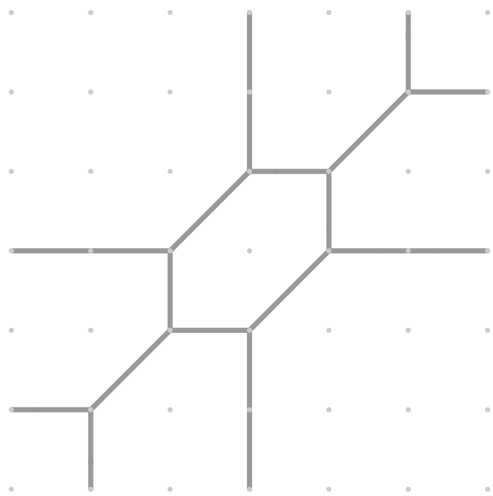
Families of Curves

and a

Lifting Problem for

Tropical Varieties



Thomas Markwig

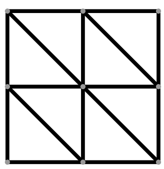
Beim Fachbereich Mathematik
der Technischen Universität Kaiserslautern
eingereichte Habilitationschrift

To my parents
Hildegard and
Eduard Keilen
and to my
wife Hannah

The picture on the titlepage shows a the tropical curve defined by the polynomial

$$f = t^2 \cdot x^2y^2 + x^2y + xy^2 + x^2 + \frac{1}{t} \cdot xy + y^2 + x + y + t^2$$

with the following Newton subdivision:



The vertices of the tropical curve are:

$$(2,2), \quad (1,1), \quad (1,0), \quad (0,1), \quad (0,-1), \quad (-1,0), \quad (-1,-1), \quad (-2,-2)$$

It should be understood as the tropicalisation of a hyperplane section of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ over the field $\mathbb{C}\{\{t\}\}$ of Puiseux series embedded into $\mathbb{P}^8$ by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)$ specified by the above Newton polygon. The picture was produced by the SINGULAR procedure `drawtropicalcurve` from the library `tropical.lib` which can be obtained via the following url:

http://www.mathematik.uni-kl.de/~keilen/en/tropical.html

Contents

Introduction

The present “Habilitationsschrift” is composed of nine scientific papers written throughout the years 2002-2007. It can be divided into three parts. The first part is considered with the study of equisingular families of curves on smooth projective surfaces. This is related to the second part which studies special fat point schemes on projective surfaces in the spirit of the Harbourne-Hirschowitz conjecture. Finally the third part is completely independent of the two former. Methods from commutative algebra and computer algebra are used to give a constructive proof of the Lifting Lemma in tropical geometry. The papers in Part A ([**Kei05b**], [**KeL05**], [**Kei06**], [**Mar05**]) have all been published in mathematical journals respectively been accepted for publication. The papers in Part B ([**Mar06**], [**ChM07a**], [**ChM07b**]) and in Part C ([**Mar07**], [**JMM07**]) are more recent work. They have been submitted for publication to mathematical journals and meanwhile they have been made available to the public on the arXive.

In this introduction we will try to give some background on the questions studied in the papers and to show how our results fit into this scheme. Doing so we will also give a brief description of the main results obtained in the papers, but we refer to the introduction of each paper for a more elaborate description of them.

Part A

The existence of singular curves, their deformations, and the structure of families of singular curves have attracted the constant attention of algebraic geometers since the late nineteenth century. The foundations of the theory were laid by mathematicians like Plücker, Severi, Segre, and Zariski. The answers to many of their questions needed, however, modern methods. Essential progress in recent years has been achieved by the work of Ciliberto, Sernesi, Chiantini, Harris, Harbourne, Hirschowitz, Greuel, Lossen, Shustin and many others. Before going into more detail let us fix some notation.

Let Σ be a smooth projective surface embedded by a very ample line bundle L, let d be a positive integer, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be singularity types – either topological or analytical. We want to consider the families $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ of irreducible curves in the linear system $|dL|$ having precisely r singular points of the prescribed type. The type of question we are concerned with is:

(a) Is $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r) \neq \emptyset$?

(b) Is $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ smooth of the expected dimension?

(c) Is $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ irreducible?

(d) What is the degree of $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ in $|dL|$?

That the dimension is the expected one means that the dimension of $|dL|$ drops for each imposed singularity type $\mathcal{S}_i$ exactly by the number of conditions imposed by $\mathcal{S}_i$ – e. g. a node imposes one condition, a cusp two. For a more detailed introduction of the concepts we refer to Section I.1.1-1.4.

The simplest possible case of nodal plane curves was more or less completely answered by Severi in the early 20th century. He showed that $V^{irr}_{|dL|}(rA_1)$, where L is a line in $\mathbb{P}^2_{\mathbb{C}}$, is non-empty if and only if

$$0 \leq r \leq \frac{(d-1)\cdot(d-2)}{2}.$$

Moreover, he showed that $V^{irr}_{|dL|}(rA_1)$ is T-smooth (i.e. smooth and of the expected dimension) whenever it is non-empty, and he claimed that the variety is always irreducible. Harris proved this claim, which had become known as the Severi Conjecture by then, in 1985 (cf. [**Har85**]). Considering more complicated singularities we may no longer expect such complete answers. Hirano provides in [**Hir92**] a series of examples of irreducible cuspidal plane curves of degree $d = 2\cdot 3^k$, $k \in \mathbb{N}$, imposing more than $\frac{d(d-3)}{2}$ conditions on $|dL|$ – that means in particular, we may hardly expect to be able to realise all smaller quantities of cusps on an irreducible curve of degree d. Moreover, we see that $V^{irr}_{|dL|}(rA_2)$ does not necessarily have the expected dimension – examples of this behaviour were already known to Segre (cf. [**Seg29**]). In 1974 Jonathan Wahl (cf. [**Wah74b**]) showed that the family $V^{irr}_{|104\cdot L|}(3636\cdot A_1, 900\cdot A_2)$ of plane curves of degree 104 is non-reduced and hence singular. However, its reduction is smooth. The first example where also the reduction is singular, is due to Luengo. In [**Lue87a**] he shows that the plane curve C given by $x^9 + z\big(xz^3+y^4\big)^2$ has a single singular point of simple type A_{35} and that $V^{irr}_{|9\cdot L|}(A_{35})$ is non-smooth, but reduced at C. Thus also the smoothness will fail in general. And finally, already Zariski (cf. [**Zar35**]) knew that the family $V^{irr}_{|6\cdot L|}(6\cdot A_2)$ of plane sextics consists of two connected components.

The best we may thus expect is to find numerical conditions, depending on the divisor L, on d, and on certain invariants of the singularities, which imply either of the properties in question. In order to see that the conditions are of the right kind - we then call them *asymptotically proper* -, they should not be too far from necessary conditions respectively they should be nearly fulfilled for series of counterexamples. Let us make this last statement a bit more precise. We are looking for conditions of the kind

$$\sum_{i=1}^{r} \alpha(\mathcal{S}_i) < p(d),$$

where α is some invariant of topological respectively analytical singularity types and $p \in \mathbb{R}[x]$ is some polynomial, neither depending on d nor on the $\mathcal{S}_i$. We say that the condition is *asymptotically proper*, if there is a necessary condition with the same invariants and a polynomial of the same degree. If instead we find an infinite series of examples not having the desired property, where, however, the above inequality is reversed for the same invariants and some other polynomial of the same degree, we say that at least for the involved subclass of singularity types, the condition is *asymptotically proper*.

