NORMAL FORMS OF HYPERSURFACE SINGULARITIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. In connection with his classification of real and complex hypersurface singularities Arnol'd introduced in the 1970's the condition A which allows to compute a normal form of a power series f with respect to right equivalence. For this he uses piecewise filtrations induced by the Newton polytope of f. Wall considered in 1999 a non-degeneracy condition which implies a weaker but sufficient form of condition A and which can be checked on the Newton polytope. We generalise Arnol'd's and Wall's results to arbitrary characteristic and modify it in order to treat also contact equivalence. We deduce thus normal forms and determinacy bounds for hypersurface singularities with respect to right and contact equivalence in arbitrary characteristic. We apply this to obtain a partial classification of hypersurface singularities in positive characteristic.

1. INTRODUCTION

Throughout this paper $\mathbb{K}$ shall be an algebraically closed field of arbitrary characteristic unless explicitly stated otherwise. By

$$\mathbb{K}[[\boldsymbol{x}]] = \mathbb{K}[[x_1, \ldots, x_n]] = \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha \cdot \boldsymbol{x}^\alpha \;\middle|\; a_\alpha \in \mathbb{K} \right\}$$

we denote the formal power series ring over $\mathbb{K}$ in $n \geq 2$ indeterminates $x_1, \ldots, x_n$ using the usual multiindex notation $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Moreover, we denote by

$$\mathfrak{m} = \langle x_1, \ldots, x_n \rangle \lhd \mathbb{K}[[\boldsymbol{x}]]$$

the unique maximal ideal of $\mathbb{K}[[\boldsymbol{x}]]$, so that the set of units in $\mathbb{K}[[\boldsymbol{x}]]$ is $\mathbb{K}[[\boldsymbol{x}]]^* = \mathbb{K}[[\boldsymbol{x}]] \setminus \mathfrak{m}$.

If we are interested in the classification of power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ there are two natural equivalence relations, right equivalence and contact equivalence.

We say, two power series $f, g \in \mathbb{K}[[\boldsymbol{x}]]$ are *right equivalent* to each other if and only if there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ such that $f = \varphi(g)$, and we denote this by $f \sim_r g$. If we replaced $\mathbb{K}$ by the complex numbers and formal power series

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by convergent ones then φ would induce an isomorphism of the zero fiber of f as well as of close by fibers. That is how we should interpret right equivalence also in this more general setting.

If we are only interested in the geometry of the zero fiber, then the second equivalence relation is the appropriate one. We call $f, g \in \mathbb{K}[[\boldsymbol{x}]]$ *contact equivalent* if and only if there is an automorphism $\varphi \in \operatorname{Aut}(\mathbb{K}[[\boldsymbol{x}]])$ and a unit $u \in \mathbb{K}[[\boldsymbol{x}]]^*$ such that $f = u \cdot \varphi(g)$, and we denote this by $f \sim_c g$. The idea here is, that φ and u still induce an isomorphism of the zero fibers of f and g.

However, we have to replace the geometric notion of the zero fiber by the algebraic counterpart of its coordinate ring. That is, for a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ we call $R_f = \mathbb{K}[[\boldsymbol{x}]]/\langle f\rangle$ the induced *hypersurface singularity*. We obviously have

$$f \sim_c g \quad\Longleftrightarrow\quad R_f \cong R_g,$$

i.e. f and g are contact equivalent if and only if the induced hypersurface singularities are isomorphic as analytic $\mathbb{K}$-algebras.

Over the complex numbers we would say that the origin is an isolated singular point of f if f is not singular at any point close-by, i.e. the origin is the only common zero of the partial derivatives of f. We have to reformulate this algebraically so that it works over any field. For a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$ we denote by

$$\mathrm{j}(f) = \langle f_{x_1}, \ldots, f_{x_n}\rangle \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$$

the *Jacobian ideal* of f, i.e. the ideal generated by the partial derivatives of f, and we call the associated algebra

$$M_f = \mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{j}(f)$$

the *Milnor algebra* of f and its dimension

$$\mu(f) = \dim_{\mathbb{K}}(M_f)$$

the *Milnor number* of f. We then call the origin an isolated singular point of f, or we simply call f an *isolated singularity* if $\mu(f) < \infty$, which is equivalent to the existence of a positive integer k such that $\mathfrak{m}^k \subseteq \mathrm{j}(f)$.

Similarly, over $\mathbb{C}$ we would call the origin an isolated singular point of the hypersurface singularity defined by f if this hypersurface singularity has no other singular point close-by, i.e. the origin is the only common zero of f and its partial derivatives. Algebraically we thus consider the *Tjurina ideal*

$$\mathrm{tj}(f) = \langle f, f_{x_1}, \ldots, f_{x_n}\rangle = \langle f\rangle + \mathrm{j}(f) \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$$

of f, the associated *Tjurina algebra*

$$T_f = \mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{tj}(f)$$

of f and its dimension

$$\tau(f) = \dim_{\mathbb{K}}(T_f),$$

the *Tjurina number* of f. We then call the origin an isolated singular point of the hypersurface singularity R_f, or we will simply call R_f an *isolated hypersurface singularity* if $\tau(f) < \infty$, or equivalently if there is a positive integer such that $\mathfrak{m}^k \subseteq \mathrm{tj}(f)$.

It is straight forward to see that the Milnor number is *invariant* under right equivalence and the Tjurina number is *invariant* under contact equivalence.

Our principle interest is the classification of power series in positive characteristic with respect to right respectively contact equivalence, where the latter is the same as to say that we are interested in classifying hypersurface singularities up to isomorphism. In order to have good finiteness conditions at hand we restrict to the case that f is an isolated singularity for right equivalence respectively that R_f is an isolated hypersurface singularity for contact equivalence. Note that these are two distinct conditions in positive characteristic (see also [BGM10]).

A first important step in the attempt to classify singularities from a theoretical point of view as well as from a practical one is to know that the equivalence class is determined by a finite number of terms of the power series f and to find the corresponding degree bound. We say that f is *right* respectively *contact k-determined* if f is right respectively contact equivalent to every g which coincides with f up to order k.

In [BGM10] we have shown that f is finitely right repectively contact determined if and only if $\mu(f)$ respectively $\tau(f)$ is finite, and we have shown that $2\cdot\mu(f)-d(f)+2$ respectively $2\cdot\tau(f)-\mathrm{ord}(f)+2$ is an upper bound for the determinacy. Here $\mathrm{ord}(f)$ denotes the order and $d(f)$ the differential order, i.e. the minimum of the orders of all partial derivatives, of f. In Corollaries 4.6 and 4.7 we show how this degree bound can be considerably improved when the singularities satisfy the conditions AA resprectively AAC introduced in Section 4 (see also the examples in Section 5).

Once we know that a finite number of terms of f suffices to determine its equivalence class then we would like to determine a *normal form* for f, i.e. an "efficient" representative for the equivalence class. This is in general a difficult task. The first classes of singularitites one comes across are those which have a *quasihomogeneous* representative. In characteristic zero they are determined by the fact that the Milnor number and the Tjurina number coincide (Theorem of K. Saito, see Theorem 2.1). The next more complicated classes of singularities are those which have a representative with a quasihomogeneous principal part (that governs its topology over the complex numbers), i.e. the *right semi-quasihomogeneous* rSQH respectively *contact semi-quasihomogeneous* cSQH singularities. These are considered in Section 2, and among others we show that they are indeed isolated (see Proposition 2.4).

When obtaining normal forms of power series which are not right semi-quasihomogeneous the only known classification method was introduced by Arnol'd in [Arn75] over the complex numbers and slightly generalised by Wall in [Wal99]. The method generalises semi-quasihomogeneity and requires the principal part $\mathrm{in}_P(f)$ of the power series (with respect to some C-polytope P) to be an *isolated* singularity and its Milnor algebra to have a *finite regular basis* – see Section 3 and 4 for the notions. At the heart lies the study of *piecewise filtrations* as introduced by Arnol'd [Arn75] and used by Kouchnirenko to study non-degeneracy conditions [Kou76]. Section 3 is devoted to these. Arnol'd actually gives a more restrictive condition than Wall but his proof shows that the weaker condition suffices as was pointed out by Wall. In Section 4 we generalise these conditions both in the strict form A of Arnol'd and in the weak form AA of Wall to the situation of contact equivalence, calling them AC and AAC respectively, and derive normal forms for right as well as for contact equivalence in arbitrary characteristic. See Theorem 4.2 and 4.3 and Corollaries 4.4 and 4.5.

The results on normal forms and degree bounds apply to large classes of examples. In Corollary 4.9 we show that all rSQH singularities satisfy AA and all cSQH singularities satisfy AAC. Moreover, a result by Wall [Wal99] over the complex numbers generalises to arbitrary characteristic and shows that all *strictly Newton non-degenerate* singularities (for the definition see Remark 4.11) satisfy both AA and AAC (see Theorem 4.12). In Section 5 we then use our results to consider normal forms for singularities of type T_{pq}, Q_{10}, $W_{1,1}$ and E_7 in Arnol'd's notation in positive characteristic.

2. Quasi- and semi-quasihomogeneous singularities

A polynomial $f \in \mathbb{K}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $n \geq 1$, is called *quasihomogeneous* with respect to the weight vector $w \in \mathbb{Z}^n_{>0}$ if all monomials $\boldsymbol{x}^\alpha$ have the same *weighted degree* $d := \deg_w(\boldsymbol{x}^\alpha) = w \cdot \alpha = \sum_{i=1}^n w_i \cdot \alpha_i$. We say for short that f is QH of type $(w; d)$. By the *Euler formula* a quasihomogeneous polynomial f of weighted degree $\deg_w(f) := d$ satisfies

$$d \cdot f = w_1 \cdot x_1 \cdot f_{x_1} + \ldots + w_n \cdot x_n \cdot f_{x_n},$$

so that

$$\mathrm{j}(f) = \mathrm{tj}(f) \quad \text{if} \quad \mathrm{char}(\mathbb{K}) \not| \, d.$$

In particular, for a quasihomogeneous polynomial f in characteristic zero the Milnor number and the Tjurina number coincide. A famous result of K. Saito states that over the complex numbers the reverse is true as well up to equivalence (see [Sai71]). His proof generalises to any algebraically closed field of characteristic zero.

Theorem 2.1 (Saito)
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and suppose that $f \in \mathbb{K}[[\boldsymbol{x}]]$ is an isolated singularity. Then the following are equivalent:

(a) *f is right equivalent to a quasihomogeneous polynomial.*

(b) *f is contact equivalent to a quasihomogeneous polynomial.*

(c) $\mu(f) = \tau(f)$.

(d) $f \in \mathrm{j}(f)$.

The Milnor and the Tjurina number are important invariants which even characterise the singularities for values 0 and 1 in any characteristic. In fact, by the implicit function theorem we have

$$\mu(f) = 0 \iff \tau(f) = 0 \iff \mathrm{ord}(f) = 1 \iff f \sim_r x_1.$$

If $\mathrm{ord}(f) \geq 3$ we have $\mu(f) \geq \tau(f) \geq n+1 \geq 2$. If $\mathrm{ord}(f) = 2$ we have the following lemma.

Lemma 2.2
For $f \in \mathfrak{m}$ the following are equivalent:

(a) $\mu(f) = 1$.

(b) $\tau(f) = 1$.

(c) (1) $f \sim_r x_1^2 + \ldots + x_n^2$ *if* $\mathrm{char}(\mathbb{K}) \neq 2$.

(2) *$n = 2k$ is even and* $f \sim_r x_1x_2 + \ldots + x_{2k-1}x_{2x}$ *if* $\mathrm{char}(\mathbb{K}) = 2$.

Proof: This follows from [GrK90, 3.5, Prop. 3]. □

The class of *quasihomogeneous singularities*, i. e. of singularities with a quasihomogeneous polynomial representative under right (or contact) equivalence, is an important class of singularities in characteristic zero.
In positive characteristic we have to be more careful, since the Euler relation is not helpful when the characteristic divides the weighted degree. E.g. $f = x^p + y^{p-1}$ is quasihomogeneous of degree $p \cdot (p-1)$ with respect to $w = (p-1, p)$ with $\tau(f) = p \cdot (p-2)$ and $\mu(f) = \infty$. However, when the characteristic does not divide the weighted degree some of the good properties still hold true.

Proposition 2.3
Let $f \in \mathbb{K}[\boldsymbol{x}] \setminus \mathbb{K}$ be QH of type $(w; d)$ with $\gcd(w_1, \ldots, w_n) = 1$.

(a) *If $f \in \mathfrak{m}^3$ then*

$$\mu(f) < \infty \quad \iff \quad \tau(f) < \infty \text{ and } \mathrm{char}(\mathbb{K}) \not| \, d.$$

In this case obviously $\mu(f) = \tau(f)$.

