# TRIPLE-POINT DEFECTIVE RULED SURFACES

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Abstract. In [ChM06] we studied triple-point defective very ample linear systems on regular surfaces, and we showed that they can only exist if the surface is ruled. In the present paper we show that we can drop the regularity assumption, and we classify the triple-point defective very ample linear systems on ruled surfaces.

Let $S$ be a smooth projective surface, $K = K_S$ the canonical class and $L$ a divisor class on $S$

We study a classical interpolation problem for the pair $(S, L)$, namely whether for a general point $p \in S$ the linear system $|L - 3p|$ has the expected dimension

$$\operatorname{expdim} |L - 3p| = \max\{-1, \dim |L| - 6\}.$$

If this is not the case we call the pair $(S, L)$ *triple-point defective*.

This paper is indeed a continuation of [ChM06], where some classification of triple point defective pairs is achieved, under the assumptions:

$$L, L - K \text{ very ample, and } (L - K)^2 > 16,$$

conditions that we will take all over the paper.

With these assumptions, the main result of [ChM06] says that all triple-point defective *regular* surfaces are rationally ruled.

We tackled the problem by considering $|L-3p|$ as fibres of the the map $\alpha$ in the following diagram,

$$|L| = \mathbb{P}(H^0(L)^*) \xleftarrow{\beta} \mathcal{L}_3 \xrightarrow{\alpha} S \tag{1}$$

where $\mathcal{L}_3$ denotes the incidence variety

$$\mathcal{L}_3 = \{(C, p) \in |L| \times S \mid \operatorname{mult}_p(C) \geq 3\}$$

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and $\alpha$ and $\beta$ are the obvious projections.
Assuming that for a general point $p \in S$ there is a curve in $L_p$ with a triple-point in $p$ – and hence $\alpha$ surjective, we considered then the *equimultiplicity scheme* $Z_p$ of a curve $L_p \in |L - 3p|$ defined by

$$\mathcal{J}_{Z_p,p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p \rangle^3.$$

One easily sees that $(S, L)$ triple-point defective necessarily implies that

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

Non–zero elements in $H^1\big(S, \mathcal{J}_{Z_p}(L)\big)$ determine by Serre duality a non–trivial extension $\mathcal{E}_p$ of $\mathcal{J}_{Z_p}(L-K)$ by $\mathcal{O}_S$, which turns out to be a rank 2 bundle on the surface. Due to the assumption $(L-K)^2 > 16$, $\mathcal{E}_p$ is Bogomolov unstable. We then exploited the destabilizing divisor $A_p$ of $\mathcal{E}_p$ in order to obtain the above mentioned result.

For non–regular surfaces, the argument of [ChM06] shows the following lemma (see [ChM06], Prop. 17 and Prop. 18):

**Proposition 1**
*Suppose that, with the notation in* (1), $\alpha$ *is surjective, and suppose as usual that* $L$ *and* $L-K$ *are very ample with* $(L-K)^2 > 16$.
*For* $p$ *general in* $S$ *and for* $L_p \in |L-3p|$ *general, call* $Z_p'$ *the minimal subscheme of the equimultiplicity scheme* $Z_p$ *of* $L_p$ *such that*

$$h^1\big(S, \mathcal{J}_{Z_p'}(L)\big) \neq 0.$$

*Then either:*

1) $\mathrm{length}(Z_p') = 3$ *and* $S$ *is ruled; or*
2) $\mathrm{length}(Z_p') = 4$ *and, for* $p \in S$ *general, there are smooth, elliptic curves* $E_p$ *and* $F_p$ *in* $S$ *through* $p$ *such that* $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ *and* $L.E_p = L.F_p = 3$. *In particular, both* $|E|_a$ *and* $|F|_a$ *induce an elliptic fibration with section on* $S$ *over an elliptic curve.*

This is our starting point. We will in this paper show that the latter case actually cannot occur, and we will classify the triple-point defective linear systems $L$ as above on ruled surfaces. It will in particular follow that the fibre of the ruling is contained exactly twice, and thus that the map $\beta$ above is generically finite.