While the study of nodal and cuspidal curves has a long tradition, the consideration of families of more complicated singularities needed a suitable description of the tangent space of the family at a point, giving a concrete meaning to "the number of conditions imposed by a singularity type", that is to the expected dimension of the family. Greuel and Karras in [**GrK89**] in the analytical case, Greuel and Lossen in [**GrL96**] in the topological case identify the tangent spaces basically with the global sections of the ideal sheaves of certain zero-dimensional schemes associated to the singularity types. This approach – in combination with a Viro gluing type method in the existence case – allows to reduce the existence, T-smoothness and irreducibility problem to the vanishing of certain cohomology groups. Various efforts in this direction culminate in asymptotically proper conditions for the existence (cf. [**GLS98b**]) and conditions for the T-smoothness and irreducibility, which are better than any previously known ones (cf. [**GLS00**]) – all for plane curves. Due to known examples the conditions for the T-smoothness are even asymptotically proper for simple singularities and ordinary multiple points. All these are results for curves in the projective plane.

Much less is known for surfaces other than the projective plane. In [**KeT02**] we studied the existence of curves with prescribed singularities, and we gave an asymptotically proper condition of the form

$$\sum_{i=1}^{r} \delta(\mathcal{S}_i) \leq \alpha \cdot d^2 + \beta \cdot d + \gamma,$$

where the coefficients α, β, and γ depend on the surface Σ and the divisor L, and where $\delta(\mathcal{S}_i)$ is the delta invariant of $\mathcal{S}_i$. In the case of topological singularity types one can replace δ by the Milnor number μ. The conditions work for any surface Σ, and even though they are asymptotically proper, when applied to the case of the projective plane the coefficients are worse than the previously known ones. The question of irreducibility for arbitrary surfaces was first considered in [**Kei03**] and then improved in [**Kei05a**], and replacing δ

by the Tjurina number τ respectively the equisingular Tjurina number τ^{es} we got a condition of the same type for the irreducibility. Due to the lack of suitable examples of reducible equisingular families of curves on general surfaces we still do not know if these conditions are asymptotically proper for general singularities, not even in the plane case.

In Part A of this "Habilitationsschrift" we study mainly the question of T-smoothness of $V_{|dL|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, and we restrict our attention mostly to general surfaces in $\mathbb{P}_{\mathbb{C}}^3$, general products of curves and geometrically ruled surfaces. In Paper I, [**Kei05b**], we give numerical conditions for the T-smoothness of such a family using a new invariant for singularities introduced in Paper II, [**KeL05**], and in Paper III, [**Kei06**], and Paper IV, [**Mar05**], we study series of equisingular families showing that on general surfaces in $\mathbb{P}_{\mathbb{C}}^3$ the conditions are asymptotically proper e.g. for ordinary multiple points – as in the plane case.

The varieties $V_{|dL|}(rA_1)$ of (not necessarily irreducible) curves in $|dL|$ with precisely r simple nodes (respectively the open subvarieties $V_{|dL|}^{irr}(rA_1)$ of reduced and irreducible nodal curves) in a fixed linear system $|dL|$ on a smooth projective surface Σ are also called *Severi varieties*. As mentioned above, when $\Sigma = \mathbb{P}_{\mathbb{C}}^2$ Severi showed that these varieties are smooth of the expected dimension, whenever they are non-empty – that is, nodes always impose independent conditions.

In [**Tan82**] Tannenbaum showed that also on K3-surfaces $V_{|dL|}(rA_1)$ is always smooth, that, however, the dimension is larger than the expected one and thus $V_{|dL|}(rA_1)$ is not T-smooth in this situation. If we restrict our attention to the subvariety $V_{|dL|}^{irr}(rA_1)$ of *irreducible* curves with r nodes, then we gain T-smoothness again whenever the variety is non-empty. That is, while on a K3-surface the conditions which nodes impose on irreducible curves are always independent, they impose dependent conditions on reducible curves.

On more complicated surfaces the situation becomes even worse. Chiantini and Sernesi study in [**ChS97**] Severi varieties on surfaces in $\mathbb{P}_{\mathbb{C}}^3$. They show that on a generic quintic Σ in $\mathbb{P}_{\mathbb{C}}^3$ with hyperplane section L the variety $V_{|dL|}^{irr}\big(\frac{5d(d-2)}{4} \cdot A_1\big)$ has a non-smooth reduced component of the expected dimension, if d is even. They construct their examples by intersecting a general cone over Σ in $\mathbb{P}_{\mathbb{C}}^4$ with a general complete intersection surface of type $\big(2, \frac{d}{2}\big)$ in $\mathbb{P}_{\mathbb{C}}^4$ and projecting the resulting curve to Σ in $\mathbb{P}_{\mathbb{C}}^3$. Moreover, Chiantini and Ciliberto give in [**ChC99**] examples showing that the Severi varieties $V_{|dL|}^{irr}(rA_1)$ on a surface in $\mathbb{P}_{\mathbb{C}}^3$ also may have components of dimension larger than the expected one.

Hence, even for nodal curves one can only ask for numerical conditions ensuring that $V^{irr}_{|dL|}(rA_1)$ is T-smooth, and Chiantini and Sernesi answer this question by showing that on a surface of degree $n \geq 5$ the condition

$$r < \frac{d(d-2n+8)n}{4} \tag{0.1}$$

implies that $V^{irr}_{|dL|}(rA_1)$ is T-smooth for $d > 2n-8$. Note that the above example shows that this bound is even sharp. Actually Chiantini and Sernesi prove a somewhat more general result for surfaces with ample canonical divisor K_Σ and curves which are in $|pK_\Sigma|_l$ for some $p \in \mathbb{Q}$. For their proof they suppose that for some curve $C \in V^{irr}_{|dL|}(rA_1)$ the cohomology group $H^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ does not vanish and derive from this the existence of a Bogomolov unstable rank-two bundle E. This bundle in turn provides them with a curve Δ of small degree realising a large part of the zero-dimensional scheme $X^*(C)$, which leads to the desired contradiction.