(b) *If* $\mathrm{char}(\mathbb{K}) \not| \, d$ *and $g \in \mathbb{K}[[\boldsymbol{x}]]$, then*

$$f \sim_r g \quad \iff \quad f \sim_c g.$$

Proof: (a) If the characteristic does not divide d and $\tau(f) < \infty$ then we are done by the Euler formula. Conversely, if $\mu(f) < \infty$ then $\tau(f) < \infty$ and we have to show that the characteristic does not divide d. Assume the contrary. The Euler formula then gives the identity

$$w_1 \cdot x_1 \cdot f_{x_1} + \ldots + w_n \cdot x_n \cdot f_{x_n} = 0.$$

Since $\gcd(w_1, \ldots, w_n) = 1$ we may assume that w_n is not divisible by the characteristic, and we thus deduce

$$x_n \cdot f_{x_n} = -\frac{w_1}{w_n} \cdot x_1 \cdot f_{x_1} - \ldots - \frac{w_{n-1}}{w_n} \cdot x_{n-1} \cdot f_{x_{n-1}}.$$

f being in $\mathfrak{m}^3$ the variable x_n is not zero in $M_f = \mathbb{K}[[\boldsymbol{x}]]/\operatorname{j}(f)$, so that f_{x_n} is a zero divisor in the $\mathbb{K}[[\boldsymbol{x}]]/\langle f_{x_1}, \ldots, f_{x_{n-1}}\rangle$. Thus $f_{x_1}, \ldots, f_{x_n}$ is not a regular sequence in the Cohen-Macaulay ring $\mathbb{K}[[\boldsymbol{x}]]$, and therefore the $\mathbb{K}$-algebra M_f is not zero-dimensional (see e.g. [GLS07, Corollary B.8.3]), i.e. we get the contradiction $\mu(f) = \infty$.

(b) The proof works as in characteristic zero since d-th roots exist in $\mathbb{K}[[\boldsymbol{x}]]^*$ if d is not divisible by $\operatorname{char}(\mathbb{K})$, see e.g. [GLS07, Lemma 2.13]).

□

Note that the condition $f \in \mathfrak{m}^3$ cannot be avoided, though as seen in the proof it can be weakened to e.g.

$$\exists\, i = 1, \ldots, n \;:\operatorname{char}(\mathbb{K}) \not| \, w_i \text{ and } \; x_i \notin \langle f_{x_1}, \ldots, f_{x_{i-1}}, f_{x_{i+1}}, \ldots, f_{x_n}\rangle.$$

To see that we cannot avoid it completely consider $f = xy \in \mathbb{K}[[x, y]]$ with $\operatorname{char}(\mathbb{K}) = 2$. It is QH of type $\big((1, 1); 2\big)$ and the Milnor number is one, yet the characteristic divides the weighted degree.

When classifying singularities with respect to right or contact equivalence the first classes one comes across have quasihomogeneous representatives. This is maybe the most important reason why they deserve attention. The next more complicated class of singularities are those which have a quasihomogeneous principal part that somehow governs its discrete part of the classification.

For a power series $f = \sum_\alpha a_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[[\boldsymbol{x}]]$ and a weight vector $w \in \mathbb{Z}^n_{>0}$ we denote by

$$\operatorname{in}_w(f) = \sum_{w\cdot\alpha \text{ minimal}} a_\alpha \boldsymbol{x}^\alpha$$

the *initial form* or *principal part* of f with respect to w. We call the power series f *right semi-quasihomogeneous* rSQH respectively *contact semi-quasihomogeneous* cSQH with respect to w if $\mu\big(\operatorname{in}_w(f)\big) < \infty$ respectively $\tau\big(\operatorname{in}_w(f)\big) < \infty$. A right resp. contact equivalence class of singularities is called *semi-quasihomogeneous* if it has a semi-quasihomogeneous representative. Note that in characteristic zero the notions rSQH and cSQH coincide.

Moreover, in characteristic zero it is known that semi-quasihomogeneous singularities are always isolated and that their Milnor number coincides with the Milnor number of the principal part, i.e. if $\mathbb{K}=\mathbb{C}$ their topology is governed by the principal part. In positive characteristic we get an analogous statement.

Proposition 2.4
Let $f\in\mathbb{K}[[\boldsymbol{x}]]$ and $w\in\mathbb{Z}^n_{>0}$.

(a) *If $\mu\big(\mathrm{in}_w(f)\big)<\infty$ and $d=\deg_w\big(\mathrm{in}_w(f)\big)$, then*

$$\mu(f)=\mu(\mathrm{in}_w(f))=\left(\frac{d}{w_1}-1\right)\cdot\ldots\cdot\left(\frac{d}{w_n}-1\right)<\infty.$$

(b) $\tau(f)\leq\tau\big(\mathrm{in}_w(f)\big)$.

In particular, semi-quasihomogeneous singularities are isolated.

Proof: (a) Let d be the degree of $\mathrm{in}_w(f)$. Then

$$f':=\frac{f\big(t^{w_1}x_1,\ldots,t^{w_n}x_n\big)}{t^d}=\mathrm{in}_w(f)+t\cdot g'\in\mathbb{K}[[\boldsymbol{x},t]]$$

for some power series $g'\in\mathbb{K}[[\boldsymbol{x},t]]$. We can use f' to define the following local $\mathbb{K}$-algebra homomorphism

$$\mathbb{K}[[\boldsymbol{z},t]]\longrightarrow\mathbb{K}[[\boldsymbol{x},t]]:t\mapsto t,z_i\mapsto f'_{x_i}.$$

This gives $\mathbb{K}[[\boldsymbol{x},t]]$ the structure of a $\mathbb{K}[[\boldsymbol{z},t]]$-algebra, and if we tensorise $\mathbb{K}[[\boldsymbol{x},t]]$ with $\mathbb{K}=\mathbb{K}[[\boldsymbol{z},t]]/\langle\boldsymbol{z},t\rangle$ we get

$$\mathbb{K}[[\boldsymbol{x},t]]\otimes_{\mathbb{K}[[\boldsymbol{x},t]]}\mathbb{K}=\mathbb{K}[[\boldsymbol{x},t]]/\langle f'_{x_1},\ldots,f'_{x_n},t\rangle\cong\mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{j}(\mathrm{in}_w(f)),$$

which by assumption is a finite dimensional $\mathbb{K}$-vector space of dimension $\mu(\mathrm{in}_w(f))$. Since $(f'_{x_1},\ldots,f'_{x_n},t)$ is a regular sequence, $\mathbb{K}[[\boldsymbol{x},t]]$ is flat as a $\mathbb{K}[[\boldsymbol{z},t]]$-module (see e.g. [Eis96, Theorem 18.16]) and thus it is free of rank $\mu\big(\mathrm{in}_w(f)\big)$ by Nakayama's Lemma. Tensoring with $\mathbb{K}[[t]]=\mathbb{K}[[\boldsymbol{z},t]]/\langle\boldsymbol{z}\rangle$ we get

$$\mathbb{K}[[\boldsymbol{x},t]]\otimes_{\mathbb{K}[[\boldsymbol{z},t]]}\mathbb{K}[[\boldsymbol{z},t]]/\langle\boldsymbol{z}\rangle\cong\mathbb{K}[[\boldsymbol{x},t]]/\langle f'_{x_1},\ldots,f'_{x_n}\rangle\tag{1}$$

as a free $\mathbb{K}[[t]]$-module of rank $\mu\big(\mathrm{in}_w(f)\big)$.

Passing to the field of fractions $\mathbb{L}=\mathbb{K}((t))$ of $\mathbb{K}[[t]]$ we have the isomorphism of local $\mathbb{L}$-algebras

$$\varphi:\mathbb{L}[[\boldsymbol{x}]]\longrightarrow\mathbb{L}[[\boldsymbol{x}]]:x_i\mapsto t^{w_i}x_i.$$

Moreover

$$f'=\frac{\varphi(f)}{t^d},$$

so that in $\mathbb{L}[[\boldsymbol{x}]]$ we have the equality of ideals

$$\langle f'\rangle=\langle\varphi(f)\rangle\quad\text{and}\quad\mathrm{j}(f')=\mathrm{j}\left(\varphi(f)\right)=\varphi\big(\mathrm{j}(f)\big).$$

Extending scalars in (1) to the field of fractions $\mathbb{L}$ we get an isomorphism of $\mathbb{L}$-vector spaces

$$\mathbb{K}[[\boldsymbol{x},t]]/\langle f'_{x_1},\ldots,f'_{x_n}\rangle\otimes_{\mathbb{K}[[t]]}\mathbb{L}\cong\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{j}(f')\cong\mathbb{L}[[\boldsymbol{x}]]/\,\mathrm{j}(f).$$

By freeness the left hand side is of dimension $\mu\big(\operatorname{in}_w(f)\big)$ while the right hand side has dimension $\mu(f)$. For the formula for $\mu(\operatorname{in}_w(f))$ see [BGM10, Prop. 3.8].

(b) It suffices to consider the case $\tau\big(\operatorname{in}_w(f)\big) < \infty$, and the proof then is similar to (a), using the map

$$\mathbb{K}[[\boldsymbol{z}, t]] \longrightarrow \mathbb{K}[[\boldsymbol{x}, t]] : t \mapsto t, z_0 \mapsto f', z_i \mapsto f'_{x_i}$$

for $i = 1, \ldots, n$, where $\boldsymbol{z} = (z_0, \ldots, z_n)$. Since $(f', f'_{x_1}, \ldots, f'_{x_n}, t)$ is not a regular sequence, $\mathbb{K}[[\boldsymbol{x}, t]]/\langle f', f'_{x_1}, \ldots, f'_{x_n}\rangle$ is finitely generated but may have torsion as a $\mathbb{K}[[t]]$-module. Tensoring this module with $\mathbb{K}$ over $\mathbb{K}[[t]]$ gives a $\mathbb{K}$-vector space of dimension $\tau(\operatorname{in}_w(f))$. Tensoring with $\mathbb{L} = \mathbb{K}((t))$ over $\mathbb{K}[[t]]$ kills the torsion and gives an $\mathbb{L}$-vector space of dimension $\tau(f) \leq \tau(\operatorname{in}_w(f))$.

□

Note that the condition on the finiteness of $\mu\big(\operatorname{in}_w(f)\big)$ in Proposition 2.4 (a) cannot be avoided, and $\tau(f)$ will in general not coincide with $\tau\big(\operatorname{in}_w(f)\big)$. Moreover, if $f \in \mathfrak{m}^3$ we get from Proposition 2.3 that $\mu(\operatorname{in}_w(f)) < \infty$ is equivalent to $\tau(\operatorname{in}_w(f)) < \infty$ and $\operatorname{char}(\mathbb{K}) \nmid d = \deg_w(\operatorname{in}_w(f))$.
Consider $f = x^7 + x^6y + y^4 \in \mathbb{K}[[x, y]]$ with $\operatorname{char}(\mathbb{K}) = 7$. f is cSQH with principal part $\operatorname{in}_w(f) = x^7 + y^4$ which is QH of type $\big((4, 7); 28\big)$ with

$$\tau\big(\operatorname{in}_w(f)\big) = 21 > 17 = \tau(f).$$

Moreover, $\mu\big(\operatorname{in}_w(f)\big) = \infty$, but $\mu(f) = 21$. Note, that here of course the characteristic of the base field divides the weighted degree of $\operatorname{in}_w(f)$.

Remark 2.5
To each power series $f = \sum_\alpha a_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[[\boldsymbol{x}]]$ we can associate its *Newton diagram* $\Gamma_+(f)$ as the convex hull of the set

$$\bigcup_{\alpha \in \operatorname{supp}(f)} \big(\alpha + \mathbb{R}^n_{\geq 0}\big)$$

where $\operatorname{supp}(f) = \{\alpha \mid a_\alpha \neq 0\}$ denotes the support of f. This is an unbounded polytope in $\mathbb{R}^n$. We call the union $\Gamma(f)$ of its compact faces the *Newton polytope* of f. Note that the Newton polytope of a QH or SQH polynomial has exactly one facet, where a facet is a face of dimension $n - 1$. For later use we denote by $\Gamma_-(f)$ the union of line segments joining points on $\Gamma(f)$ with the origin. (See Figure 1 for an example.)