Our main result is:

**Theorem 2**
*Suppose that the pair $(S, L)$ is triple-point defective where $L$ and $L-K$ are very ample with $(L-K)^2 > 16$. Then $S$ admids a ruling $\pi : S \to C$.*

For the classification, call $C_0$ a section of the ruled surface $S$, $\mathfrak{e}$ the line bundle on the base curve given by the determinant of the defining bundle, and call $E_i$ the exceptional divisors (see pp. 9 and 13 for a more precise setting of the notation):

**Theorem 3**
*Assume that $\pi : S \to C$ is a ruled surface and that the pair $(S, L)$ is triple-point defective, where $L$ and $L - K$ are very ample with $(L - K)^2 > 16$.*
*Then $\pi$ is minimal, i.e. $S$ is geometrically ruled, and for a general point $p \in S$ the linear system $|L - 3p|$ contains a fibre of the ruling as fixed component with multiplicity two.*
*Moreover, in the previous notation, the line bundle $L$ is of type $C_0+\pi^*\mathfrak{b}$ for some divisor $\mathfrak{b}$ on $C$ such that $\mathfrak{b} + \mathfrak{e}$ is very ample.*

In Section 1 we will first show that a surface $S$ admitting two elliptic fibrations as required by Proposition 1 would necessarily be a product of two elliptic curves and the triple-point defective linear system would be of type $(3, 3)$. We then show that such a system is never triple-point defective, setting the first part of the main theorem.
In Section 2 we classify the triple-point defective linear systems on ruled surfaces, thus arriving at our main results.

## 1. Products of Elliptic Curves

In the above setting, consider a triple-point defective tuple $(S, L)$ where the equimultiplicity scheme $Z_p$ (see [ChM06]) of a general element $L_p \in |L-3p|$ admitted a complete intersection subscheme $Z'_p$ of length *four* with

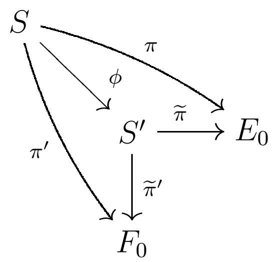
$$h^1\big(S, \mathcal{J}_{Z'_p}(L)\big) \neq 0.$$

As explained in the introduction, Prop. 1, after [ChM06] we know that, for $p \in S$ general, there are smooth, elliptic curves $E_p$ and $F_p$ in $S$ through $p$ such that $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ and $L.E_p = L.F_p = 3$.

In particular, both $|E|_a$ and $|F|_a$ induce an elliptic fibration with section on $S$ over an elliptic curve.
We will now show that this situation indeed cannot occur. Namely, for general $p$ and $L_p$ there cannot exist such a scheme $Z'_p$.

**Lemma 4**
*Suppose that the surface $S$ has two elliptic fibrations $\pi : S \longrightarrow E_0$ and $\pi' : S \longrightarrow F_0$ with general fibre $E$ respectively $F$ satisfying $E.F = 1$. Then $E_0$ and $F_0$ are elliptic curves, and $S$ is the blow-up of a product of two elliptic curves $S' = E \times E_0 \cong E \times F$.*

**Proof:** Since $E.F = 1$ we have that $F$ is a section of $\pi$, and thus $F \cong E_0$ via $\pi$. In particular, $E_0$ and, similarly, $F_0$ are elliptic curves.
It is well known that there are no non–constant maps from a rational curve to a curve of positive genus ([Har77], IV.2.5.4). Thus any exceptional curve of $S$ sits in some fiber. Thus we can reach relatively minimal models of $\pi$ and $\pi'$ by successively blowing down exceptional $-1$-curves which belong to fibres of both $\pi$ and $\pi'$, i.e. we have the following commutative diagram




where $S'$ is actually a minimal surface. Since a general fibre of $\pi$ or $\pi'$ is not touched by the blowing-down $\phi$ we may denote the general fibres of $\widetilde{\pi}$ and $\widetilde{\pi}'$ again by $E$ respectively $F$, and we still have $E.F = 1$.
We will now try to identify the minimal surface $S'$ in the classification of minimal surfaces.
By [Fri98] Ex. 7.9 the canonical divisor $K_{S'}$ is numerically trivial, since $S'$ is a minimal surface admitting two elliptic fibrations over elliptic curves.
But then we can apply [Fri98] Ex. 7.7, and since the base curve $E_0$ of the fibration $\widetilde{\pi}$ is elliptic we see that the invariant $d = \deg(L) = \deg\left((R^1\pi_*\mathcal{O}_{S'})^{-1}\right)$ of the relatively minimal fibration $\widetilde{\pi}$ mentioned in