This is basically the same approach used in **[GLS97]**. However, they allow arbitrary singularities rather than only nodes, and get in the case of a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree n

$$\sum_{i=1}^{r} \big(\tau^*_{ci}(\mathcal{S}_i)+1\big)^2 < d \cdot \big(d-(n-4)\cdot \max\big\{\tau^*_{ci}(\mathcal{S}_i)+1 \;\big|\; i=1,\ldots,r\big\}\big) \cdot n$$

as main condition for T-smoothness of $V^{irr}_{|dL|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ – see Section I.1 for the definition of τ^*_{ci} –, which for nodal curves coincides with (0.1) – here τ^* is the (equisingular) Tjurina number and K_Σ is the canonical divisor on Σ. Moreover, for families of plane curves of degree d their result gives

$$\sum_{i=1}^{r} \big(\tau^*_{ci}(\mathcal{S}_i)+1\big)^2 < d^2+6d$$

as sufficient condition for T-smoothness, which is weaker than the sufficient condition

$$\sum_{i=1}^{r} \gamma_1^*(\mathcal{S}_i) \leq (d+3)^2 \tag{0.2}$$

derived in **[GLS00]** and **[GLS01]** using the Castelnuovo function in order to provide a curve of small degree which realises a large part of $X^*(C)$. The advantage of the γ_1^*-invariant (introduced and studied in Paper II) is that, while always bounded from above by $(\tau^*_{ci}+1)^2$, in many cases it is substantially smaller – e. g. for an ordinary m-fold point M_m, $m \geq 3$, we have $\gamma_1^{es}(M_m) = 2m^2$, while

$$\big(\tau^{es}_{ci}(M_m)+1\big)^2 \geq \frac{(m^2+2m+4)^2}{16}.$$

In Paper I we combine the methods of **[GLS00]** and the method of Bogomolov instability to reproduce the result (0.2) in the plane case, and to derive a similar sufficient condition,

$$\sum_{i=1}^{r}\gamma_\alpha^*(\mathcal{S}_i)<\gamma\cdot(d\cdot L-K_\Sigma)^2,$$

for T-smoothness on other surfaces – involving a generalisation γ_α^* of the γ_1^*-invariant which is always bounded from above by the latter one – see Theorems I.2.1, I.2.5 and I.2.6.

In Paper II we compute the new invariant not only in the case of ordinary m-fold points but also for simple singularities, and we give an upper bound in the case of semi-quasihomogeneous singularities – see Propositions II.1.11, II.1.12 and II.1.13.

In Paper III we produce series of equisingular families of curves with ordinary multiple points and in Paper IV series of equisingular families with simple singularities both on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and both showing that for these families the above conditions are asymptotically proper – see Theorem III.1.1 and Examples IV.1.1 and IV.1.2.

Part B

The papers in Part B are, in some sense, concerned with a weak version of the problems considered in Part A. We still fix some linear system $|dL|$ on a smooth projective surface Σ embedded via L and we consider curves in this linear system. Fixing, moreover, r points $p_1,\ldots,p_r$ and positive integers $m_1,\ldots,m_r$ we want these curves to pass through point p_i at least with multiplicity m_i, i.e. instead of prescribing the singularity type in precise terms we only prescribe the multiplicity. This is a much weaker condition. However, at the same time the family has the disadvantage that it is harder to describe the tangent space to it in a satisfying way, as we will see in Paper V, [**Mar06**].

Let us look at the problem outlined above in a more naive way in the case where Σ is the projective plane. Then the data which we prescribe are points $p_1,\ldots,p_r$ and positive integers $m_1,\ldots,m_r$ and d, and we are looking for homogeneous polynomials of degree d such that at the point p_i in local coordinates all partial derivatives up to order m_i-1 vanish. Since we have two local coordinates the point p_i induces $\frac{m_i\cdot(m_i+1)}{2}$ conditions. A naive count therefore predicts that we should expect a family $V=V_{|dL|}(m_1,\ldots,m_r)$ of (projective) dimension

$$\operatorname{expdim}\big(V_{|dL|}(m_1,\ldots,m_r)\big)=\max\left\{\frac{d\cdot(d-3)}{2}-\sum_{i=1}^{r}\frac{m_i\cdot(m_i+1)}{2},-1\right\},$$

and a straight forward example shows that the actual dimension of course depends on the position of the points, e.g. if $d=m_1=m_2=m_3=1$ then V is empty as expected if and only if the points p_1, p_2 and p_3 do not lie on a line. This shows that the interpolation problem for several variables is harder than it is for only one. However, the dimension of $V_{|dL|}(m_1,\ldots,m_r)$ is semi-continuous in the p_i, so that for a generic choice it actually is constant and minimal. This is the justification for omitting the p_i among the data attributed to $V_{|dL|}(m_1,\ldots,m_r)$. From now on we will assume that the p_i are chosen sufficiently general for the problem considered, and the main question is:

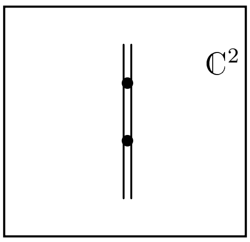
$$\text{Is } \dim\big(V_{|dL|}(m_1,\ldots,m_r)\big) = \operatorname{expdim}\big(V_{|dL|}(m_1,\ldots,m_r)\big)?$$

We call the linear system $V_{|dL|}(m_1,\ldots,m_r)$ *special* if its dimension and its expected dimension do not coincide.

Staying in the plane case and choosing two points, then sufficiently general certainly just means that the points are distinct. If we now consider plane conics passing through two distinct points with multiplicity at least 2, then the expected dimension is

$$\operatorname{expdim}\big(V_{|2L|}(2,2)\big) = 5-3-3 = -1,$$

i.e. we would not expect any such curve to exist. However, there is a unique line through the given two points and this line counted twice is a conic which has multiplicity two at each of the two points.



Of course in some sense this example is degenerate, since the curve contains a whole component with higher multiplicity while one in general would expect only a finite number of points in which the multiplicity is not one. One can rephrase this by considering the incidence variety

$$\mathcal{L}_{2,2} = \{(C,(p_1,p_2)) \in |2L| \times \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \mid \operatorname{mult}_{p_i}(C) \geq 2\}$$

together with the canonical projection

$$\mathcal{L}_{2,2} \xrightarrow{\beta} |2L| = \mathbb{P}^5_{\mathbb{C}}. \tag{0.3}$$

Then the fibre of β over the double line is not finite, as expected, but one-dimensional.

Segre, [**Seg62**], conjectured that this is indeed not an exceptional behaviour but the general one if the dimension of $V_{|dL|}(m_1,\ldots,m_r)$ is not the expected one for plane curves. More precisely:

Conjecture (Segre, 1961)
If the linear system $V_{|dL|}(m_1,\ldots,m_r)$ is special then it has a multiple fixed component.

Since then a large number of people have worked on this problem, several more precise respectively related conjectures have been formulated, and many partial results have been achieved. The conjecture in its whole, however, still withstands a proof.

A usual way to resolve singular, i.e. multiple, points of a curve is to blow up the surface in these points. That way the linear system $V_{|dL|}(m_1,\ldots,m_r)$ – for fixed generic points p_i – would be transformed into a complete linear system $|dL - m_1\cdot E_1 - \ldots - m_r \cdot E_r|$ on the the blown up surface. Harbourne, [**Har86**], and Hirschowitz, [**Hir89**], conjectured that:

Conjecture (Harbourne–Hirschowitz, 1986/89)
$V_{|dL|}(m_1,\ldots,m_r)$ is special if and only if the linear system $|dL-m_1\cdot E_1-\ldots-m_r\cdot E_r|$ contains a smooth rational curve of self-intersection -1 with multiplicity at least two.