3. Piecewise filtrations and graded algebras

Fixing a weight vector $w \in \mathbb{Z}^n_{>0}$ we get in a natural way a filtration on $\mathbb{K}[[\boldsymbol{x}]]$. If a singularity is semi-quasihomogeneous with respect to w then this filtration is perfectly suited to study the singularity and in general w singles out a unique facet of the Newton polytope of the defining power series. However, in general we will have to consider more complicated filtrations since there is no single facet of the

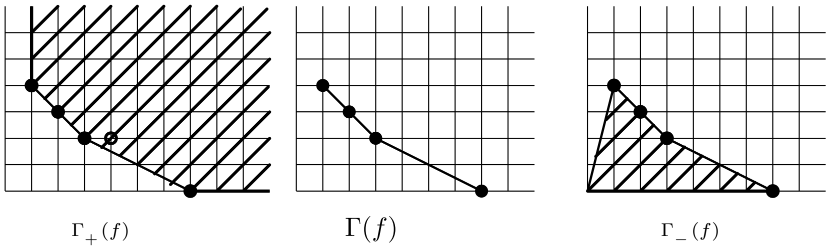

FIGURE 1. The Newton polytope of $x \cdot (y^4 + xy^3 + x^2y^2 - x^3y^2 + x^6)$.

Newton polytope which captures enough information on the singularity. This was noted by Arnold and he introduced in [Arn75] piecewise filtrations which are used to study non-degeneracy conditions by Kouchnirenko in [Kou76].
Given weight vectors $w_i \in \mathbb{Q}^n_{>0}$ with positive entries, $i = 1, \ldots, k$, they define linear functions

$$\lambda_i : \mathbb{R}^n \longrightarrow \mathbb{R} : r \mapsto w_i \cdot r := \sum_{j=1}^{n} w_{i,j} \cdot r_j,$$

and their minimum defines a convex piecewise linear function

$$\lambda : \mathbb{R}^n \longrightarrow \mathbb{R} : r \mapsto \min\{\lambda_1(r), \ldots, \lambda_k(r)\}.$$

We will always assume that the set of weights is *irredundant*, i.e. that none of the λ_i is superfluous in the definition of λ. The set

$$P_\lambda = \{r \in \mathbb{R}^n_{\geq 0} \mid \lambda(r) = 1\}$$

is a compact rational polytope of dimension $n-1$ in the positive orthant $\mathbb{R}^n_{\geq 0}$, and its facets are given by

$$\Delta_i = \{r \in P_\lambda \mid \lambda_i(r) = 1\}.$$

P_λ has the property that each ray in $\mathbb{R}^n_{\geq 0}$ emanating from the origin meets P_λ in precisely one point and that the region in $\mathbb{R}^n_{\geq 0}$ lying above P_λ is convex. Following the convention of Wall (see [Wal99]) we call such polytopes *C-polytopes*. Thus, irredundant sets of weight vectors define C-polytopes.
Conversely, given a C-polytope P the suitably scaled inner normal vectors of its facets define an irredundant set of weight vectors such that $P = P_\lambda$ for the corresponding piecewise linear function λ. We denote by λ_P the piecewise linear function defined by P, and by λ_Δ the linear function corresponding to a facet Δ of P.

If $f \in \mathbb{K}[[\boldsymbol{x}]]$ is a *convenient* power series, i.e. if the support of f contains a point on each coordinate axis, then the *Newton polytope* $\Gamma(f)$ is a C-Polytope. C-Polytopes should thus be thought of as generalising the Newton polytope, and in our applications they will basically arise by extending Newton polytopes of non-convenient power series in a suitable way.
For a C-polytope P we denote by N_P the lowest common multiple of the denominators of all entries in the weight vectors corresponding to P, so that $N_P \cdot \lambda_P$ takes

non-negative integer values on $\mathbb{N}^n$. We then define a *piecewise valuation* on $\mathbb{K}[[\boldsymbol{x}]]$ by

$$v_P(f) := \min\{N_P \cdot \lambda_P(\alpha) \mid a_\alpha \neq 0\} \in \mathbb{N}$$

for $0 \neq f = \sum_\alpha a_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[[\boldsymbol{x}]]$ and $v_P(0) := \infty$. v_P satisfies

$$v_P(f \cdot g) \geq v_P(f) + v_P(g) \quad \text{and} \quad v_P(f+g) \geq \min\{v_P(f), v_P(g)\}.$$

Indeed we should like to point out that

$$v_P(f \cdot g) = v_P(f) + v_P(g) \quad \Longleftrightarrow \quad v_P(f) = v_\Delta(f) \quad \text{and} \quad v_P(g) = v_\Delta(g) \tag{2}$$

for some facet Δ of P. The sets

$$F_d := F_{P,d} := \{f \in \mathbb{K}[[\boldsymbol{x}]] \mid v_P(f) \geq d\}$$

with $d \in \mathbb{N}$ are thus ideals in $\mathbb{K}[[\boldsymbol{x}]]$ and satisfy

$$F_d \cdot F_e \subseteq F_{d+e},$$

i.e. they form a *filtration* on $\mathbb{K}[[\boldsymbol{x}]]$. Note also that $F_0 = \mathbb{K}[[\boldsymbol{x}]]$ and $F_1 = \mathfrak{m}$. Moreover, since all weight vectors corresponding to P have only positive entries for each d there is a positive integer m such that

$$\mathfrak{m}^m \subseteq F_d, \tag{3}$$

and also for any k there is a d such that no monomial of degree less than k can have a valuation of degree d, i.e. such that

$$F_d \subseteq \mathfrak{m}^k. \tag{4}$$

Given any C-polytope P and a power series $f \in \mathbb{K}[[\boldsymbol{x}]]$, we call the polynomial

$$\operatorname{in}_P(f) = \sum_{\lambda_P(\alpha) \text{ minimal}} a_\alpha \boldsymbol{x}^\alpha$$

the *initial form* or the *principal part* of f with respect to P. f is said to be *piecewise homogeneous* PH of degree $d \in \mathbb{Q}_{\geq 0}$ with respect to P if $\lambda_P(\alpha) = d$ for all $\alpha \in \operatorname{supp}(f)$. Note that then $f = \operatorname{in}_P(f)$ is a polynomial. The power series f is called *right semi-piecewise homogeneous* rSPH respectively *contact semi-piecewise homogeneous* cSPH with respect to P if $\mu\big(\operatorname{in}_P(f)\big) < \infty$ respectively $\tau\big(\operatorname{in}_P(f)\big) < \infty$.

Even though PH, rSPH and cSPH are straight forward generalisations of QH, rSQH and cSQH things get more complicated. One of the reasons is that the product of two PH polynomials need no longer be so, as Example 3.1 shows.

Example 3.1

Consider the weights $w_1 = (1,2)$ and $w_2 = (3,1)$ together with the polynomials $f = x^7 + y^7$ and $g = x$. The corresponding C-polytope P is the black polygon shown in Figure 2. Both f and g are PH with respect to P of degree 7 respectively

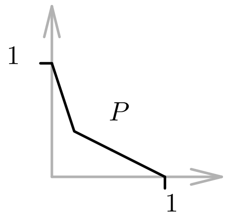

FIGURE 2. The C-polytope to $w_1 = (1,2)$ and $w_2 = (3,1)$.

1. However,

$$\mathrm{in}_P(f \cdot g) = x^8 \neq x^8 + xy^7 = f \cdot g \in F_{P,8},$$

so that $f \cdot g$ is no longer piecewise homogeneous.
This example shows also that there cannot be any *monomial ordering* $>$ which refines the piecewise degree with respect to P if P has more than one side. In fact, suppose there is, then either x^7 or y^7 is the leading term of f. However, since $>$ refines the piecewise degree, xf definitely will have leading term x^8 and yf will have leading term y^8, in contradiction to the fact that the leading term must be compatible with the multiplication by monomials. This makes computations with piecewise filtrations difficult, in particular, we cannot use Gröbner basis methods. □

We also should like to point out, that a polynomial can be PH with respect to many different C-polytopes. E.g. consider for $f = x^5 + x^2y^2 + y^5$ the two C-polytopes shown in Figure 3.

FIGURE 3. Two C-polytopes w.r.t. which $x^5 + x^2y^2 + y^5$ is PH.

If $I \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ is an ideal in $\mathbb{K}[[\boldsymbol{x}]]$ and P is a C-polytope then the filtration induced by P on $\mathbb{K}[[\boldsymbol{x}]]$ leads to the filtration

$$F_0 + I/I \supseteq F_1 + I/I \supseteq F_2 + I/I \supseteq \ldots$$

on $\mathbb{K}[[\boldsymbol{x}]]/I$, and induces thus the *associated graded* $\mathbb{K}$*-algebra*

$$\mathrm{gr}_P\left(\mathbb{K}[[\boldsymbol{x}]]/I\right) = \bigoplus_{d\in\mathbb{N}} (F_d + I)/(F_{d+1} + I)\big) \cong \bigoplus_{d\in\mathbb{N}} F_d/\big((I \cap F_d) + F_{d+1}\big).$$

The product of the classes of two monomials $\boldsymbol{x}^\alpha$ and $\boldsymbol{x}^\beta$ in $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)$ satisfies

$$\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\beta = \begin{cases} \boldsymbol{x}^{\alpha+\beta}, & \text{if } v_P(\boldsymbol{x}^{\alpha+\beta}) = v_P(\boldsymbol{x}^\alpha) + v_P(\boldsymbol{x}^\beta), \\ 0, & \text{else.} \end{cases} \tag{5}$$

We will show next that there are isomorphisms of vector spaces,

$$M_f \cong \mathrm{gr}_P(M_f) \quad \text{respectively} \quad T_f \cong \mathrm{gr}_P(T_f),$$

if the graded algebras are finite dimensional. Therfore, these graded algebras are natural means to study the singularity defined by f. Arnol'd [Arn75] has shown how to use a monomial basis of $\mathrm{gr}_P(M_f)$ under suitable conditions on f to compute a normal form for f. We will generalise this in Section 4.

Proposition 3.2

Let $I \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ be an ideal and let P be a C-polytope.

(a) *Then*

$$\dim_{\mathbb{K}}(\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)) = \dim_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]]/I).$$

(b) *If $\dim_{\mathbb{K}}(\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)) < \infty$, then any monomial basis of $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)$ is a basis $\mathbb{K}[[\boldsymbol{x}]]/I$ as $\mathbb{K}$-vector space.*

Proof: (a) The sequence of ideals

$$\mathbb{K}[[\boldsymbol{x}]] = F_0 + I \supseteq F_1 + I \supseteq \ldots \supseteq F_d + I \supseteq F_{d+1} + I \supseteq \ldots \supseteq I$$

shows that $\dim_K(\mathbb{K}[[\boldsymbol{x}]]/I) < \infty$ if and only if there are only finitely many d such that $0 < \dim_K(F_d + I/F_{d+1} + I) < \infty$, and this is equivalent to $\dim_K\big(\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)\big) < \infty$. In this case the dimensions obviously coincide.

(b) Let B be any set of monomials whose residue classes in $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)$ form a $\mathbb{K}$-vector space basis. We have to show that the residue classes of the elements of B in $\mathbb{K}[[\boldsymbol{x}]]/I$ generate $\mathbb{K}[[\boldsymbol{x}]]/I$ as a $\mathbb{K}$-vector space.

Let $f \in \mathbb{K}[[\boldsymbol{x}]]$ be given and let $d = v_P(f)$ be its piecewise valuation. We then can write f as

$$f = \sum_{\substack{\boldsymbol{x}^\alpha \in B \\ v_P(\boldsymbol{x}^\alpha) = d}} c_\alpha \boldsymbol{x}^\alpha + g_d + h_d$$

with $c_\alpha \in \mathbb{K}$, $g_d \in I \cap F_d$ and $h_d \in F_{d+1}$.

We continue with h_d in the same way, and thus for any $k \geq d$ there are $c_\alpha \in \mathbb{K}$, $g_k \in I \cap F_k$ and $h_k \in F_{k+1}$ such that

$$f = \sum_{\substack{\boldsymbol{x}^\alpha \in B \\ d \leq v_P(\boldsymbol{x}^\alpha) \leq k}} c_\alpha \boldsymbol{x}^\alpha + (g_d + g_{d+1} + \ldots + g_k) + h_k,$$

where $g = g_d + g_{d+1} + \ldots + g_k \in I$. Since $\dim_{\mathbb{K}}\big(\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)\big) < \infty$ there is a d_0 such that

$$(I \cap F_d) + F_{d+1} = F_d \quad \text{for all } d \geq d_0,$$

i.e. $B \cap F_{d_0} = \emptyset$. For $k \geq d_0$ we thus have

$$f - \sum_{\substack{\boldsymbol{x}^\alpha \in B \\ d \leq v_P(\boldsymbol{x}^\alpha) < d_0}} c_\alpha \boldsymbol{x}^\alpha = (g_d + g_{d+1} + \ldots + g_k) + h_k \in I + F_{k+1},$$

where the left hand side does not depend on k. Using Krull's Intersection Theorem this shows that

$$f - \sum_{\substack{\boldsymbol{x}^\alpha \in B \\ d \leq v_P(\boldsymbol{x}^\alpha) < d_0}} c_\alpha \boldsymbol{x}^\alpha \in \bigcap_{k \geq 0} (I + F_k) \overset{(3),(4)}{=} \bigcap_{k \geq 0} (I + \mathfrak{m}^k) = I,$$

and hence the claim.