[Fri98] Cor. 7.17 is zero, so that the same corollary implies that the fibration has at most multiple fibres with smooth reduction as singular fibres. However, since $\widetilde{\pi}$ has a section $F$ there are no multiple fibres, and thus all fibres of $\widetilde{\pi}$ are smooth.

Moreover, since the canonical divisor of $S'$ is numerically trivial it is in particular nef, and by [Fri98] Thm. 10.5 we get that the Kodaira dimension $\kappa(S')$ of $S'$ is zero.

Moreover, by [Fri98] Cor. 7.16 the surface $S'$ has second Chern class $c_2(S') = 0$, since the invariant $d = \deg\left((R^1\pi_*\mathcal{O}_{S'})^{-1}\right) = 0$ as already mentioned above. Thus by the Enriques-Kodaira Classification (see e.g. [BHPV04] Thm. 10.1.1) $S'$ must either be a torus or hyperelliptic (where the latter is sometimes also called bielliptic). A bielliptic surface has precisely two elliptic fibrations, but one of them is a fibration over a $\mathbb{P}^1$ and only one is over an elliptic curve (see e.g. [Rei97] Thm. E.7.2). Thus $S'$ is not bielliptic. Moreover, if $S'$ is a torus then $K_{S'}$ is trivial and thus so is $(R^1\pi_*\mathcal{O}_{S'})^{-1}$, which by [Fri98] Cor. 7.21 implies that $S'$ is a product of the base curve with a fibre. $\square$

Lemma 4 implies that in order to show that the situation of Proposition 1 cannot occur, we have to understand products of elliptic curves.

Let us, therefore, consider a surface $S = C_1 \times C_2$ which is the product of two smooth elliptic curves.

Let us set some notation. We will use some results by [Kei01] Appendices G.b and G.c in the sequel.

The surface $S$ is naturally equipped with two projections $\pi_i : S \longrightarrow C_i$. If $\mathfrak{a}$ is a divisor on $C_2$ of degree $a$ and $\mathfrak{b}$ is a divisor on $C_1$ of degree $b$ then the divisor $\pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b} \sim_a aC_1 + bC_2$, where by abuse of notation we denote by $C_1$ a fixed fibre of $\pi_2$ and by $C_2$ a fixed fibre of $\pi_1$. Moreover, $K_S$ is trivial, and given two divisors $D \sim_a aC_1 + bC_2$ and $D' \sim_a a'C_1 + b'C_2$ then the intersection product is

$$D.D' = (aC_1 + bC_2).(a'C_1 + b'C_2) = a \cdot b' + a' \cdot b.$$

We will consider first the case

$$L = \pi_2^*(\mathfrak{a}) + \pi_1^*(\mathfrak{b})$$

where both $\mathfrak{b}$ on $C_1$ and $\mathfrak{a}$ on $C_2$ are divisors of degree 3. The dimension of the linear system $|L|$ is $\dim|L| = 8$, and thus for a point $p \in S$ the expected dimension is $\operatorname{expdim}|L - 3p| = \dim|L| - 6 = 2$.
Notice that a divisor of degree three on an elliptic curve is always very ample and embeds the curve as a smooth cubic in $\mathbb{P}^2$. Since the smooth plane cubics are classified by their normal forms $xz^2 - y \cdot (y-x) \cdot (y - \lambda \cdot x)$ with $\lambda \neq 0$ the following example reflects the behaviour of any product of elliptic curves embedded via a linear system of bidegree $(3,3)$.