Moreover, they make very precise predictions about the structure of $V_{|dL|}(m_1,\ldots,m_r)$ and about its general elements if it is non-special. A similar conjecture was made by Gimiliano, [**Gim87**]. It is evident that the Harbourne–Hirschowitz Conjecture implies the Segre Conjecture, but it turns out that they are actually equivalent – even including the extra statements on the structure of $V_{|dL|}(m_1,\ldots,m_r)$ and its general elements. This was proven by Ciliberto and Miranda, [**CiM01**], but it can also be deduced from some results by Nagata, [**Nag59**]. In [**CiM01**] Ciliberto and Miranda do not only show the equivalence of the above mentioned conjectures, but they yet specify them further by classifying the special systems with reducible general curves and the so called *homogeneous* systems, i.e. those for which all m_i coincide. Finally, they show that the Segre Conjecture implies a famous conjecture by Nagata, [**Nag59**], on plane curves:

Conjecture (Nagata, 1959)
If $r\geq 10$ and $C\in V_{|dL|}(m_1,\ldots,m_r)$ then $\sum_{i=1}^r m_i < d\cdot\sqrt{r}$.

Nagata, [**Nag59**], himself proved the conjecture for r being a square. In the homogeneous case Evain, [**Eva98**], proved the conjecture when m is small compared to r, and Xu, [**Xu94**], and Roe, [**Roé01**], were able to prove slightly weaker inequalities. Szemberg, [**Sze01**], and Roe, [**Roé03**], finally relate Nagata's Conjecture to Seshadri constants, that way generalising the conjecture to other surfaces than the projective plane.

As already indicated, many special cases of the conjecture have been treated and solved when $\Sigma=\mathbb{P}^2_{\mathbb{C}}$. It is classically known that the statement holds if

$r \leq 9$, see e.g. [**Nag59**]. The case when all multiplicities are 2 was treated by Arbarello and Cornalba, [**ArC81**], and the homogeneous case for multiplicities up to 20 and the quasi-homogeneous case – i.e. where all but the largest multiplicity coincide – was in several steps done by many authors, see e.g. [**Hir85**], [**CiM98**] to mention only some. Similarly the case of small multiplicities has been solved, see e.g. [**Mig00**], [**Mig01**], [**Yan04**], [**Roé01**], or when $r = 4^k$ is of particular type, see [**Eva99**]. The conjectures have then been extended to and studied on other surfaces – e.g. K3 surfaces, rational scrolls –, see e.g. [**Laf02**], [**Laf06**], [**?**], [**?**], [**LaU03a**], respectively in other varieties – e.g. $\mathbb{P}^n_{\mathbb{C}}$, toric varieties –, see e.g. [**LaU03b**], [**?**].

The problem of speciality can be reduced to an h^1-vanishing, and the method proposed by Alexander and Hirschowitz to tackle the problem was to specialise points in such a way that fixed curves split off and an induction is possible, see e.g. [**Hir85**], [**Eva99**], [**Mig01**], [**AlH00**]. Ran, [**Ran89**], proposed a different approach. He degenerates the plane containing the linear system rather than the curves. In the limit the projective plane splits and so does the corresponding linear system, again allowing some kind of inductive procedure, see e.g. [**CiM98**], [**Yan04**]. Arbarello and Cornalba used for their result techniques from deformation theory, see [**ArC81**], [**Mir00**], and Miranda proposed an approach using the so called interpolation matrix, see [**BoM04**].

The situation when all multiplicities are two has its own and rather particular flavour, since it can be reinterpreted as the *Waring Problem*. This originates in the question whether a given integer $z \in \mathbb{Z}$ can be written as a sum of $k+1$ d-th integer powers, i.e. $z = z_0^d + \cdots + z_k^d$ with $z_i \in \mathbb{Z}$. Replacing the integers by linear forms we may ask whether a given homogeneous polynomial $f \in \mathbb{C}[x,y,z]_d$ of degree d can be written as a sum of $k+1$ d-th powers of linear forms, i.e. if there are linear forms $f_0, \ldots, f_k \in \mathbb{C}[x,y,z]_1$ such that

$$f = f_0^d + \ldots + f_k^d. \tag{0.4}$$

Identifying $\mathbb{C}[x,y,z]_1/\mathbb{C}^*$ with $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{C}[x,y,z]_d/\mathbb{C}^*$ with $\mathbb{P}^N_{\mathbb{C}}$, $N = \frac{d\cdot(d+3)}{2}$, the "$d$-th power" map

$$\nu_d : \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \mathbb{P}^N_{\mathbb{C}} : f \mapsto f^d$$

is just the d-tuple Veronese embedding of the projective plane. Moreover, f satisfies an equation of the form (0.4) if and only if there is a k-secant plane of $X_d = \nu_d\big(\mathbb{P}^2_{\mathbb{C}}\big)$ spanned by points $\nu_d(f_0) = f_0^d, \ldots, \nu_d(f_k) = f_k^d$ containing f – here we use of course that in $\mathbb{C}$ we have d-th roots. The question, if every f has a decomposition as in (0.4) is therefore equivalent to asking whether the k-secant variety $S_k(X_d)$ of X_d fills the whole $\mathbb{P}^N_{\mathbb{C}}$. For a k-secant we choose $k+1$ points in X_d, which is two dimensional, and for a general choice they span a

k-dimensional plane so that a straight forward dimension count says that we should expect

$$\operatorname{expdim}\big(S_k(X_d)\big) = \min\{3k+2, N\}$$

as the dimension of $S_k(X_d)$. We thus would expect a solution to (0.4) as soon as $3k+2 \geq N$. However, the actual dimension of $S_k(X_d)$ might be less than expected, and this can be checked by choosing a general point $u \in S_k(X_d)$ and computing the dimension of the tangent space $T_u\big(S_k(X_d)\big)$. Terracini's Lemma now is the key to connect the Waring Problem to linear systems as above being special. It states that if $u \in S_k(X_d)$ is general and $u \in \langle p_0, \ldots, p_k\rangle$ for some $p_0, \ldots, p_k \in X_d$, then we have

$$T_u\big(S_k(X_d)\big) = \langle T_{p_0}(X_d), \ldots, T_{p_k}(X_d)\rangle\,,$$

i.e. the tangent space to the k-secant variety at u is spanned by the tangent spaces to X_d at the p_i. But we thus we have that

$$\dim\big(S_k(X_d)\big) < \operatorname{expdim}\big(S_k(X_d)\big) \tag{0.5}$$

– we then say X_d is k-defective – if and only if for general points $p_0, \ldots, p_k \in X_d$

$$\dim\big\langle T_{p_0}(X_d), \ldots, T_{p_k}(X_d)\big\rangle < \operatorname{expdim}\big(S_k(X_d)\big),$$

which is equivalent to

$$\dim\big\{H \text{ Hyperplane} \;\big|\; T_{p_i}(X_d) \subset H \;\forall\, i\big\} > N - \max\{N+1, 3\cdot(k+1)\}.$$