□

If we apply Proposition 3.2 to M_f and T_f we get the following corollary.

Corollary 3.3
Let $f \in \mathbb{K}[[\boldsymbol{x}]]$ be a power series and P a C-polytope.

(a) $\mu(f) = \dim_{\mathbb{K}}(\mathrm{gr}_P(M_f))$.

(b) $\tau(f) = \dim_{\mathbb{K}}(\mathrm{gr}_P(T_f))$.

v_P induces a $\mathbb{K}$-linear decomposition of the polynomial ring $\mathbb{K}[\boldsymbol{x}] = \bigoplus_{d \geq 0} \mathbb{K}[\boldsymbol{x}]_d$ with

$$\mathbb{K}[\boldsymbol{x}]_d = \langle \boldsymbol{x}^\alpha \mid v_P(\boldsymbol{x}^\alpha) = d \rangle_{\mathbb{K}},$$

and for any ideal $J \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ we can consider

$$J_d = \mathbb{K}[\boldsymbol{x}]_d \cap J.$$

Note that in general $J \neq \bigoplus_{d \geq 0} J_d$ if the C-polytope P has more than one facet, even if J is generated as an ideal by piecewise homogeneous elements. See e.g. $J = \langle \mathrm{in}_P(h \cdot f) \mid h \in \mathbb{K}[[x,y]] \rangle$ with P and f as in Example 3.1 then it is easy to see that $x \cdot f = x^8 + xy^7 = f_8 + f_{10} \in J$ but $f_{10} = xy^7 \notin J$.

Proposition 3.4
Let $I \trianglelefteq \mathbb{K}[[\boldsymbol{x}]]$ be an ideal in $\mathbb{K}[[\boldsymbol{x}]]$ and let P be a C-polytope. Then there is a natural isomorphism of $\mathbb{K}$-vector spaces

$$\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I) \cong \bigoplus_{d \geq 0} \mathbb{K}[\boldsymbol{x}]_d / \mathrm{in}_P(I)_d,$$

where $\mathrm{in}_P(I) = \langle \mathrm{in}_P(f) \mid f \in I \rangle$ is the initial ideal of I with respect to P.

Proof: Consider the $\mathbb{K}$-linear map

$$\varphi_d : \mathbb{K}[\boldsymbol{x}]_d \longrightarrow F_d / \big((I \cap F_d) + F_{d+1}\big) : f \mapsto \overline{f}$$

sending a polynomial f to its residue class. This map is obviously surjective, and we claim that $\ker(\varphi_d) = \mathrm{in}_P(I)_d$. If $f = \mathrm{in}_P(g) \in \mathrm{in}_P(I)_d$ with $g \in I$ then $f - g \in F_{d+1}$ and thus $\varphi_d(f) = \overline{g + (f-g)} = 0$. If $f \in \ker(\varphi_d)$ then $f = g + h$ with $g \in I \cap F_d$

and $h \in F_{d+1}$, so that $f = \mathrm{in}_P(g) \in \mathrm{in}_P(I)_d$. Thus φ_d induces an isomorphism as desired. $\square$

Remark 3.5
If the C-polytope P in Proposition 3.4 has only *one* facet, i.e. P induces a weighted filtration, then we have a natural isomorphism

$$\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I) \cong \mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{in}_P(I).$$

The reason for this is that if P has only one facet then

$$\mathrm{in}_P(I) = \bigoplus_{d\geq 0} \mathrm{in}_P(I)_d,$$

since a weighted homogeneous polynomial lies in the weighted homogeneous ideal $\mathrm{in}_P(I)$ if and only if its weighted homogeneous summands belong to $\mathrm{in}_P(I)$. The isomorphism is thus induced by

$$\mathbb{K}[[\boldsymbol{x}]] \longrightarrow \mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I) : f = \sum_d f_d \mapsto \sum_d \overline{f_d},$$

where $f = \sum_d f_d$ is the decomposition of f into its weighted homogeneous parts and $\overline{f_d}$ is the the residue class of f_d in $F_d/\big((I \cap f_d) + F_{d+1}\big)$.
This fact can be used to compute a monomial basis for $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)$. P determines a weight vector w and we can fix a local weighted degree ordering with respect to this weight vector w. If we then compute a standard basis of I with respect to this ordering, the w-initial forms of the basis elements generate $\mathrm{in}_P(I)$. Moreover, we can compute the standard monomials of of $\mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{in}_P(I)$ via the leading ideal and they are a monomial basis of both, $\mathbb{K}[[\boldsymbol{x}]]/\,\mathrm{in}_P(I)$ and of $\mathrm{gr}_P(\mathbb{K}[[\boldsymbol{x}]]/I)$.

For any C-polytope P the piecewise valuation v_P on $\mathbb{K}[[\boldsymbol{x}]]$ can easily be extended to the $\mathbb{K}[[\boldsymbol{x}]]$-module $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])$ of derivations on $\mathbb{K}[[\boldsymbol{x}]]$. For this we define

$$v_P(\xi) = \min\{\lambda_P(\alpha - e_i) \mid a_{i,\alpha} \neq 0\}$$

where

$$\xi = \sum_{i=1}^{n} \sum_{\alpha\in\mathbb{N}^n} a_{i,\alpha} \cdot \boldsymbol{x}^\alpha \cdot \partial_{x_i} \neq 0$$

and where e_i is the i-th standard basis vector of $\mathbb{Z}^n$, i.e. we naturally extend

$$v_P(\boldsymbol{x}^\alpha \partial_{x_i}) = \lambda_P(\alpha - e_i)$$

where the derivation ∂_{x_i} lowers the exponent of x_i in $\boldsymbol{x}^\alpha$ by one. Note that $v_P(\partial_{x_i})$ is negative.
Straight forward computations show that v_P then satisfies (see e.g. [Bou09, Lemma 2.2.3])

$$v_P(\xi f) \geq v_P(\xi) + v_P(f) \tag{6}$$

for any $0 \neq f \in \mathbb{K}[[\boldsymbol{x}]]$ and any $0 \neq \xi \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])$. Moreover (see [Bou09, Lemma 2.2.5] or [Arn75, Lemma 6.6]), if $f \in \mathfrak{m}^2$ and $g_1, \ldots, g_n \in \mathfrak{m}$ with $v_P(g_i) > v_P(x_i)$ then $\varphi : \mathbb{K}[[\boldsymbol{x}]] \longrightarrow \mathbb{K}[[\boldsymbol{x}]] : x_i \mapsto x_i + g_i$ is an isomorphism and

$$\varphi(f) = f + \xi f + h \tag{7}$$

where

$$\xi = \sum_{i=1}^{n} g_i \partial_{x_i} \quad \text{and} \quad v_P(h) > v_P(\xi) + v_P(f).$$

The fact that we do not always have $v_P(\xi f) = v_P(\xi) + v_P(f)$ is somewhat annoying and forces us to adapt the filtrations induced by v_P on the ideals

$$\mathrm{j}(f) = \{\xi f \mid \xi \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])\}$$

respectively

$$\mathrm{tj}(f) = \{g \cdot f + \xi f \mid g \in \mathbb{K}[[\boldsymbol{x}]], \xi \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])\}.$$

In the following definitions we will restrict our attention in $\mathrm{j}(f) \cap F_d$ respectively $\mathrm{tj}(f) \cap F_d$ to those elements whose valuation is *expected* to be at least d, avoiding those who do so simply by bad luck.
For $d \geq 0$ we define the ideals

$$\mathrm{j}_P^A(f)_d := \{\xi f \mid v_P(\xi) + v_P(f) \geq d\} \lhd \mathbb{K}[[\boldsymbol{x}]]$$

and

$$\mathrm{tj}_P^{AC}(f)_d := \{g \cdot f + \xi f \mid \min\{v_P(g) + v_P(f), v_P(\xi) + v_P(f)\} \geq d\} \lhd \mathbb{K}[[\boldsymbol{x}]].$$

Replacing $\mathrm{j}(f) \cap F_d$ resp. $\mathrm{tj}(f) \cap F_d$ in the definition of $\mathrm{gr}_P(M_f)$ resp. $\mathrm{gr}_P(T_f)$ by $\mathrm{j}_P^A(f)_d$ resp. $\mathrm{tj}_P^{AC}(f)_d$ we get the graded $\mathbb{K}$-algebras

$$\mathrm{gr}_P^A(M_f) := \bigoplus_{d \geq 0} F_d \big/ \big(\mathrm{j}_P^A(f)_d + F_{d+1}\big)$$

respectively

$$\mathrm{gr}_P^{AC}(T_f) := \bigoplus_{d \geq 0} F_d \big/ \big(\mathrm{tj}_P^{AC}(f)_d + F_{d+1}\big).$$

We obviously have the inclusions

$$\mathrm{tj}(f) \cap F_d \supseteq \mathrm{tj}_P^{AC}(f)_d \supseteq \mathrm{j}_P^A(f)_d \subseteq \mathrm{j}(f) \cap F_d,$$

and hence canonical surjections

$$\mathrm{gr}_P^A(M_f) \twoheadrightarrow \mathrm{gr}_P(M_f), \quad \mathrm{gr}_P^A(T_f) \twoheadrightarrow \mathrm{gr}_P(T_f).$$

Due to Proposition 3.2 this yields together with Corollary 3.3 the following result.

Corollary 3.6
Let $f \in \mathbb{K}[[\boldsymbol{x}]]$ be a power series and let P be a C-polytope.

(a) *Any monomial basis B of* $\mathrm{gr}_P^A(M_f)$ *generates* $\mathrm{gr}_P(M_f)$*, and if* $\mu(f)<\infty$ *then* B *also generates* M_f*. In particular,*

$$\mu(f)=\dim_{\mathbb{K}}(\mathrm{gr}_P(M_f))\leq \dim_K(\mathrm{gr}_P^A(M_f)).$$

(b) *Any monomial basis B of* $\mathrm{gr}_P^{AC}(T_f)$ *generates* $\mathrm{gr}_P(T_f)$*, and if* $\tau(f)<\infty$ *then* B *also generates* T_f*. In particular,*

$$\tau(f)=\dim_{\mathbb{K}}(\mathrm{gr}_P(T_f))\leq \dim_K(\mathrm{gr}_P^A(T_f)).$$

Following Arnol'd [Arn75] and Wall [Wal99], who considered this notion for M_f, we call a monomial basis of $\mathrm{gr}_P^A(M_f)$ respectively $\mathrm{gr}_P^{AC}(T_f)$ a *regular basis* for M_f respectively T_f.

We should point out that the finiteness of $\mu(f)$ respectively of $\tau(f)$ does *not* suffice in general to guarantee the finite dimensionality of $\mathrm{gr}_P^A(M_f)$ respectively of $\mathrm{gr}_P^{AC}(T_f)$. The reason for this is that elements of valuation d in $\mathrm{j}(f)$ respectively in $\mathrm{tj}(f)$ may not be contained in $\mathrm{j}_P^A(f)_d$ respectively in $\mathrm{tj}_P^A(f)_d$, as in the following example.

Example 3.7 (T_{45}-Singularity in characteristic 2)
Let $\mathrm{char}(\mathbb{K})=2$ and let the C-polytope P be defined by the weights $w_1=(4,6)$ and $w_2=(5,5)$. The polynomial $f=x^5+x^2y^2+y^4$ is PH of degree 20 with respect to P with Tjurina number $\tau(f)=16$. For $n\geq 4$ we have

$$y^{4n}=g\cdot f+\xi f\in \mathrm{tj}(f),$$

where $g=y^{4n-4}+x^2y^{4n-6}+x^4y^{4n-8}$ is QH of degree $20n-20=v_P(y^4n)-v_P(f)$ and $\xi=(x\cdot g+x^2y^{4n-6})\cdot\partial_x$. Thus g guarantees that y^{4n} is indeed in $\mathrm{tj}_P^{AC}(f)_{20n}$, however,

$$v_P(\xi)=v_P(x^2y^{4n-6}\partial_x)=20n-25\ <\ v_P(y^{4n})-v_P(f)$$

has a valuation which is too small. Moreover, we cannot do any better, i.e.

$$y^{4n}\not\in \mathrm{tj}_P^{AC}(f)+F_{20n+1}$$

and thus

$$\dim_{\mathbb{K}}\left(\mathrm{gr}_P^{AC}(T_f)\right)=\infty.$$

See also Example 4.10. □

Lemma 3.8
If P is a C-polytope and $f\in\mathbb{K}[[\boldsymbol{x}]]$ *then*

$$\mathrm{gr}_P^A(M_f)=\mathrm{gr}_P^A(M_{\mathrm{in}_P(f)})\quad\textit{and}\quad \mathrm{gr}_P^{AC}(T_f)=\mathrm{gr}_P^{AC}(T_{\mathrm{in}_P(f)}).$$

Proof: For this we write $f=\mathrm{in}_P(f)+h$ for some $h\in\mathbb{K}[[\boldsymbol{x}]]$ with $v_P(h)\geq v_P(f)+1$, and we note that for any $\xi\in\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])$

$$v_P(\xi h)\geq v_P(\xi)+v_P(h)\geq v_P(\xi)+v_P(f)+1. \tag{8}$$

In order to show $\operatorname{gr}_P^A(M_f) = \operatorname{gr}_P^A(M_{\mathrm{in}_P(f)})$ we have to show

$$\mathrm{j}_P^A(f)_d + F_{d+1} = \mathrm{j}_P^A\big(\mathrm{in}_P(f)\big)_d + F_{d+1} \tag{9}$$

for all $d \geq 0$.