**Example 5**
Consider two smooth plane cubics

$$C_1 = V\big(xz^2 - y \cdot (y-z) \cdot (y - az)\big)$$

and

$$C_2 = V\big(xz^2 - y \cdot (y-z) \cdot (y - bz)\big).$$

The surface $S = C_1 \times C_2$ is embedded into $\mathbb{P}^8$ via the Segre embedding

$$\phi : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8 : \big((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)\big) \mapsto (x_0y_0 : \ldots : x_2y_2).$$

We may assume that both curves contain the point $p = (1 : 0 : 0)$ as a general non-inflexion point, and the point $(p,p)$ is mapped by the Segre embedding to $\phi(p,p) = (1 : 0 : \ldots : 0)$. If we denote by $z_{i,j}$, $i,j \in \{0,1,2\}$, the coordinates on $\mathbb{P}^8$ as usual, then the maximal ideal locally at $\phi(p,p)$ is generated by $z_{0,2}$ and $z_{2,0}$, i.e. these are local coordinates of $S$ at $\phi(p,p)$. A standard basis computation shows that locally at $\phi(p,p)$ the coordinates $z_{i,j}$ satisfy modulo the ideal of $S$ and up to multiplication by a unit the following congruences (note, $z_{0,0} = 1$)

$$\begin{aligned} z_{0,1} &\equiv \frac{1}{b} \cdot z_{0,2}^2, & z_{1,0} &\equiv \frac{1}{a} \cdot z_{2,0}^2, & z_{1,1} &\equiv \frac{1}{ab} \cdot z_{0,2}^2 \cdot z_{2,0}^2, \\ z_{1,2} &\equiv \frac{1}{a} \cdot z_{0,2} \cdot z_{2,0}^2, & z_{2,1} &\equiv \frac{1}{b} \cdot z_{0,2}^2 \cdot z_{2,0}, & z_{2,2} &\equiv z_{0,2} \cdot z_{2,0}. \end{aligned}$$

Thus a hyperplane section $H = a_{0,0}z_{0,0} + \ldots + a_{2,2}z_{2,2}$ of $S$ is locally in $\phi(p,p)$ modulo $\mathfrak{m}^3 = \langle z_{0,2}, z_{2,0}\rangle^3$ given by

$$H \equiv a_{0,0} + a_{0,2}z_{0,2} + a_{2,0}z_{2,0} + \frac{a_{0,1}}{b} \cdot z_{0,2}^2 + \frac{a_{1,0}}{a} \cdot z_{2,0}^2 + a_{2,2}z_{0,2}z_{2,0},$$

and hence the family of hyperplane sections having multiplicity at least three in $\phi(p,p)$ is given by

$$a_{0,0} = a_{0,1} = a_{1,0} = a_{0,2} = a_{2,0} = a_{2,2} = 0.$$

But then the family has parameters $a_{1,1}, a_{1,2}, a_{2,1}$, and its dimension coincides with the expected dimension 2. Moreover, the 3-jet of a hyperplane section $H$ through $\phi(p,p)$ with multiplicity at least three is

$$\mathrm{jet}_3(H) \equiv z_{0,2} \cdot z_{2,0} \cdot \left( \frac{a_{1,2}}{a} \cdot z_{2,0} + \frac{a_{2,1}}{b} \cdot z_{0,2} \right),$$

which shows that for a general choice of $a_{2,1}$ and $a_{1,2}$ the point $\phi(p,p)$ is an ordinary triple point.

**Remark 6**

We actually can say very precisely what it means that $p$ is general in the product, namely that neither $\pi_1(p)$ is a inflexion point of $C_1$, nor $\pi_2(p)$ is a inflexion point of $C_2$.

Indeed, since $\mathfrak{a}$ is very ample of degree three, for each point $p \in S$ there is a unique point $q_a \in C_2$ such that $q_a + 2 \cdot \pi_2(p) \sim_l \mathfrak{a}$. When $\pi_2(p)$ is a inflexion point of $C_2$, then $q_a = \pi_2(p)$ and thus the two-dimensional family

$$3C_{1,\pi_2(p)} + |\pi^*(\mathfrak{b})| \subset |L - 3p|$$

gives a superabundance of the dimension of $|L - 3p|$ by one.