The latter again is equivalent to

$$\dim\big\{H \;\big|\; H \cap X_d \text{ singular in } p_i \;\forall\, i\big\} > \max\{-1, N-3k-3\},$$

which means that

$$V_{|dL|}(2, \ldots, 2) \text{ is special},$$

where L is the class of a line in $\mathbb{P}^2_{\mathbb{C}}$ and we have $k+1$-times the multiplicity 2. That is the Waring Problem (0.4) has a solution if and only if $6k+4 \geq d\cdot(d+3)$ and $V_{|dL|}(2, \ldots, 2)$ is non-special. In particular, since $V_{|2L|}(2,2)$ is special, as we have seen above, this shows that one needs at least the sum of three squares of linear forms in order to write a general quadratic form. The classification of special linear systems on $\mathbb{P}^2_{\mathbb{C}}$ which are homogeneous of multiplicity 2 by Alexander and Hirschowitz shows that the case $d = 4$ and $k = 5$ is the only other defective case of a Veronese embedding. Similar interpretations have the linear systems which are homogeneous of multiplicity 2 for other surfaces, and this situation is quite well understood. More precisely, a general curve in such a system will always have a double component through all the p_i, see [**Ter22**], [**ArC81**], and see also [**Laf06**].

Much less is known for other multiplicities on surfaces other than the projective plane, even if we prescribe only one multiple point of multiplicity three. In $\mathbb{P}^2_{\mathbb{C}}$ such a linear system will of course be non-special, but on $\mathbb{F}_0 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

this need not be the case any more. Consider the linear system $|L| = |\mathcal{O}_{\mathbb{F}_0}(1,2)|$ on $\mathbb{F}_0$. Its dimension is $\dim|L| = 5$ and a single triple point gives 6 conditions, so that one does not expect any curve in $|L|$ to pass through a general point with multiplicity 3. However, $\mathbb{F}_0$ has the ruling $|\mathcal{O}_{\mathbb{F}_0}(0,1)|$ which has a line, say F_p, passing through each point p of $\mathbb{F}_0$, and this ruling has through each point $p \in \mathbb{F}_0$ a section, say C_p, in $|\mathcal{O}_{\mathbb{F}_0}(1,0)|$. But then $C_p + 2 \cdot F_p \in |L|$ and it passes through the indeed general point p with multiplicity two. This shows $V_{|L|}(3)$ is special, and we call a tuple (Σ, L) of a surface and a very ample line bundle *triple-point defective* if $V_{|L|}(3)$ is special. One can of course generalise this immediately to the other Hirzebruch surfaces $\mathbb{F}_e$ (see Example VI.1.1) in the sense that $V_{|L|}(3)$ is special with $L = C_0 + (2+e) \cdot F$, where F is a fibre of the ruling and C_0 is the unique section with negative self-intersection $C_0^2 = -e$.

In Paper VI, **[ChM07a]**, and VII, **[ChM07b]**, we show that these examples are actually *the* typical triple-point defective surfaces. We first study the situation where the surface Σ is regular and we show that (Σ, L) can only be triple-point defective if Σ is ruled and $V_{|L|}(3) = |L - 3p|$, for p general, contains a fibre of the ruling with multiplicity two – see Theorem VI.1.4. In particular, Σ is a Hirzebruch surface or, potentially, a blow-up thereof. For this result, however, we have to impose some technical restriction on the very ample line bundles L considered, namely that $L - K_\Sigma$ is very ample as well and that $(L - K_\Sigma)^2 > 16$. These assumptions are imposed by the method that we use for the proof of the statement. In the case that Σ is a Hirzebruch surface $\mathbb{F}_e$ the condition $(L - K_\Sigma)^2 > 16$ actually comes for free if we only assume that $L - K_\Sigma$ is very ample as well. In the second paper we then show that we can actually drop the regularity assumption on Σ and that under the given technical restrictions on L the surface Σ will be minimal. More precisely, we show that if L and $L - K_\Sigma$ are very ample and $(L - K_\Sigma)^2 > 16$ then (Σ, L) can only be triple-point defective if Σ is geometrically ruled and $V_{|L|}(3) = |L - 3p|$, with $p \in \Sigma$ general, contains the fibre through p of the ruling twice – see Theorem VII.1.2. Moreover, we classify the triple-point defective linear systems on geometrically ruled surfaces completely subject to our technical restriction – see Theorem VII.1.3. For the Hirzebruch surfaces it turns out that the above mentioned examples are their only triple-point defective linear systems.

The main idea of the proof is a technique involving Bogomolov instability of certain rank two bundles which with certain modifications was already used in **[ChS97]** and in **[Kei05b]**. For this we consider the incidence variety

$$\mathcal{L}_3 = \left\{ (C,p) \in |L| \times \Sigma \;\middle|\; \operatorname{mult}_p(C) \geq 3 \right\}$$

together with the two canonical projections

$$|L| \xleftarrow{\beta} \mathcal{L}_3 \xrightarrow{\alpha} \Sigma \;.$$

The linear system $|L - 3p|$ in which we are interested for a general point p is just

$$|L - 3p| = \beta\big(\alpha^{-1}(p)\big).$$

Unless the general element in $|L - 3p|$ has a triple component for a general point p the map β will be generically finite, and since the situation where this is not the case is completely understood due to **[Cas22]**, **[FrI01]** and **[BoC05]**, we restrict our attention to the case when β is generically finite. But then the dimension of $|L - 3p|$ can be read off from the dimension of $\mathcal{L}_3$, and it suffices to check if $\mathcal{L}_3$ has the expected dimension. For this we can choose a generic point $(L_p, p) \in \mathcal{L}_3$ and compute the dimension of the tangent space. When we studied equisingular deformations in Part A the tangent space to the family in question was described by the global sections of some twisted ideal sheaf. Here the situation is not quite as good any more. It turns out, however, that we still can compute the dimension of the tangent space of $\mathcal{L}_3$ at (L_p, p) by considering global sections of what we call the *equimultiplicity ideal sheaf*, say $\mathcal{J}_{Z_p}$, twisted by L – see Section VI.3 and Proposition V.1.11. This is proven in Paper V, **[Mar06]**, working along the lines of the classical proof for the equisingular case. As long as the singularity of L_p at p is not unitangential everything works fine and the tangent space will coincide with the global sections as usual. If, however, the singularity is unitangential then the tangent space only surjects onto the global sections and the kernel will be one dimensional. But this is good enough to deduce the dimension of the tangent space from $h^0\big(\Sigma, \mathcal{J}_{Z_p}(L)\big)$, and in particular if it is not the expected one then $H^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big)$ does not vanish. By Serre's construction, this yields the existence of a rank two bundle $\mathcal{E}_p$ with first Chern class $L - K_\Sigma$, with a global section whose zero-locus is a subscheme of length at most 4, supported at p. Moreover the assumption $(L - K_\Sigma)^2 > 16$ implies that $\mathcal{E}_p$ is *Bogomolov unstable*, thus it has a destabilising divisor A. By exploiting the properties of A and $B = L - K_\Sigma - A$, we obtain the results in Paper VI.