If $g \in \mathrm{j}_P^A(f)_d$ then there is a derivation ξ such that $g = \xi f$ with

$$d \leq v_P(\xi) + v_P(f) = v_P(\xi) + v_P\big(\mathrm{in}_P(f)\big).$$

In view of (8) we thus have

$$g = \xi\,\mathrm{in}_P(f) + \xi h \in \mathrm{j}_P^A\big(\mathrm{in}_P(f)\big)_d + F_{d+1},$$

which shows that the left hand side in (9) is contained in the right hand side.

On the other hand, if $g \in \mathrm{j}_P^A\big(\mathrm{in}_P(f)\big)_d$ then there is a derivative ξ such that $g = \xi\,\mathrm{in}_P(f)$ with

$$d \leq v_P(\xi) + v_P\big(\mathrm{in}_P(f)\big) = v_P(\xi) + v_P(f).$$

Again, in view of (8) we thus have

$$g = \xi f - \xi h \in \mathrm{j}_P^A\big(\mathrm{in}_P(f)\big)_d + F_{d+1},$$

which shows that the left hand side in (9) is contained in the right hand side.

The proof for $\operatorname{gr}_P^{AC}(T_f) = \operatorname{gr}_P^{AC}(T_{\mathrm{in}_P(f)})$ works analogously. □

Corollary 3.9

Let $f \in \mathbb{K}[\boldsymbol{x}]$ be QH of type $(w; d)$ and let P be the C-polytope defined by the single weight vector w.

(a) *Then $\Gamma(f) \subseteq P$, $\operatorname{gr}_P^A(M_f) = \operatorname{gr}_P(M_f)$ and $\operatorname{gr}_P^{AC}(T_f) = \operatorname{gr}_P(T_f)$*

(b) *If moreover $\mu(f) < \infty$ respectively $\tau(f) < \infty$, then a set of monomials is a $\mathbb{K}$-vector space basis for $\operatorname{gr}_P^A(M_f)$ respectively for $\operatorname{gr}_P^{AC}(T_f)$ if and only if it is one for M_f respectively for T_f.*

Proof: Since P has only one side it induces a grading on $\mathbb{K}[\boldsymbol{x}]$ and a homogeneous filtration on $\mathbb{K}[[\boldsymbol{x}]]$.

(a) We note that the partial derivative f_{x_i} is QH of type $(w; d - w_i)$ if it does not vanish. Thus the ideals $\mathrm{j}(f)$ and $\mathrm{tj}(f)$ are generated by weighted homogeneous elements. This implies that

$$(\mathrm{j}(f) \cap F_k) + F_{k+1} = \mathrm{j}_P^A(f)_k + F_{k+1}$$

and

$$(\mathrm{tj}(f) \cap F_k) + F_{k+1} = \mathrm{tj}_P^{AC}(f)_k + F_{k+1}$$

for all $k \geq 0$ as required. Let us elaborate this argument for the ideal $\mathrm{j}(f)$. Suppose that $h = \sum_{i=1}^n g_i \cdot f_{x_i} \in \mathrm{j}(f)$ is given. We can decompose the g_i into their quasihomogeneous parts

$$g_i = \sum_{j \geq 0} g_{i,j}$$

with $g_{i,j}$ QH of type $(w;j)$. Then h decomposes into quasihomogeneous parts $h=\sum_{j\geq 0} h_j$ with

$$h_j = \sum_{f_{x_i}\neq 0} g_{i,j-d+w_i}\cdot f_{x_i}.$$

If we now suppose that $h\in F_k$ then we can replace $g_{i,j}$ by zero for $j<k+d-w_i$, i.e. we may assume that

$$g_i=\sum_{j\geq k+d-w_i} g_{i,j}\in F_{k+d-w_i}.$$

Setting

$$\xi=\sum_{f_{x_i}\neq 0} g_i\cdot\partial_{x_i}$$

we have $h=\xi f$ and necessarily $v_P(\xi)+v_P(f)=v_P(h)=k$. Thus $h\in \mathrm{j}(f)_k$.

(b) By Corollary 3.6 any monomial basis B of $\mathrm{gr}_P^A(M_f)$ is a generating set of M_f. However, by (a) and Corollary 3.3 these vector spaces have the same dimension. Hence, B is a basis of M_f.
For the converse we note that $\mathrm{j}(f)$ is generated by weighted homogeneous polynomials. If B is a monomial basis M_f and $\boldsymbol{x}^\beta$ is any monomial, then there are $c_\alpha\in\mathbb{K}$ and a $g\in\mathrm{j}(f)$ such that

$$\boldsymbol{x}^\beta=\sum_{\boldsymbol{x}^\alpha\in B} c_\alpha\cdot\boldsymbol{x}^\alpha+g,$$

and all $\boldsymbol{x}^\alpha$ as well as g are weighted homogeneous polynomials of the same weighted degree as $\boldsymbol{x}^\beta$. In particular $g\in\mathrm{j}(f)\cap F_d$ with $d=v_P(\boldsymbol{x}^\beta)$, and thus $\boldsymbol{x}^\beta$ is a linear combination of the elements of B in $\mathrm{gr}_P(M_f)$. This shows that B generates $\mathrm{gr}_P(M_f)=\mathrm{gr}_P^A(M_f)$, and since M_f and $\mathrm{gr}_P(M_f)$ have the same dimension by Corollary 3.3 B must be a basis of $\mathrm{gr}_P^A(M_f)$.
The proof for $\mathrm{gr}_P^{AC}(T_f)$ and T_f works in the same way.

□

4. Normal forms

When obtaining normal forms of power series which are not right semi-quasihomogeneous the only known method was introduced by Arnol'd in [Arn75] over the complex numbers and slightly generalised by Wall in [Wal99]. It requires the principal part $\mathrm{in}_P(f)$ of the power series (with respect to some C-polytope P) to be an *isolated* singularity and its Milnor algebra to have a *finite regular basis*. Arnol'd actually gives a more restrictive condition but his proof shows that this suffices as was pointed out by Wall. We generalise Arnol'd's condition both in the strict and in the weak form to the situation of contact equivalence and derive normal forms for right as well as for contact equivalence in arbitrary characteristic.

Definition 4.1 (a) Let P be a C-polytope and let $f \in \mathbb{K}[[\boldsymbol{x}]]$ be a power series. Following Arnol'd [Arn75] we say that f satisfies *condition* A with respect to P if for any $g \in \mathrm{j}(f)$ there exists a derivation $\xi \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])$ such that

$$v_P(g) = v_P(\xi) + v_P(f) < v_P(g - \xi f),$$

i.e. if

$$\big(\mathrm{j}(f) \cap F_d\big) + F_{d+1} = \mathrm{j}_P^A(f)_d + F_{d+1}, \quad \text{for all } d \geq 0,$$

i.e. if

$$\mathrm{gr}_P(M_f) = \mathrm{gr}_P^A(M_f).$$

i.e. if

$$\mu(f) = \dim_{\mathbb{K}}\big(\mathrm{gr}_P^A(M_f)\big).$$

(b) Following Wall we modify the condition, and say f satisfies *condition* AA – for *almost A* – if

$$\dim_{\mathbb{K}}\big(\mathrm{gr}_P^A(M_f)\big) < \infty,$$

i.e. if the Milnor algebra of f has a finite regular basis.

(c) We modify these two conditions, which are meant to deal with right equivalence to the situation of contact equivalence, and say f satisfies *condition* AC – for *A for contact equivalence* – if for all $h \in \mathrm{tj}(f)$ there exists a $g \in \mathbb{K}[[\boldsymbol{x}]]$ and a derivation $\xi \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]])$ such that

$$v_P(h) = \min\{v_P(g) + v_P(f), v_P(\xi) + v_P(f)\} \; < \; v_P(h - g \cdot f - \xi f),$$

i.e. if

$$\big(\mathrm{tj}(f) \cap F_d\big) + F_{d+1} = \mathrm{tj}_P^{AC}(f)_d + F_{d+1} \quad \text{for all } d \geq 0,$$

i.e. if

$$\mathrm{gr}_P(T_f) = \mathrm{gr}_P^{AC}(T_f).$$

i.e. if

$$\tau(f) = \dim_{\mathbb{K}}\big(\mathrm{gr}_P^{AC}(T_f)\big).$$

(d) Again we modify the condition, and say f satisfies *condition* AAC – for *almost AC* – if

$$\dim_{\mathbb{K}}\big(\mathrm{gr}_P^{AC}(T_f)\big) < \infty,$$

i.e. if the Tjurina algebra of f has a finite regular basis.

The naming of the above conditions should explain the *superscripts* in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$ and in the corresponding ideals.
The above equivalence of the characterisations of condition A respectively AC uses Corollary 3.6. Corollary 3.3 and 3.6 together with Example 5.9 show that for isolated singularities the *almost* conditions are indeed strictly weaker. Moreover,

$$\mu(f) < \infty \text{ and } f \text{ satisfies A} \quad \Longrightarrow \quad f \text{ satisfies AA} \quad \Longrightarrow \quad \mu(f) < \infty \tag{10}$$

and

$$\tau(f) < \infty \text{ and } f \text{ satisfies AC} \implies f \text{ satisfies AAC} \implies \tau(f) < \infty. \tag{11}$$

We point out that by Lemma 3.8

$$f \text{ satisfies AA resp. AAC} \iff \mathrm{in}_P(f) \text{ satisfies AA resp. AAC}, \tag{12}$$

i.e. the conditions AA and AAC only depend on the principal part of f.

We now formulate our main result on normal forms without refering to the conditions AA respectively AAC, and we will see later how they come in useful. The statement for right equivalence in this form without refering to condition A or AA was first stated over the complex numbers in [Wal99, Theorem 2.1], but was already proved by Arnol'd in [Arn75, Theorem 9.5]. We generalise the statement to contact equivalence and give a different proof which works for any algebraically closed field making an "Ansatz" with power series. Recall that $\mathrm{ord}(f)$ denotes the order of the power series f and that $d(f)$ denotes its *differential order*, i.e.

$$d(f) := \min\{\mathrm{ord}(f_{x_1}), \ldots, \mathrm{ord}(f_{x_n})\}.$$

Theorem 4.2 (Normal forms with respect to right equivalence)
Let $f \in \mathfrak{m}$, P be a C-polytope and $B = \{\boldsymbol{x}^\alpha \mid \alpha \in \Lambda\}$ a regular basis for $M_{\mathrm{in}_P(f)}$. If $\mathfrak{m}^{k+2} \subseteq \mathfrak{m}^2 \cdot \mathrm{j}(f)$ then

$$f \sim_r \mathrm{in}_P(f) + \sum_{\alpha \in \Lambda_f} c_\alpha \boldsymbol{x}^\alpha. \tag{13}$$

for suitable $c_\alpha \in \mathbb{K}$, where Λ_f is the finite set

$$\Lambda_f = \big\{\alpha \in \Lambda \;\big|\; \deg(\boldsymbol{x}^\alpha) \leq 2k - d(f) + 1,\ v_P(\boldsymbol{x}^\alpha) \geq v_P\big(f - \mathrm{in}_P(f)\big)\big\}.$$

Theorem 4.3 (Normal forms with respect to contact equivalence)
Let $f \in \mathfrak{m}$, P be a C-polytope and $B = \{\boldsymbol{x}^\alpha \mid \alpha \in \Lambda\}$ a regular basis for $T_{\mathrm{in}_P(f)}$. If $\mathfrak{m}^{k+2} \subseteq \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot \mathrm{j}(f)$ then

$$f \sim_c \mathrm{in}_P(f) + \sum_{\alpha \in \Lambda_f} c_\alpha \boldsymbol{x}^\alpha. \tag{14}$$

for suitable $c_\alpha \in \mathbb{K}$, where Λ_f is the finite set

$$\Lambda_f = \big\{\alpha \in \Lambda \;\big|\; \deg(\boldsymbol{x}^\alpha) \leq 2k - \mathrm{ord}(f) + 2,\ v_P(\boldsymbol{x}^\alpha) \geq v_P\big(f - \mathrm{in}_P(f)\big)\big\}.$$

We will only prove Theorem 4.3 since the proof of Theorem 4.2 works along the same lines.