Similarly one can argue when $\pi_1(p)$ is a inflexion point of $C_1$.

Now we are ready for the proof of Theorem 2.

**Proof of Theorem 2:** By Proposition 1, it is enough to prove that when $S$ has two elliptic fibrations as in the proposition, then $S$ is not triple–point defective.

By Lemma 4, $S$ is the blow-up $\pi : S \longrightarrow S'$ of a product $S' = C_1 \times C_2$ of two elliptic curves, and we may assume that the curves $E_p$ and $F_p$ in Proposition 1 are the fibres of $\pi_1$ respectively $\pi_2$.

Our first aim will be to show that actually $S = S'$. For this note that

$$\mathrm{Pic}(S) = \bigoplus_{i=1}^{k} E_i \oplus \pi^* \mathrm{Pic}(S'),$$

where the $E_i$ are the total transforms of the exceptional curves arising throughout the blow-up, i.e. the $E_i$ are (not necessarily irreducible) rational curves with self-intersection $E_i^2 = -1$ and such that $E_i.E_j = 0$ for $i \neq j$ and $E_i.\pi^*(C) = 0$ for any curve $C$ on $S'$. In particular, since $K_{S'}$ is trivial we have that $K_S = \sum_{i=1}^k E_i$, and if $L = \pi^*L' - \sum_{i=1}^k e_iE_i$ then $L - K = \pi^*L' - \sum_{i=1}^k (e_i + 1)E_i$. We therefore have

$$16 < (L-K)^2 = (L')^2 - \sum_{i=1}^k (e_i+1)^2,$$

or equivalently

$$(L')^2 \geq 17 + \sum_{i=1}^k (e_i+1)^2 \geq 17 + 4k, \tag{2}$$

where the latter inequality is due to the fact that $e_i = L.E_i > 0$ since $L$ is very ample. By the assumption of Proposition 1 we know that $L'.C_1 = L.E_p = 3$ and $L'.C_2 = L.F_p = 3$, and therefore by [Har77] Ex. V.1.9

$$(L')^2 \leq 2 \cdot (L'.C_1) \cdot (L'.C_2) = 18. \tag{3}$$

But (2) and (3) together imply that no exceptional curve exists, i.e. $S = S'$.

Since now $S$ is a product of two elliptic curves, by [LaB92] we know that the Picard number $\rho = \rho(S)$ satisfies $2 \leq \rho \leq 4$, and the Néron-Severi group can be generated by the two general fibres $C_1$ and $C_2$ together with certain graphs $C_j$, $3 \leq j \leq \rho$, of morphisms $\varphi_j : C_1 \longrightarrow C_2$. In particular, $C_j.C_2 = 1$ and $C_j.C_1 = \deg(\varphi_j) \geq 1$ for $3 \leq j \leq \rho$. Moreover, these graphs have self intersecting zero. If we now assume that $L \sim_a \sum_{j=1}^\rho a_iC_i$ then

$$L^2 = 2 \cdot \sum_{i<j} a_i \cdot a_j \cdot (C_i.C_j)$$

is divisible by 2, and since $L = L - K$ with $(L-K)^2 > 16$ we deduce with [Har77] Ex. V.1.9 that

$$L^2 = (L-K)^2 = 18 = 2 \cdot (L.C_1) \cdot (L.C_2),$$

and thus that

$$L \sim_a 3C_1 + 3C_2,$$

or in equivalently, that

$$L = \pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b}$$

for some divisors $\mathfrak{a}$ on $C_2$ and $\mathfrak{b}$ on $C_1$, both of degree 3. That is, we are in the situation of Example 5, and we showed there that $(S, L)$ then is not triple-point defective. □

**Remark 7**
Notice that, in practice, since

$$h^1(S, L) = h^0(C_1, \mathfrak{b}) \cdot h^1(C_2, \mathfrak{a}) + h^0(C_2, \mathfrak{a}) \cdot h^1(C_1, \mathfrak{b}) = 0,$$

the non-triple-point defectiveness shows that for general $p \in S$ and $L_p \in |L - 3p|$ no $Z_p'$ as in the assumptions of Proposition 1 can have length 4.