Part C

In the third part of this "Habilitationsschrift" we are concerned with computational aspects of tropical geometry. The mathematical area of tropical geometry is still very young and when it comes to defining what one actually means by a *tropical variety* the ideas vary quite a bit. Not all the present definitions lead to the same class of geometrical objects, even though there always is a large overlap. Paper IX is concerned with showing that at least two of the definitions absolutely coincide – one more geometric, the other more combinatorial and computational. We will come back to this after we introduced the necessary notation.

All available definitions of tropical varieties agree that they should, at least locally, be piece wise linear subsets of some $\mathbb{R}^n$ carrying maybe some additional structure. One way to get such an object is to start with an algebraic variety over an algebraically closed field with a non-archimedian valuation to the real numbers, e.g. with the field of Puiseux series

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{N=1}^{\infty} \operatorname{Quot}\left(\mathbb{C}\left[\left[t^{\frac{1}{N}}\right]\right]\right)$$

whose elements are formal Laurent series in $t^{\frac{1}{N}}$ for all possible $N > 0$, i.e. series of the form

$$a = \sum_{k=m}^{\infty} a_k \cdot t^{\frac{k}{N}}$$

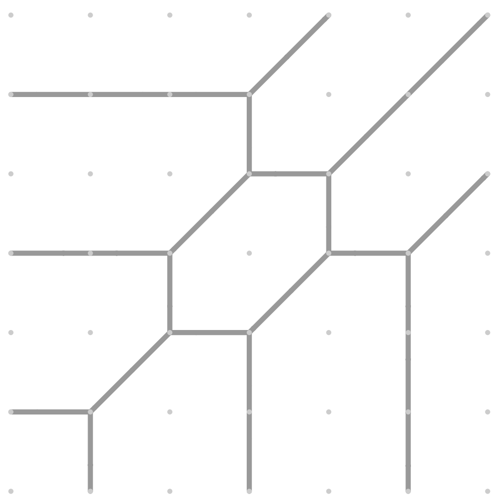
for some $m \in \mathbb{Z}$, $N > 0$ and $a_k \in \mathbb{C}$. K admits a non-archimedian valuation by $\operatorname{val}(a) = m$ if $a_m \neq 0$. If we now consider an ideal in $J \lhd K[x_1, \ldots, x_n]$ and the algebraic variety which it defines in the torus $(K^*)^n$, say $X = V(J) \cap (K^*)^n$, then negative of the component wise valuation defines a map

$$-\operatorname{val} : X \to \mathbb{R}^n : (p_1, \ldots, p_n) \mapsto \big(-\operatorname{val}(p_1), \ldots, -\operatorname{val}(p_n)\big).$$

The topological closure of the image of X is what we call the *tropical variety* $\operatorname{Trop}(J)$ of J for the purposes of this work. E.g. if

$$J = \left\langle t \cdot (x_1^3 + x_2^3 + 1) + \frac{1}{t} \cdot (x_1^2 + x_2^2 + x + y + x_1^2 x_2 + x_1 x_2^2) + \frac{1}{t^2} \cdot x_1 x_2 \right\rangle \tag{0.6}$$

then $\operatorname{Trop}(J)$ looks like[1]



where the vertices are just

$$(2,0),\ \ (1,1),\ \ (1,0),\ \ (0,2),\ \ (0,1),\ \ (0,-1),\ \ (-1,0),\ \ (-1,-1),\ \ (-2,-2).$$

[1]The picture was produced by the SINGULAR procedure `drawtropicalcurve` from the library `tropical.lib` which can be obtained via the url http://www.mathematik.uni-kl.de/~keilen/en/tropical.html.

Tropical geometry evolved rather fast over the past years and the main idea is as follows: even though the valuation map is very crude and one looses a lot of information, many properties of algebraic varieties carry over to the tropical world. Since tropical varieties as piece wise linear objects are easier to deal with than algebraic varieties, and since new methods e.g. from discrete mathematics and combinatorics can be applied, results in the algebraic world can be derived easier on the tropical side. To exploit the tropical world in this sense, however, a sophisticated machinery for the translation of concepts has to be developed.

Among the properties of algebraic varieties that are surprisingly well preserved under tropicalisation are for example the numbers $N(d,g)$ of genus g degree d plane nodal curves through $3d+g-1$ points in general position (also referred to as Gromov-Witten invariants of $\mathbb{P}^2_{\mathbb{C}}$). In his celebrated work **[Mik05]**, Mikhalkin develops a concept how properties as degree and genus have to be translated to the tropical world and proves the Correspondence Theorem stating that the numbers $N(d,g)$ are equal to the numbers of tropical genus g degree d curves through $3d+g-1$ points, counted with multiplicity. He also discovered a way to determine Welschinger invariants for real curves (which can be thought of as analogues of Gromov-Witten invariants) by means of tropical geometry. There is no algorithm within algebraic geometry to compute those numbers. The Correspondence theorem was the start for many new developments in enumerative geometry. For example, the Caporaso-Harris Algorithm to determine relative Gromov-Witten invariants for $\mathbb{P}^2_{\mathbb{C}}$ was reproven using tropical methods in **[GaM07a]**. This tropical proof was used in **[IKS06]** to find a new and much faster algorithm to compute Welschinger invariants. These are only some of many achievements in this new area (see e.g. **[GaM07b]**, **[GaM05]**, **[KeM06]**, **[IKS03]**, **[IKS04]**, **[Shu04]**, **[SpS04b]**, **[SpS04a]**, **[Spe05]**, **[BJS⁺07]**, **[PaS04]**, **[DFS07]**, **[StY07]**, **[Abo06]**). Since the area is young and very active recently, it is particularly important that the foundations are laid carefully to which Paper IX, **[JMM07]**, contributes.

Note that it is not a coincidence that the tropical variety corresponding to the plane curve $V(J)$ in (0.6) is a "curve". It is indeed a theorem that if X is an irreducible d-dimensional variety in the torus then $\mathrm{Trop}(J)$ is a rational polyhedral complex of pure dimension d which is connected in codimension one (cf. **[BiG84]**, **[Stu02**, Thm 9.6]). The proof given by Sturmfels, however, uses a different description of $\mathrm{Trop}(J)$ in terms of initial ideals and term orders.