Proof of Theorem 4.3: In the proof we will write f_P instead of $\mathrm{in}_P(f)$ to shorten the notation. The basic idea is to construct a finite sequence $(f_i)_{i=0}^m$ with $f_0 = f$ such that $f_i \sim_c f$ for all i and that

$$f_m \equiv f_P + \sum_{\alpha \in \Lambda_f} c_\alpha \boldsymbol{x}^\alpha \ \big(\mathrm{mod}\ \mathfrak{m}^{2k - \mathrm{ord}(f) + 3}\big).$$

We try to do so by eliminating terms in f (piecewise) degree by degree. If we succeed then by [BGM10, Theorem 2.1] we have

$$f \sim_c f_m \sim_c f_P + \sum_{\alpha\in\Lambda_f} c_\alpha \boldsymbol{x}^\alpha$$

as desired since $2k - \mathrm{ord}(f) + 2$ is a bound for the determinacy of f.
We start our construction by denoting by $g = \mathrm{in}_P(f - f_P)$ the principal part of $g_1 = f - f_P$ with respect to P and setting $h = f - f_P - g$. If we set $d_0 = v_P(f) = v_P(f_P)$ and $d_1 = v_P(f - f_P) > d_0$ then $f_P \in F_{d_0}$ is PH of degree d_0, $g \in F_{d_1}$ is PH of degree d_1 and $h \in F_{d_1+1}$. Moreover, since B is a $\mathbb{K}$-vector space basis of $\mathrm{gr}_P^{AC}(T_{f_P})$ we have

$$g = \sum_{\substack{\alpha\in\Lambda\\ \lambda_P(\alpha)=d_1}} c_\alpha \boldsymbol{x}^\alpha + b_0 \cdot f_P + \xi f_P + h'$$

for suitable

$$c_\alpha \in \mathbb{K},\ b_0 \in \mathbb{K}[[\boldsymbol{x}]],\ \xi = \sum_{i=1}^n b_i \cdot \partial_{x_i} \in \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[[\boldsymbol{x}]]),\quad \text{and } h' \in \mathbb{K}[[\boldsymbol{x}]]$$

satisfying

$$d_1 = \min\{v_P(b_0) + d_0, v_P(\xi) + d_0\} \quad \text{and} \quad v_P(h') > d_1. \tag{15}$$

Form (15) we deduce that

$$v_P(b_0) \geq d_1 - d_0 > 0$$

and thus

$$b_0 \in \mathfrak{m}, \tag{16}$$

and also

$$v_P(b_i) - v_P(x_i) = v_P(b_i \partial_{x_i}) \geq v_P(\xi) \geq d_1 - d_0 > 0. \tag{17}$$

From (7) we know that then

$$\varphi : \mathbb{K}[[\boldsymbol{x}]] \longrightarrow \mathbb{K}[[\boldsymbol{x}]] : x_i \mapsto x_i - b_i$$

is a local $\mathbb{K}$-algebra isomorphism of $\mathbb{K}[[\boldsymbol{x}]]$.
Moreover, applying (7) to $\varphi(f_P)$ and to $\varphi(g)$ we get

$$\begin{aligned}
\varphi(f) =& \varphi(f_P) + \varphi(g) + \varphi(h)\\
=& f_P - \xi f_P + h_1 + g - \xi g + h_2 + \varphi(h)\\
=& (1 + b_0) \cdot f_P + \sum_{\substack{\alpha\in\Lambda\\ \lambda_P(\alpha)=d_1}} c_\alpha \boldsymbol{x}^\alpha + (h' + h_1 + h_2 + \varphi(h) - \xi g)
\end{aligned}$$

with $h', h_1, h_2, \varphi(h), \xi g \in F_{d_1+1}$ taking (6) and (17) into account.

Since by (16) $b_0 \in \mathfrak{m}$ we can multiply by the inverse of the unit $1 + b_0$ which is of the form $1 + b$ with $v_P(b) \geq d_1 - d_0 > 0$ so that we get

$$(1+b)\cdot\varphi(f) = f_P + \sum_{\substack{\alpha\in\Lambda\\ \lambda_P(\alpha)=d_1}} c_\alpha \boldsymbol{x}^\alpha + g_2 \quad \text{with} \quad v_P(g_2) > d_1.$$

Setting $f_1 = (1+b)\cdot\varphi(f)$ we have $f_1 \sim_c f$, and we can go on inductively treating g_2 as we have treated $g_1 = f - f_P$ before. That way we construct power series

$$f_m = f_P + \sum_{\substack{\alpha\in\Lambda\\ d_1\leq\lambda_P(\alpha)\leq d_m}} c_\alpha \boldsymbol{x}^\alpha + g_{m+1}$$

with $g_{m+1} \in F_{d_m}$ and $d_1 < d_2 < \ldots < d_m$, $m \geq 0$. By (4) we eventually have that

$$F_{d_m} \subseteq \mathfrak{m}^{2k-\operatorname{ord}(f)+3},$$

and we are done. □

Theorem 4.2 has as easy corollary the result of Arnol'd which he proved over the complex numbers in [Arn75, Theorem 9.5], even though he used condition A and $\mu\big(\operatorname{in}_P(f)\big) < \infty$ (see also [Wal99, Theorem 2.1]).

Corollary 4.4 (Normal forms for right equivalence)
Let P be a C-polytope and $f \in \mathfrak{m}$ be a power series such that $\operatorname{in}_P(f)$ satisfies AA then f is finitely right determined, rSPH and

$$f \ \sim_r \ \operatorname{in}_P(f) + \sum_{\substack{\boldsymbol{x}^\alpha\in B\\ v_P(\boldsymbol{x}^\alpha)>d}} c_\alpha \boldsymbol{x}^\alpha$$

for suitable $c_\alpha \in \mathbb{K}$, where B is a finite regular basis for $M_{\operatorname{in}_P(f)}$ and $d = v_P(\operatorname{in}_P(f))$.

Proof: If $\operatorname{in}_P(f)$ satisfies AA then f does so as well by Lemma 3.8. By (10) thus $\mu(f)$ is finite and therefore $\mathrm{j}(f)$ contains some power of the maximal ideal. Hence we are done by Theorem 4.2 since a regular basis for $M_{\operatorname{in}_P(f)}$ is finite due condition AA. □

Using Lemma 3.8, (11) and Theorem 4.3 we get the analogous statement for contact equivalence.

Corollary 4.5 (Normal forms for contact equivalence)
Let P be a C-polytope and $f \in \mathfrak{m}$ be a power series such that $\operatorname{in}_P(f)$ satisfies AAC then f is finitely contact determined, cSPH and

$$f \ \sim_c \ \operatorname{in}_P(f) + \sum_{\substack{\boldsymbol{x}^\alpha\in B\\ v_P(\boldsymbol{x}^\alpha)>d}} c_\alpha \boldsymbol{x}^\alpha$$

for suitable $c_\alpha \in \mathbb{K}$, where B is a finite regular basis for $T_{\operatorname{in}_P(f)}$ and $d = v_P(\operatorname{in}_P(f))$.

The proof of Theorem 4.2 and 4.3 actually gives a more precise bound on the determinacy if AA respectively AAC is fulfilled.

Corollary 4.6 (Finite determinacy bound for right equivalence)
Let P be a C-polytope, $f \in \mathfrak{m}$ be a power series such that $\mathrm{in}_P(f)$ satisfies AA and let B be a regular basis for $M_{\mathrm{in}_P(f)}$. Then

$$d := \max\left\{v_P\big(\mathrm{in}_P(f)\big), v_P\big(\boldsymbol{x}^\alpha\big) \;\big|\; \boldsymbol{x}^\alpha \in B\right\}$$

is finite and $f \sim_r g$ for any $g \in \mathbb{K}[[\boldsymbol{x}]]$ with $v_P(f-g) > d$.
In particular, if $\mathfrak{m}^{k+1} \subseteq F_{d+1}$, then f is right k-determined.

We will explain the proof only in the case of contact equivalence.

Corollary 4.7 (Finite determinacy bound for contact equivalence)
Let P be a C-polytope, $f \in \mathfrak{m}$ be a power series such that $\mathrm{in}_P(f)$ satisfies AAC and let B be a regular basis for $T_{\mathrm{in}_P(f)}$. Then

$$d := \max\left\{v_P\big(\mathrm{in}_P(f)\big), v_P\big(\boldsymbol{x}^\alpha\big) \;\big|\; \boldsymbol{x}^\alpha \in B\right\}$$

is fintie and $f \sim_c g$ for any $g \in \mathbb{K}[[\boldsymbol{x}]]$ with $v_P(f-g) > d$.
In particular, if $\mathfrak{m}^{k+1} \subseteq F_{d+1}$, then f is contact k-determined.

Proof: By Corollary 4.5 f is finitely determined, and thus by [BGM10, Theorem 2.5] some power of $\mathfrak{m}$ lies in $\langle f\rangle + \mathfrak{m}\cdot \mathrm{j}(f)$, so that we are in the situation of Theorem 4.3. The proof of Theorem 4.3 shows that

$$f \;\sim_c\; \mathrm{in}_P(f) + \sum_{\substack{\boldsymbol{x}^\alpha \in B \\ v_P(\boldsymbol{x}^\alpha) > v_P(\mathrm{in}_P(f))}} c_\alpha \boldsymbol{x}^\alpha + g_l$$

for suitable $c_\alpha \in \mathbb{K}$ and with $g_l \in F_l$ for l arbitrarily large. Moreover, in the process of constructing the transformations we see that terms of piecewise valuation larger than d do not have any influence on the coefficients c_α of the above normal form. Thus any power series g which coincides with f up to valuation d will give the same normal form and is thus contact equivalent to f. □

The determinacy bounds from Corollaries 4.6 resp. 4.7 for power series satisfying conditions AA resp. AC are in general better than those for arbitrary isolated singularities given in [BGM10] (see Example 5.1). This shows that the conditions AA and AAC are desirable. In the following we give lots of examples of power series satisfying these conditions. We will first consider quasihomogeneous polynomials.

Proposition 4.8
If $f \in \mathbb{K}[\boldsymbol{x}]$ is QH of type $(w; d)$ and P is the C-polytope defined by the single weight vector w, then f satisfies the conditions A and AC with respect to P.

Proof: This was proved in from Corollary 3.9 (a). □

From (10) and (11) together with (12) it follows that any power series with an isolated quasihomogeneous principal part satisfies AA and AAC.

Corollary 4.9
If $f \in \mathbb{K}[[\boldsymbol{x}]]$ is rSQH respectively cSQH w.r.t. w then f is AA respectively AAC w.r.t. the C-polytope defined by w.

Example 4.10 (T_{45}-Singularity in characteristic 2)
The condition in Corollary 4.9 that the C-polytope has only *one* facet, i.e. that the principal part is quasihomogeneous, is essential. Let $\mathrm{char}(\mathbb{K}) = 2$ and $f = x^5 + x^2y^2 + y^4 + x^3y^2 \in \mathbb{K}[[x, y]]$. Then f is cSPH with respect to $P = \Gamma(f)$ with principal part $\mathrm{in}_P(f) = x^5 + x^2y^2 + y^4$ and $\tau\big(\mathrm{in}_P(f)\big) = 16$. However, $\tau(f) = \infty$, which is an alternative proof of the fact f is not AAC, as we have already seen in Example 3.7. □

Remark 4.11 ([Wal99])
In [Wal99] Wall introduces the notion of strict Newton non-degeneracy which turns out to be a sufficient condition for AA and AAC. Let us recall the definition here. A face Δ of P is called an *inner face* if it is not contained in any coordinate hyperplane. Each point $q \in \mathbb{K}^n$ determines a coordinate hyperspace $H_q = \bigcap_{q_i=0}\{x_i = 0\} \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$. We call f *strictly non-degenerate SND along* Δ if for no common zero q of $\mathrm{j}(\mathrm{in}_\Delta(f))$ the polytope Δ contains a point on H_q, and we call f *strictly Newton non-degenerate* SNND w.r.t. P if f is non-degenerate of type SND along each *inner* face of P.
Strict Newton non-degeneracy can be formulated differently so that the connection to $\mathrm{gr}_P^A(M_f)$ is more evident. Each face Δ of the Newton polytope of f determines a finitely generated semigroup C_Δ in $\mathbb{Z}^n$ by considering those lattice points which lie in the cone over Δ with the origin as base. This semigroup then determines a finitely generated $\mathbb{K}$-algebra $\mathbb{K}[C_\Delta] = \mathbb{K}[\boldsymbol{x}^\alpha|\alpha \in C_\Delta]$ and a $\mathbb{K}[C_\Delta]$-module

$$D_\Delta = \langle \boldsymbol{x}^\alpha \cdot \partial_{x_i} \mid \boldsymbol{x}^\alpha \cdot \partial_{x_i}\boldsymbol{x}^\gamma \in K[C_\Delta] \ \ \forall\, \gamma \in C_\Delta \rangle_{\mathbb{K}[\mathbb{C}_\Delta]}$$

generated by monomial derivations which leave $\mathbb{K}[C_\Delta]$ invariant. Applying all elements in D_Δ to $\mathrm{in}_\Delta(f)$ leads to an ideal J_Δ in $\mathbb{K}[C_\Delta]$, and Wall then shows that (see [Wal99, Prop. 2.2])

$$\dim_{\mathbb{K}}(\mathbb{K}[C_\Delta]/J_\Delta) < \infty \quad\Longleftrightarrow\quad f \text{ is SND along all } \textit{inner} \text{ faces of } \Delta.$$

The rings $\mathbb{K}[C_\Delta]/J_\Delta$ can be stacked neatly in an exact sequence of complexes whose homology Wall uses to show that (see [Wal99, Prop. 2.3])

$$f \text{ is SNND} \quad\Longrightarrow\quad \dim_{\mathbb{K}}\left(\mathrm{gr}_P^A(M_f)\right) < \infty.$$

Wall's arguments use only standard facts from toric geometry and homolgoical algebra and do not depend on the characteristic of the base field. This proves Theorem 4.12, which shows that strictly Newton non-degenerate singularities possess good normal forms w.r.t. right equivalence and also w.r.t. contact equivalence (see Corollaries 4.4 and 4.5).