## 2. Geometrically Ruled Surfaces

Let $S = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalized bundle $\mathcal{E}$ (in the sense of [Har77] V.2.8.1). The Néron-Severi group of $S$ is

$$\mathrm{NS}(S) = C_0\mathbb{Z} \oplus f\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $f \cong \mathbb{P}^1$ is a fixed fibre of $\pi$, $C_0$ a fixed section of $\pi$ with $\mathcal{O}_S(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $e = -\deg(\mathfrak{e}) \geq -g$ where $\mathfrak{e} = \Lambda^2\mathcal{E}$. If $\mathfrak{b}$ is a divisor on $C$ we will write $\mathfrak{b}f$ for the divisor $\pi^*(\mathfrak{b})$ on $S$, and so for the canonical divisor we have

$$K_S \quad \sim_l \quad -2C_0 + (K_C + \mathfrak{e}) \cdot f \quad \sim_a \quad -2C_0 + (2g - 2 - e)f,$$

where $g = g(C)$ is the genus of the base curve $C$.

**Example 8**
Let $\mathfrak{b}$ be a divisor on $C$ such that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are both very ample and such that $\mathfrak{b}$ is non-special. If $C$ is rational we should in addition assume that $\deg(\mathfrak{b}) + \deg(\mathfrak{b} + \mathfrak{e}) \geq 6$. Then the divisor $L = C_0 + \mathfrak{b}f$ is very ample (see e.g. [FuP00] Prop. 2.15) of dimension

$$\dim |L| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 1$$

Moreover, for any point $p \in S$ we then have (see [FuP00] Cor. 2.13)

$$\dim |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f| = \dim |\, C_0 + \mathfrak{b} f| - 4 = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 5,$$

and we have for $p$ general

$$\dim |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f - p\,| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 6.$$

For this note that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ very ample implies that this number is non-negative – in the rational case we need the above degree bound. If we denote by $f_p = \pi^*\big(\pi(p)\big)$ the fibre of $\pi$ over $\pi(p)$, then by Bézout and since $L.f_p = (L - f_p).f_p = 1$ we see that $2f_p$ is a fixed component of $|L - 3p|$ and we have

$$|L - 3p| = 2f_p + |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f - p\,|,$$

so that

$$\dim |L - 3p| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 6 = \dim |L| - 5$$
$$> \dim |L| - 6 = \operatorname{expdim} |L - 3p|.$$

This shows that $(S, L)$ is triple-point defective and $|L - 3p|$ contains a fibre of the ruling as double component. Moreover, for a general $p$ the linear series $|L - 3p|$ cannot contain a fibre of the ruling more than twice due to the above dimension count for $|\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f - p\,|$.

Next we are showing that a geometrically ruled surface is indeed triple-point defective with respect to a line bundle $L$ which fulfills our assumptions, and in Corollary 13 we will see that this is not the case for non-geometrically ruled surfaces.

**Proposition 9**
*On every geometrically ruled surface $S = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ there exists some very ample line bundle $L$ such that the pair $(S, L)$ is triple–point defective, and moreover also $L - K$ is very ample with $(L - K)^2 > 16$.*

**Proof:** It is enough to take $L = C_0 + \mathfrak{b} f$, with $b = \deg(\mathfrak{b}) = 3a$ such that $a, a - e, a + e, a - 2g + 2 + e, a - 2g + 2 - e$ are all bigger or equal than $2g + 1$.
Indeed in this case $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are both very ample. For $p \in C$ general, we also have that both $\mathfrak{b} - p$ and $\mathfrak{b} + \mathfrak{e} - p$ are non-special. It follows that $L$ is very ample (by [Har77] Ex. V.2.11.b) and $(S, L)$ is triple

point defective, by the previous example. Moreover, in this situation we have:

$$L - K \sim_l 3C_0 + \big(\mathfrak{b} - K_C - \mathfrak{e}\big) \cdot f.$$

Hence

$$(L-K)^2 = \big(3C_0 + (\deg(\mathfrak{b}) - 2g + 2 + e) \cdot f\big)^2 \geq 18 > 16.$$