Given a vector $\omega \in \mathbb{R}^n$ and a "monomial" $t^\alpha \cdot x_1^{\beta_1} \cdots x_n^{\beta_n}$ we can assign to it the ω-weight or ω-degree

$$\deg_\omega\left(t^\alpha \cdot x_1^{\beta_1} \cdots x_n^{\beta_n}\right) = -\alpha + \omega_1 \cdot \beta_1 + \ldots + \omega_n \cdot \beta_n.$$

Doing so, we can define the ω-initial form $\text{in}_\omega(f)$ of an element f in $K[x_1, \ldots, x_n]$ to be the sum of those terms of f which have the maximal ω-degree. Note, that in this definition we treat t as if it was a variable even though it is an element of the base field K. E.g. if

$$f = \frac{4}{t^3} \cdot x_1 \cdot x_2 + (t - 5t^2) \cdot x_1^5 \cdot x_2 - \sum_{k=0}^{\infty} t^k \cdot x_1^4$$

and $\omega = (1, 1)$ then

$$\text{in}_\omega(f) = \frac{4}{t^3} \cdot x_1 \cdot x_2 + t \cdot x_1^5 \cdot x_2.$$

Doing this for all elements of J the resulting initial forms generate the ω-initial ideal $\text{in}_\omega(J)$ of J. However, since the generators are all weighted homogeneous with respect to the weight $(-1, \omega)$ we may as well dehomogenise with respect to t and pass to the *t-initial form* $\text{t-in}_\omega(f) = \text{in}_\omega(f)_{|t=1}$ respectively the *t-initial form* $\text{t-in}_\omega(J) = \text{in}_\omega(J)_{|t=1}$ without loosing any information. That way we get rid of the additional variable t again. It is now an easy observation that if a point $p = (p_1, \ldots, p_n)$ with valuations $\text{val}(p_i) = -\omega_i$ and $p_i = a_i \cdot t^{\omega_i} + h.o.t.$ is in $V(J)$ then the leading term $\overline{p} = (a_1 \cdot t^{\omega_1}, \ldots, a_n \cdot t^{\omega_n})$ is a zero of $\text{in}_\omega(f)$ for each $f \in J$ – or alternatively, $(a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$ is a zero of $\text{t-in}_\omega(f)$ for each $f \in J$. Since this zero $(a_1, \ldots, a_n)$ is in the torus $(\mathbb{C}^*)^n$ necessarily $\text{t-in}_\omega(J)$ does not contain any monomial. And indeed it turns out that

$$\text{Trop}(J) = \big\{\omega \in \mathbb{R}^n \;\big|\; \text{t-in}_\omega(J) \text{ is monomial-free}\big\}, \tag{0.7}$$

which is known as the *Lifting Lemma* (see Theorem IX.2.13 and IX.3.1) and is a description allowing a computational approach for its solution. In the case that X is a hypersurface the proof basically goes back to Newton and was formulated for more general valuation fields in [**EKL04**]. A constructive proof can be found in [**Tab05**]. The general case was proven in [**SpS04b**], but the proof contained a gap which led to a series of papers repairing the prove using different methods and applying to various types of non-archimedian valued fields – in [**Dra06**] affinoid algebras are used, in [**Kat06**] flat deformations over valuation rings are used, and recently in [**Pay07**] the general problem is reduced to the hypersurface case using intersections with and projections to tori and the last proof works for any algebraically closed non-archimedian valued field. Paper IX gives a *constructive* proof of the statement over the Puiseux series field reducing the general case to the zero dimensional case and using a space curve version of the Newton-Puiseux Algorithm proposed in [**Mau80**]. "Constructive" here means that given a point ω in the right hand side of (0.7) which has only rational entries, then we are able to construct a point p in $V(J)$ with $-\text{val}(p) = \omega$. The algorithms (see Algorithm IX.3.8 and IX.4.8) deduced from the proof are implemented using the computer algebra system SINGULAR and the program `gfan` for computing tropical varieties in the SINGULAR library `tropical.lib` which can be obtained via the following url:

`http://www.mathematik.uni-kl.de/~keilen/en/tropical.html`

Of course, the input data for `Singular` procedures have to be restricted to polynomials in $\mathbb{Q}(t)[x_1,\ldots,x_n]$ instead of $\mathbb{C}\{\{t\}\}[x_1,\ldots,x_n]$ and we can only construct p up to a – actually any – finite number of terms, but this is sufficient for most purposes. Where necessary field extensions of $\mathbb{Q}$ will be computed.

The algorithm basically consists of two steps. If $\dim(X) = d$ then in a first step we choose d generic hyperplanes in K^n whose tropicalisation passes through ω and cut X with these hyperplanes so that we reduce to the zero-dimensional case. Then considering t as a variable we have the germ at the origin of a space curve and we can use a space curve version of the Newton-Puiseux algorithm. We can, however, not work over the Puiseux series field for this – not even theoretically, and much less computationally. We have instead to pass to power series and polynomials.

An easy coordinate transformation allows to assume that ω is the origin and we may anyhow assume that the generators are actually contained in

$$R_N = \mathbb{C}\big[\big[t^{\frac{1}{N}}\big]\big],$$

the ring of formal power series in $t^{\frac{1}{N}}$. Replacing then K by $\mathrm{Quot}(R_N)$ the ideal $J\cap R_N[x_1,\ldots,x_n]$ defines a flat, surjective family of curves whose general fibre are just the $\mathrm{Quot}(R_N)$-points of $V(J)$ and whose special fibre is the zero locus of $\text{t-in}_\omega(J)$. Thus, if $V(J\cap R_N[x_1,\ldots,x_n])$ is zero-dimensional so is $V(\text{t-in}_\omega(J))$ – in particular, it is non-empty. Now adding a generic ω-quasi homogeneous linear form a straight forward application of Krull's Principle Ideal Theorem shows that the dimension drops by one. It is slightly tricky though to ensure that this happens simultaneously in J, $J\cap R_N[x_1,\ldots,x_n]$ and $\text{t-in}_\omega(J)$. However, even though the ideas are rather straight forward the proof that the dimension behaves well when passing from J to $J\cap R_N[x_1,\ldots,x_n]$ and finally to $\text{t-in}_\omega(J)$ needs quite a bit of consideration – see e.g. Section IX.6.

Moreover, in order to do computations we need to compute $\text{t-in}_\omega(J)$ from a given generator set of J, and it is in general not enough just to take the t-initial forms of the generators as one knows from general Gröbner basis theory. The computations are done by extending the weight vector $(-1,\omega)$ on the monomials in $(t,x_1,\ldots,x_n)$ to an actual monomial ordering, say $>_\omega$. Note, since the input in t need not be polynomial the ordering will be local with respect to t. The idea then is that the t-initial forms of a standard basis computed from the given generators with respect to $>_\omega$ will generate $\text{t-in}_\omega(J)$ – see Theorem VIII.6.10. In particular, if the input data is actually polynomial in t as well, then the computations can be done using SINGULAR – see Corollary

VIII.6.11. This was previously known for the case where there are no x_i (see [**GrP96**]).

In order to show that this works out it is necessary to first develop the theory of standard bases for mixed power series and polynomial rings, i.e. rings of the form

$$K[[\underline{t}]][\underline{x}].$$

with $\underline{t} = (t_1, \ldots, t_m)$ and $\underline{x} = (x_1, \ldots, x_n)$. This is done the Paper VIII. We follow along the lines of [**GrP02**] and [**DeS07**] generalising the results where necessary. Basically, the main results for this standard bases theory are the proofs of the Division Theorems VIII.2.1 and VIII.3.3, and they can be seen as easy generalisations of Grauert-Hironaka's respectively Mora's Division Theorem (the latter in the form stated and proven first by Greuel and Pfister, see [**GGM⁺94**], [**GrP96**]; see also [**Mor82**], [**Grä94**]). From there we can develop the theory of standard bases using Buchberger's Criterion (see Theorem VIII.4.5) and Schreyer's Theorem (see Theorem VIII.5.3) basically translating the standard proofs word by word with only very few modifications. We treat only formal power series, while Grauert (see [**Gra72**]) and Hironaka (see [**Hir64**]) considered convergent power series with respect to certain valuations which includes the formal case. It should be rather straight forward how to adjust Theorem VIII.2.1 accordingly. Many authors contributed to the further development (see e.g. [**Bec90**] for a standard basis criterion in the power series ring) and to generalisations of the theory, e.g. to algebraic power series (see e.g. [**Hir77**], [**AMR77**], [**ACH05**]) or to differential operators (see e.g. [**GaH05**]). This list is by no means complete.