We refer to [Wal99] and [BGM10, Sec. 3] for more information on strict Newton non-degeneracy. □

Theorem 4.12 (Wall, [Wal99])
If $f \in \mathbb{K}[[\boldsymbol{x}]]$ is SNND w.r.t. P, then f is AA and AAC w.r.t. P, and hence

$$\tau(f) \leq \mu(f) < \infty.$$

5. Examples

In this section we apply the results of the previous sections to the classification of singularities of low modality in positive characteristic. A full classification of hypersurface singularities of right modality at most 2 and of contact modality at most 1 is still missing in positive characteristic, although a big part of this classification was achieved in [GrK90] and [Bou02].

Example 5.1 (Q_{10}-Singularity in characteristic 2)
Let $\mathrm{char}(\mathbb{K}) = 2$ and assume that $f \in \mathbb{K}[[x, y, z]]$ is cSQH with respect to the C-polytope P containing $\Gamma(f)$ and with principal part $\mathrm{in}_P(f) = x^2z + y^3 + z^4$. Using Singular ([DGPS10]) we see that

$$B = \{1, x, y, z, xy, xz, yz, z^2, xyz, xz^2, yz^2, z^3, xyz^2, xz^3, yz^3, xyz^3\}$$

is a $\mathbb{K}$-vector space basis of $T_{\mathrm{in}_P(f)}$. By Proposition 4.8 we see that B is indeed a regular basis for T_f and that f is AC with

$$\dim_{\mathbb{K}} \left(\mathrm{gr}_P^{AC}(T_f) \right) = \tau\big(\mathrm{in}_P(f) \big) = 16.$$

Corollary 4.5 then shows that

$$f \ \sim_c \ x^2z + y^3 + z^4 + c_1 \cdot xyz^2 + c_2 \cdot xz^3 + c_3 \cdot yz^3 + c_4 \cdot xyz^3$$

for some $c_1, \ldots, c_4 \in \mathbb{K}$. Moreover, using the weight vector $w = (9, 8, 6)$ to determine the filtration induced by P then

$$\max\{v_P(f), v_P(b) \mid b \in B\} = 35$$

and an easy computation shows that $\mathfrak{m}^6 \in F_{36}$. Thus f is contact 5-determined, and this bound of determinacy is better than the one obtained from [BGM10, Theorem 2.1], which would be 11. □

When checking if certain monomials are zero in $\mathrm{gr}_P^A(M_f)$ respectively $\mathrm{gr}_P^{AC}(T_f)$ the following lemma is very helpful.

Lemma 5.2
Let P be a C-polytope, Δ a facet of P and $f \in \mathbb{K}[[\boldsymbol{x}]]$. Moreover, denote by C_Δ the cone over Δ, and assume that $\alpha, \beta \in C_\Delta \cap \mathbb{Z}^n$. If $\boldsymbol{x}^\alpha$ is zero in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$ then $\boldsymbol{x}^{\alpha+\beta}$ is so.

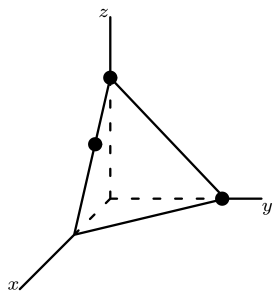

FIGURE 4. The Newton polytope of $xz^2 + y^3 + z^4$.

Proof: Since α and β belong to the same cone C_Δ Equation (2) shows that

$$v_P\big(\boldsymbol{x}^{\alpha+\beta}\big) = v_P\big(\boldsymbol{x}^{\alpha}\big) + v_P\big(\boldsymbol{x}^{\beta}\big).$$

Thus in the graded algebra $\mathrm{gr}_P^A(M_f)$ respectively $\mathrm{gr}_P^{AC}(T_f)$ the class of $\boldsymbol{x}^{\alpha+\beta}$ is the product of the classes of $\boldsymbol{x}^\alpha$ and $\boldsymbol{x}^\beta$ (see (5)). Since the former is zero by assumption so is the product. □

Remark 5.3
With the notation and assumptions of Lemma 5.2 it follows that all monomials corresponding to the cone $\alpha + C_\Delta$ vanish in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$.

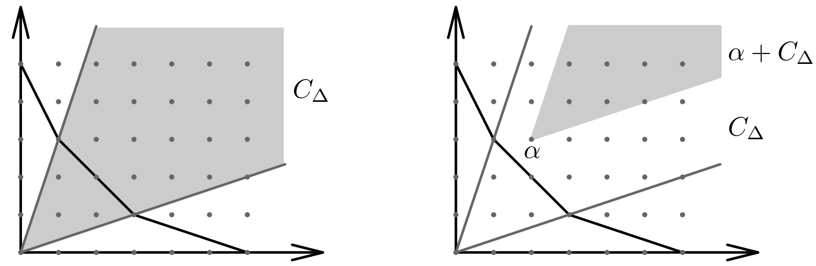

FIGURE 5. The lattice points in $\alpha + C_\Delta$ correspond to monomials x^γ vanishing in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$.

Example 5.4 (T_{pq}-Singularities)
Arnol'd considered in [Arn75, Example 9.6] the power series $f = x^p + \lambda x^2y^2 + y^q \in \mathbb{C}[[\boldsymbol{x}]]$ with $\lambda \neq 0$ and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ or equivalently

$$pq - 2p - 2q > 0.$$

f is PH with respect to its Newton polygon $P = \Gamma(f)$ depicted in Figure 6. If we scale the corresponding weight vectors to length $2pq$ instead of one they are

$$w_1 = (2q, pq - 2q) \quad \text{and} \quad w_2 = (pq - 2p, 2p),$$

and the piecewise degree of f is then $\deg_P(f) = 2pq$. Arnol'd describes in his paper a geometric procedure to compute a regular basis for M_f if the partial derivatives

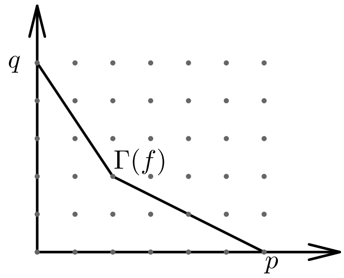

FIGURE 6. The Newton polygon of $x^p + \lambda x^2 y^2 + y^q$.

have only two terms as in the example, and he deduces that f satisfies condition A and AA and that

$$B = \{1, x, \dots, x^p, y, y^2, \dots, y^{q-1}, xy\}$$

is a regular basis for M_f. In particular,

$$\mu(f) = \dim_{\mathbb{K}}\left(\mathrm{gr}_P^A(M_f)\right) = p + q + 1.$$

Since no monomial in B lies above $\Gamma(f)$ it follows as seen in Corollary 4.4 and 4.6 that any power series whose principal part with respect to the above P coincides with T_{pq} actually is right equivalent to T_{pq}.

Arnol'd's arguments actually work for any field $\mathbb{K}$ where the *characteristic* is neither two, nor divides p or q. If the characteristic divides p or q then $\mu(f) = \infty$, and in the characteristic two case the Jacobian ideal is generated by the monomials x^{p-1} and y^{q-1}, so that $\mu(f) = pq$.

We now want to investigate $f = T_{pq}$ with respect to contact equivalence and the condition AC, and we first want to show that

$$\mathrm{char}(\mathbb{K}) \neq 2 \quad \Longrightarrow \quad f \text{ satisfies AC and hence AAC}.$$

Assume first that in addition $\mathrm{char}(\mathbb{K})$ *does neither divide* p *nor* q *nor* $pq - 2\cdot(p+q)$*.* Then $\mu(f) < \infty$ and thus also $\tau(f) < \infty$. Moreover, by Corollary 3.6 the above B generates $\mathrm{gr}_P^{AC}(T_f)$ and

$$T_f = \mathbb{K}[[x, y]]/\langle f, f_x, f_y\rangle.$$

It is clear that other than x^p the monomials in B will stay linearly independent modulo $\mathrm{tj}(f)$, and all monomials $x^i y^j$ which are in $\mathrm{j}(f)_d + F_{d+1}$ with $d = v_P(x^i y^j)$ are also in $\mathrm{tj}(f)_d + F_{d+1}$ (since they are a regular basis for M_f, see also [Bou09, Proposition 3.2.14]). To see that f satisfies AC it thus suffices to show that there are $a, b, c \in \mathbb{K}$ such that

$$x^p = a \cdot f + b \cdot x \cdot f_x + c \cdot y \cdot f_y$$

since then

$$x^p \in \mathrm{tj}(f)_{2pq} \subseteq \mathrm{tj}(f).$$

Considering the coefficients for x^p, x^2y^2 and y^p this leads to a linear system of equations with extended coefficient matrix

$$M = \left(\begin{array}{ccc|c} 1 & p & 0 & 1 \\ \lambda & 2\lambda & 2\lambda & 0 \\ 1 & 0 & q & 0 \end{array}\right).$$

This system is solvable if and only if the equation

$$\lambda \cdot \big(pq - 2 \cdot (p+q)\big) = 2\lambda$$

has a solution, i.e. if the first 3×3-Minor $-\lambda \cdot \big(pq - 2 \cdot (p+q)\big) \neq 0$. This shows that

$$p + q = \tau(f) = \dim_{\mathbb{K}} \left(\operatorname{gr}_P^{AC}(T_f)\right),$$

where for the latter equality we take into account that $\tau(f)$ is a lower bound for the dimension. Moreover,

$$B' = \{1, x, \ldots, x^{p-1}, y, y^2, \ldots, y^{q-1}, xy\}$$

is a regular basis for T_f and f satisfies AC.

Assume next that $\operatorname{char}(\mathbb{K})$ *does neither divide* p *nor* q*, but it divides* $pq - 2 \cdot (p+q)$*.* We have already seen in the first case that the system of linear equations with extended coefficient matrix M is not solvable under the given hypotheses. It follows that x^p does not lie in $\operatorname{tj}(f)$. Therefore, B is a regular basis for T_f and

$$p + q + 1 = \tau(f) = \dim_{\mathbb{K}} \left(\operatorname{gr}_P^{AC}(T_f)\right).$$

Assume now that $\operatorname{char}(\mathbb{K})$ *divides* p *but not* q*.* Then it is straight forward to see that

$$\operatorname{tj}(f) = \langle x^p, y^q, xy^2, qy^{q-1} - 2\lambda x^2 y\rangle.$$

and thus B' is a $\mathbb{K}$-vector space basis of T_f. We claim that B' also generates $\operatorname{gr}_P^{AC}(T_f)$, so that f satisfies AC with Tjurina number $\tau(f) = p + q$. By Lemma 5.2 it suffices to check that the monomials in

$$B^c = \left\{x^2y^2, xy^2, x^2y, x^p, y^q\right\}$$

are zero in $\operatorname{gr}_P^{AC}(T_f)$ (see Figure 7). However, we have that

$$x^2y^2 = \frac{1}{2\lambda} \cdot x \cdot \partial_x f \in \operatorname{tj}(f)_{2pq}$$

and

$$x^p = f - \left(\frac{1}{2} + \frac{1}{q}\right) \cdot x \cdot \partial_x f - \frac{1}{q} \cdot y \cdot \partial_y f \in \operatorname{tj}(f)_{2pq}.$$

Moreover, $v_P(xy^2) = pq + 2q$ and $v_P(\partial_x) = 2p - pq$ so that

$$xy^2 = \frac{1}{2\lambda} \cdot \partial_x f \in \operatorname{tj}(f)_{pq+2p}.$$

Similar arguments hold for x^2y and y^q. This finishes this case.