Finally, if we fix a divisor $\mathfrak{a}$ of degree $a$ on $C$, then $L - K$ is the sum of the divisors $C_0 + \big(\mathfrak{a} - K_C\big) \cdot f$, $C_0 + \big(\mathfrak{a} - \mathfrak{e}\big) \cdot f$, $C_0 + \mathfrak{a}f$, which are very ample ([Har77] Ex. V.2.11). Thus $L - K$ is very ample. □

Next, let us describe which linear systems $L$ on a ruled surface $S$ determine a triple-point defective pair $(S, L)$.
We will show that example 8 describes, in most cases, the only possibilities. In order to do so we first have to consider the possible algebraic classes of irreducible curves with self-intersection zero on a ruled surface.

**Lemma 10**
*Let $B \in |bC_0+b'f|_a$ be an irreducible curve with $B^2 = 0$ and $\dim |B|_a \geq 0$, then we are in one of the following cases:*

(a.1) *$B \sim_a f$,*
(a.2) *$e = 0$, $b \geq 1$, $B \sim_a bC_0$, and $|B|_a = |B|_l$, or*
(a.3) *$e < 0$, $b \geq 2$, $b' = \frac{b}{2}e < 0$, $B \sim_a bC_0 + \frac{b}{2}ef$ and $|B|_a = |B|_l$.*

*Moreover, if $b = 1$, then $S \cong C_0 \times \mathbb{P}^1$.*

**Proof:** See [Kei01] App. Lemma G.2. □

We can now classify the triple-point defective linear systems on a geometrically ruled surface. In order to do so we should recall the result of [ChM06] Prop. 18.

**Proposition 11**
*Suppose that, with the notation in (1), $\alpha$ is surjective, and suppose that $L$ and $L - K$ are very ample with $(L - K)^2 > 16$. Moreover, suppose that for $p \in S$ general and for $L_p \in |L - 3p|$ general the equimultiplicity scheme $Z_p$ of $L_p$ has a subscheme $Z'_p$ of length 3 such that $h^1\big(S, \mathcal{J}_{Z'_p}(L)\big) \neq 0$.*
*Then for $p \in S$ general there is an irreducible, smooth, rational curve $B_p$ in a pencil $|B|_a$ with $B^2 = 0$, $(L - K).B = 3$ and $L - K - B$ big.*

*In particular, $S \to |B|_a$ is a ruled surface and $2B_p$ is a fixed component of $|L-3p|$.*

**Theorem 12**

*With the above notation let $\pi : S \to C$ be a geometrically ruled surface, and let $L$ be a line bundle on $S$ such that $L$ and $L-K$ are very ample. Suppose that $(L-K)^2 > 16$ and that for a general $p \in S$ the linear system $|L-3p|$ contains a curve $L_p$ such that $h^1\big(S, \mathcal{J}_{Z_p}(L)\big) \neq 0$ where $Z_p$ is the equimultiplicity scheme of $L_p$ at $p$.*

*Then $L = C_0 + \mathfrak{b}\cdot f$ for some divisor $\mathfrak{b}$ on $C$ such that $\mathfrak{b}+\mathfrak{e}$ is very ample and $|L-3p|$ contains a fibre of $\pi$ as fixed component with multiplicity two. Moreover, if $e \geq -1$ then $\deg(\mathfrak{b}) \geq 2g+1$ and we are in the situation of Example 8.*

**Proof:** As in the proof of [ChM06] Thm. 19, since the case in which the length of $Z_p$ is 4 has been ruled out in Remark 7, we only have to consider the situations in Proposition 11 above.

