)$. The proof of [BeP00, Prop. 3.17] shows then that f has indeed $l(\Delta_i)$ branches corresponding to Δ_i, if f is WND along Δ_i (see also [BeP00, Prop. 3.18]). This shows that f has exactly $r_N(f)$ branches, counting the branches x and y with multiplicity, if f is WNND. □

Lemma 4.11
If $f \in \mathbb{K}[[x,y]]$, then $\mu_N(f) = 2 \cdot \nu_N(f) - r_N(f) + 1$.

Proof: If x^2 or y^2 divides f, then both sides of the equation are infinite, and we may thus assume that this is not the case.
Suppose now that $\Gamma(f)$ has the facets $\Delta_1, \ldots, \Delta_k$ and let m be very large. Then $\Gamma(f_m)$ also has the facets $\Delta_1, \ldots, \Delta_k$, and it has an additional facet of lattice length one if x divides f and the same for y. In particular, $r_N(f) = r_N(f_m)$.
Since f_m is convenient the definition of μ_N, ν_N and r_N gives right away

$$\mu_N(f_m) = 2 \cdot \nu_N(f_m) - r_N(f_m) + 1.$$

Moreover, for sufficiently large m we have $\mu_N(f_m) = \mu_N(f)$ and $\nu_N(f_m) = \nu_N(f)$, and hence

$$\mu_N(f) = 2 \cdot \nu_N(f) - r_N(f) + 1.$$

□

Combining the last three results we get the following generalisation of the result of Beelen and Pellikan.

Theorem 4.12
If $f \in \mathbb{K}[[x,y]]$ is WNND, then $\mu_N(f) = 2 \cdot \delta(f) - r(f) + 1$.

Proof: The result follows from Lemma 4.11, Proposition 4.9 and Lemma 4.10. □

Together with Kouchnirenko's formula for the Milnor number in Proposition 4.5 we deduce then that Milnor's formula $\mu(f) = 2 \cdot \delta(f) - r(f) + 1$ in characteristic zero (see [Mil68] or [GLS07, Prop. 3.35]) holds in arbitrary characteristic for Newton non-degenerate singularities, even without the condition of convenience.

Theorem 4.13
If $f \in \mathbb{K}[[x,y]]$ is NND, then $\mu(f) = 2 \cdot \delta(f) - r(f) + 1$.

Without the assumption of Newton non-degeneracy one has at least an inequality as was proved by Melle and Wall [MHW01, Formula (14)] based on a result by Deligne [Del73, Theorem 2.4]. We are grateful to Alejandro Melle for pointing out this result.

Proposition 4.14 (Deligne, Melle-Wall)
If $f \in \mathbb{K}[[x,y]]$, then $\mu(f) \geq 2 \cdot \delta(f) - r(f) + 1$.

The difference of the two sides is measured by the so called Swan character which counts wild vanishing cycles that can only occur in positive characteristic. For details we refer to [MHW01] and [Del73].

Note that we always have the inequalities

$$\mu_N(f) \leq 2 \cdot \nu_N(f) - r(f) + 1 \leq 2 \cdot \delta(f) - r(f) + 1 \leq \mu(f).$$

It is easy to see that the equality may be violated in positive characteristic, and that the above inequalities may be strict. E.g. $\mathrm{char}(\mathbb{K}) = 2$, $f = (x-y)^2 + x^5$ then $\mu_N(f) = 1$, $\nu_N(f) = 1$, $\delta(f) = 2$, $r(f) = 1$, $\mu(f) = \infty$, so that

$$\mu_N(f) < 2 \cdot \nu_N(f) - r(f) + 1 < 2 \cdot \delta(f) - r(f) + 1 < \mu(f).$$

Note that the first two inequalities hold in characteristic zero as well.
We can now use the above results to measure the difference between $\mu(f)$ and $\mu_N(f)$ better and thereby generalise a result of Płoski [Pło99], who proved this for $\mathbb{K} = \mathbb{C}$ and f convenient.

Proposition 4.15
If $f \in \mathbb{K}[[x,y]]$, then $\mu(f) - \mu_N(f) \geq r_N(f) - r(f) \geq 0$.

Proof: Combining Proposition 4.14 with Lemma 4.8, Lemma 4.10 and Lemma 4.11 we get

$$\begin{aligned} \mu(f) &\geq 2 \cdot \delta(f) - r(f) + 1 \geq 2 \cdot \nu(f) - r(f) + 1 \\ &= 2 \cdot \nu_N(f) - r(f) + 1 = 2 \cdot \nu_N(f) - r_N(f) + 1 + (r_N(f) - r(f)) \\ &= \mu_N(f) + (r_N(f) - r(f)) \geq \mu_N(f), \end{aligned}$$

which proves the claim. □

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