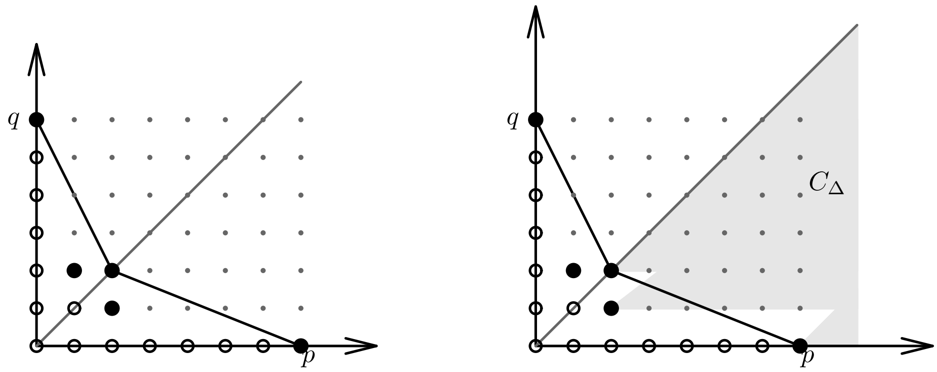

FIGURE 7. On the left hand side the elements of B' are depicted by large white dots and the elements in B^c are depicted by large black dots. The right hand side shows the union of $(2,2)+C_\Delta$, $(2,1)+C_\Delta$ and $(p,0)+C_\Delta$ which covers all lattice points in C_Δ which are not in B'.

Assume now that $\mathrm{char}(\mathbb{K})$ *divides* q *but not* p. This case follows by symmetry from the previous case, i.e. f is AC with Tjurina number $\tau(f)=p+q$.
Assume finally that $\mathrm{char}(\mathbb{K})$ *divides both* p *and* q. Then $\mathrm{tj}(f)=\langle xy^2, x^2y, x^p+y^q, x^{p+1}, y^{q+1}\rangle$ and B is a $\mathbb{K}$-vector space basis of T_f. Moreover, we claim that it is a regular basis for T_f as well. By Lemma 5.2 it suffices to show that the monomials x^{p+1}, y^{q+1}, x^2y^2, xy^2 and x^2y as well as the binomial x^p+y^q are zero in $\mathrm{gr}_P^{AC}(T_f)$. This can be achieved in the same way as above. In particular we have

$$p+q+1=\tau(f)=\dim_{\mathbb{K}}\left(\mathrm{gr}_P^{AC}(T_f)\right)$$

and f satisfies AC.
Conclusion: In each of the above cases the regular basis B respectively B' consists of monomials on or below the Newton polygon $P=\Gamma(f)$. Therefore, the normal form algorithm shows that any power series with principal part f with respect to P has indeed f as normal form. □

Corollary 5.5 (Normal form of T_{pq}-Singularities)
Suppose that $\mathrm{char}(\mathbb{K})\neq 2$ *and let* $f\in\mathbb{K}[[x,y]]$ *be a power series with* $\mathrm{in}_{\Gamma(f)}=x^p+\lambda\cdot x^2y^2+y^q$, $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$ *and* $\lambda\neq 0$. *Then* f *is AC and*

$$f\ \sim_c\ x^p+\lambda\cdot x^2y^2+y^q.$$

The contact determinacy of f *is* $\max\{p,q\}$.

Proof: That f satisfies AC and is contact equivalent to its principal part was shown in Example 5.4. It is obvious that any monomial above the Newton polygon has stricly larger piecewise valuation than f. Using the notation from Example 5.4 it follows that $\mathfrak{m}^{k+1}\subseteq F_{2pq+1}$ for $k=\max\{p,q\}$, so that by Corollary 4.7 the degree of contact determinacy is at most k. To see that it cannot be less we may assume the contrary and we may assume moreover that $p \geq q$. Then $f \sim_c \mathrm{in}_{\Gamma(f)}(f) - x^p = \lambda \cdot x^2y^2 + y^q$, but the latter is non-reduced and has thus infinite Tjurina number. This is clearly a contradiction. □

Remark 5.6
If $\mathrm{char}(\mathbb{K}) = 2$ neither the conclusion in Corollary 5.5 nor the investigation in Example 5.4 hold in general as we can see from Example 3.7.

For the T_{pq}-Singularities we considered the conditions A and AC and deduced a normal form. However, for normal forms we only need a good way to choose a *small* regular basis for $M_{\mathrm{in}_P(f)}$ respectively $T_{\mathrm{in}_P(f)}$. There the following observations are useful.

Remark 5.7
Each C-polytope P has only finitely many zero-dimensional faces and each facet is the convex hull of some of these. The cones over these zero-dimensional faces are rays, and for each facet Δ of P the cone C_Δ is spanned by a finite number of these rays, none of which is superfluous, i.e. they are the extremal rays of the cone.
Then $f \in \mathbb{K}[[\boldsymbol{x}]]$ satisfies condition AA respectively AAC w.r.t. P if and only if on each ray spanned by a zero-dimensional face of P there is a lattice point α such that $\boldsymbol{x}^\alpha$ is zero in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$.

Proof: Consider first the two-dimensional situation such that each cone C_Δ is spanned by two rays. Suppose that a cone C_Δ is given and it is spanned by the rays r and s, and suppose that α is a lattice point on r and β is a lattice point on s. The shifted rays $\alpha + s$ and $\beta + r$ will intersect, since r and s are not parallel,

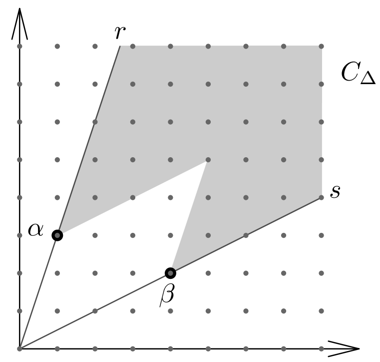

FIGURE 8. C_Δ almost filled by two shifted copies of C_Δ.

and thus the rays r, s, $\alpha + s$ and $\beta + r$ bound a finite region in the cone C_Δ (see Figure 8). By Lemma 5.2 the lattice points which are not inside the bounded region will be zero in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$. We can play this game for each facet of P, and thus there are only finitely many monomials whose class is not zero.

The argument generalises right away to higher dimensions. □

Corollary 5.8
Let $f = x^a + y^b + \lambda \cdot x^c y^d \in \mathbb{K}[[x,y]]$ with $\lambda \in \mathbb{K}$, $a > c \geq 1$, $b > d \geq 1$ and $ad + bc < ab$, and let $P = \Gamma(f)$. Then f satisfies AA respectively AAC w.r.t. P if and only if there are natural numbers k, m, n such that x^m, y^n and $x^{ck}y^{dk}$ are zero in $\mathrm{gr}_P^A(M_f)$ respectively in $\mathrm{gr}_P^{AC}(T_f)$.

Proof: The Newton polygon of f is schematically shown in Figure 9, and the result follows from Remark 5.7. □

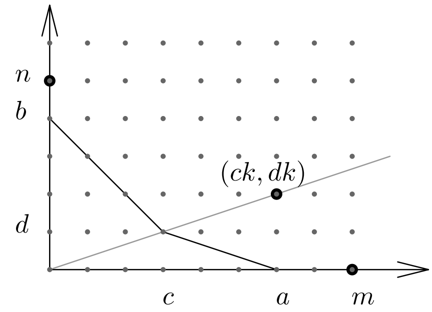

FIGURE 9. The Newton polygon of $x^a + y^b + \lambda \cdot x^c y^d$.

Example 5.9 ($E_{3,3}$-Singularity in characteristic 3)
Let $\mathrm{char}(\mathbb{K}) = 3$ and consider the equation

$$f = x^{12} + x^3y^2 + y^3 \in \mathbb{K}[[x,y]].$$

f is piecewise homogeneous with respect to its Newton polygon and using the procedure `isAC` from the SINGULAR library `gradalg.lib` we can check that x^{15}, y^{15} and x^9y^6 are zero in $\mathrm{gr}_P^{AC}(T_f)$. Thus f is AAC with respect to $\Gamma(f)$ by Corollary 5.8. Moreover, using the procedure `ACgrbase` from the same library we can compute the regular basis

$$B = \{1, x, \ldots, x^{12}, y, xy, x^2y, y^2, xy^2, x^2y^2, xy^3, x^2y^3, x^2y^4\}$$

for T_f. Hence, $\dim_{\mathbb{K}}\left(\mathrm{gr}_P^{AC}(T_f)\right) = |B| = 22$ while $\tau(f) = 21$. This shows that f is *not* AC.
Theorem 4.3 shows that any power series g whose principal part with respect to $\Gamma(f)$ is f satisfies

$$g \sim_c f + c_1 \cdot xy^3 + c_2 \cdot x^2y^3 + c_3 \cdot x^2y^4$$

for suitable $c_1, c_2, c_3 \in \mathbb{K}$.
Scaling the weight vectors corresponding to the facets of $\Gamma(f)$ suitably they are $w_1 = (6, 27)$ and $w_2 = (8, 24)$, and f is PH of piecewise degree 72. Moreover, the maximum of the piecewise degree of the monomials in B is $d = 112$, and an easy

computation shows that $\mathfrak{m}^{19} \subseteq F_{113}$. By Corollary 4.7 we therefore know that the contact determinacy bound of f is at most 18. That is much better than the bound

$$2 \cdot \tau(f) - \operatorname{ord}(f) + 2 = 41$$

which [BGM10, Theorem 2.1] gives. □

Example 5.10 (E_7-Singularity)
Let $f \in \mathbb{K}[[x,y,z]]$ and let P be a C-polytope containing $\Gamma(f)$ and suppose that the principal part of f is $\operatorname{in}_P(f) = x^3 + xy^3 + z^2$. Then f is cSQH and our methods show the following (for the details we refer to [Bou09, Example 3.3.22]):
1st Case: $\operatorname{char}(\mathbb{K}) \notin \{2,3\}$: Then $f \sim_c \operatorname{in}_P(f)$, $\tau(f) = \tau\big(\operatorname{in}_P(f)\big) = 7$ and the contact determinacy is 4.
2nd Case: $\operatorname{char}(\mathbb{K}) = 3$: Then $f \sim_c \operatorname{in}_P(f) + c \cdot x^2y^2$ for some $c \in \mathbb{K}$, $\tau\big(\operatorname{in}_P(f)\big) = 9$ and the contact determinacy is again 4. If $c \neq 0$ then $\tau(f) = 7$.
3rd Case: $\operatorname{char}(\mathbb{K}) = 2$: Then $f \sim_c \operatorname{in}_P(f) + c_1 \cdot y^3z + c_4 \cdot y^4z$, $\tau\big(\operatorname{in}_P(f)\big) = 14$ and the determinacy is 5.

Example 5.11 ($W_{1,1}$-Singularities)
Let $f \in \mathbb{K}[[x,y]]$ be such that the principal part with respect to $P = \Gamma(f)$ is $\operatorname{in}_P(f) = x^7 + x^3y^2 + y^4$. Then f is cSPH and our methods give the following normal forms (for the details we refer to [Bou09, Example 3.3.9]):
1st Case: $\operatorname{char}(\mathbb{K}) \notin \{2,3\}$: Then $f \sim_c \operatorname{in}_P(f)$.
2nd Case: $\operatorname{char}(\mathbb{K}) = 3$: Then $f \sim_c \operatorname{in}_P(f) + c_1 \cdot xy^4 + c_2 \cdot x^2y^3 + c_3 \cdot x^2y^4 + c_4 \cdot x^2y^5$ for some $c_1, \ldots, c_4 \in \mathbb{K}$. However, considering parametrisations it can be shown that actually $f \sim_c \operatorname{in}_P(f) + c \cdot x^2y^3$ for some $c \in \mathbb{K}$ (see [Bou02]).
3rd Case: $\operatorname{char}(\mathbb{K}) = 2$: Then $f \sim_c \operatorname{in}_P(f) + c \cdot x^6y$ for some $c \in \mathbb{K}$.

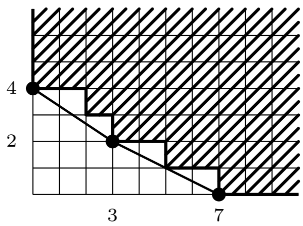

FIGURE 10. The Newton polygon of $x^7 + x^3y^2 + y^4$ for $\operatorname{char}(\mathbb{K}) \neq 2,3,7$.

□

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