Using the notation there we have a divisor $A := L-K-B \sim_a aC_0+a'f$ and a curve $B \sim_a bC_0 + b'f$ satisfying certain numerical properties, in particular $p_a(B) = 0$, $B^2 = 0$, and $a > 0$ since $A$ is big. Moreover,

$$3 = A.B = -eab + ab' + a'b \tag{4}$$

and

$$a\cdot(2a'-ae) = A^2 = (L-K)^2 - 2\cdot A.B - B^2 \geq 17 - 2\cdot A.B - B^2 = 11. \tag{5}$$

By Lemma 10 there are three possibilities for $B$ to consider. If $e < 0$ and $B \sim_a bC_0 + \frac{eb}{2}\cdot f$ with $b \geq 2$, then Riemann-Roch leads to the impossible equation

$$-2 = 2p_a(B) - 2 = B.K = (2g-2)\cdot b.$$

If $e = 0$ and $B \sim_a bC_0$, then similarly Riemann-Roch shows

$$-2 = B.K = (2g-2)\cdot b,$$

which now implies that $b = 1$ and $g = 0$. In particular, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $L \sim_a A + B + K \sim_a (a-1)\cdot C_0 + f$, since $3 = A.B = a'$. But this is then one of the cases of Example 8.

Finally, if $B \sim_a f$ then (4) gives $a = 3$, and thus

$$L \sim_a A + B + K \sim_a C_0 + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) \cdot f,$$

where $A = 3C_0 + \mathfrak{a}' \cdot f$. Moreover, by the assumptions of Case (b) the linear system $|L - 3p|$ contains the fibre of the ruling over $p$ as double fixed component, and since $L$ is very ample it induces on $C$ the very ample divisor $\mathfrak{e} + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e})$. Note also, that (5) implies that

$$a' - 2 - e \geq \frac{e}{2},$$

and thus for $e \geq -1$ we have

$$\deg(\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) = 2g + 1 + (a' - 2 - e) \geq 2g + 1,$$

so that then the assumptions of Example 8 are fulfilled. This finishes the proof. $\square$

If $\pi : S \longrightarrow C$ is a ruled surface, then there is a (not necessarily unique (if $g(C) = 0$)) minimal model

$$\begin{array}{ccc} S & & \\ & \overset{\phi}{\searrow} & \overset{\pi}{\searrow} \\ & S' & \overset{\widetilde{\pi}}{\longrightarrow} C, \end{array}$$

and the Néron-Severi group of $S$ is

$$\mathrm{NS}(S) = C_0 \cdot \mathbb{Z} \oplus f \cdot \mathbb{Z} \oplus \bigoplus_{i=1}^{k} E_i \cdot \mathbb{Z},$$

where $f$ is a general fibre of $\pi$, $C_0$ is the total transform of section of $\widetilde{\pi}$, and the $E_i$ are the total transforms of the exceptional divisors of the blow-up $\phi$. Moreover, for the Picard group of $S$ we just have to replace $f \cdot \mathbb{Z}$ by $\pi^* \operatorname{Pic}(C)$. We may, therefore, represent a divisor class $A$ on $S$ as

$$L = a \cdot C_0 + \pi^* \mathfrak{b} - \sum_{i=1}^{k} c_i E_i. \qquad (6)$$

**Corollary 13**

*Suppose that $(S, L)$ is a tuple as in Proposition 1 with ruling $\pi : S \to C$, and suppose that the Néron-Severi gruop of $S$ is as described before with general fibre $f = B_p$.*

*Then $S$ is minimal, $L = C_0 + \pi^*\mathfrak{b}$ for some divisor $\mathfrak{b}$ on $C$ such that $\mathfrak{b}+\mathfrak{e}$ is very ample and $|L-3p|$ contains a fibre of $\pi$ as fixed component with multiplicity two.*

**Proof:** Let $L = C_0 + \pi^*\mathfrak{b} - \sum_{i=1}^k c_iE_i$, as described in (6). Then

$$L - K = (a+2)\cdot C_0 + \pi^*(\mathfrak{b} - K_C - \mathfrak{e}) - \sum_{i=1}^k (c_i+1)\cdot E_i,$$

and thus considering Proposition 11

$$3 = (L-K).B = a+2.$$

The very ampleness of $L$ implies thus that $c_i > 0$ for all $i$. But then, if $S$ is not minimal and $f'$ is the strict transform of a fiber of the minimal model, meeting some $E_i$, then $L \cdot f' \leq 0$, a contradiction. □

By [ChM06] we get Theorem 3 as an immediate corollary.

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