Part A

PAPER I

Smoothness of Equisingular Families of Curves

Abstract: Francesco Severi (cf. [**Sev21**]) showed that equisingular families of plane nodal curves are T-smooth, i. e. smooth of the expected dimension, whenever they are non-empty. For families with more complicated singularities this is no longer true. Given a divisor D on a smooth projective surface Σ it thus makes sense to look for conditions which ensure that the family $V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ of irreducible curves in the linear system $|D|_l$ with precisely r singular points of types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ is T-smooth. Considering different surfaces including the projective plane, general surfaces in $\mathbb{P}^3_{\mathbb{C}}$, products of curves and geometrically ruled surfaces, we produce a sufficient condition of the type

$$\sum_{i=1}^{r}\gamma_\alpha(\mathcal{S}_i) < \gamma\cdot(D-K_\Sigma)^2,$$

where γ_α is some invariant of the singularity type and γ is some constant. This generalises the results in [**GLS01**] for the plane case, combining their methods and the method of Bogomolov instability, used in [**ChS97**] and [**GLS97**]. For many singularity types the γ_α-invariant leads to essentially better conditions than the invariants used in [**GLS97**], and for most classes of geometrically ruled surfaces our results are the first known for T-smoothness at all.

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1. Introduction

The varieties $V_{|D|}(rA_1)$ (respectively the open subvarieties $V_{|D|}^{irr}(rA_1)$) of reduced (respectively reduced and irreducible) nodal curves in a fixed linear system $|D|_l$ on a smooth projective surface Σ are also called *Severi varieties*. When $\Sigma=\mathbb{P}^2_{\mathbb{C}}$ Severi showed that these varieties are smooth of the expected dimension, whenever they are non-empty – that is, nodes always impose independent conditions. It seems natural to study this question on other surfaces, but it is not surprising that the situation becomes harder.

Tannenbaum showed in [**Tan82**] that also on K3-surfaces $V_{|D|}(rA_1)$ is always smooth, that, however, the dimension is larger than the expected one and thus $V_{|D|}(rA_1)$ is not T-smooth in this situation. If we restrict our attention to the subvariety $V_{|D|}^{irr}(rA_1)$ of *irreducible* curves with r nodes, then we gain T-smoothness again whenever the variety is non-empty. That is, while on a K3-surface the conditions which nodes impose on irreducible curves are always independent, they impose dependent conditions on reducible curves.

On more complicated surfaces the situation becomes even worse. Chiantini and Sernesi study in **[ChS97]** Severi varieties on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. They show that on a generic quintic Σ in $\mathbb{P}^3_{\mathbb{C}}$ with hyperplane section H the variety $V_{|dH|}^{irr}\big(\frac{5d(d-2)}{4}\cdot A_1\big)$ has a non-smooth reduced component of the expected dimension, if d is even. They construct their examples by intersecting a general cone over Σ in $\mathbb{P}^4_{\mathbb{C}}$ with a general complete intersection surface of type $\big(2,\frac{d}{2}\big)$ in $\mathbb{P}^4_{\mathbb{C}}$ and projecting the resulting curve to Σ in $\mathbb{P}^3_{\mathbb{C}}$. Moreover, Chiantini and Ciliberto give in **[ChC99]** examples showing that the Severi varieties $V_{|dH|}^{irr}(rA_1)$ on a surface in $\mathbb{P}^3_{\mathbb{C}}$ also may have components of dimension larger than the expected one.

Hence, one can only ask for numerical conditions ensuring that $V_{|dH|}^{irr}(rA_1)$ is T-smooth, and Chiantini and Sernesi answer this question by showing that on a surface of degree $n \geq 5$ the condition

$$r < \frac{d(d-2n+8)n}{4} \tag{1.1}$$

implies that $V_{|dH|}^{irr}(rA_1)$ is T-smooth for $d > 2n-8$. Note that the above example shows that this bound is even sharp. Actually Chiantini and Sernesi prove a somewhat more general result for surfaces with ample canonical divisor K_Σ and curves which are in $|pK_\Sigma|_l$ for some $p \in \mathbb{Q}$. For their proof they suppose that for some curve $C \in V_{|dH|}^{irr}(rA_1)$ the cohomology group $H^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ does not vanish and derive from this the existence of a Bogomolov unstable rank-two bundle E. This bundle in turn provides them with a curve Δ of small degree realising a large part of the zero-dimensional scheme $X^*(C)$, which leads to the desired contradiction.

This is basically the same approach used in **[GLS97]**. However, they allow arbitrary singularities rather than only nodes, and get in the case of a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree n

$$\sum_{i=1}^{r}\big(\tau_{ci}^*(\mathcal{S}_i)+1\big)^2 < d\cdot\big(d-(n-4)\cdot\max\big\{\tau_{ci}^*(\mathcal{S}_i)+1 \;\big|\; i=1,\ldots,r\big\}\big)\cdot n$$

as main condition for T-smoothness of $V_{|dH|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$, which for nodal curves coincides with (1.1). Moreover, for families of plane curves of degree d their result gives

$$\sum_{i=1}^{r}\big(\tau_{ci}^*(\mathcal{S}_i)+1\big)^2 < d^2+6d$$

as sufficient condition for T-smoothness, which is weaker than the sufficient condition

$$\sum_{i=1}^{r} \gamma_1^*(\mathcal{S}_i) \leq (d+3)^2 \tag{1.2}$$

derived in [**GLS00**] and [**GLS01**] using the Castelnuovo function in order to provide a curve of small degree which realises a large part of $X^*(C)$. The advantage of the γ_1^*-invariant is that, while always bounded from above by $(\tau_{ci}^*+1)^2$, in many cases it is substantially smaller – e. g. for an ordinary m-fold point M_m, $m \geq 3$, we have $\gamma_1^{es}(M_m) = 2m^2$, while

$$\big(\tau_{ci}^{es}(M_m)+1\big)^2 \geq \frac{(m^2+2m+4)^2}{16}.$$

In this paper we combine the methods of [**GLS00**] and the method of Bogomolov instability to reproduce the result (1.2) in the plane case, and to derive a similar sufficient condition,

$$\sum_{i=1}^{r} \gamma_\alpha(\mathcal{S}_i) < \gamma \cdot (D-K_\Sigma)^2,$$

for T-smoothness on other surfaces – involving a generalisation γ_α^* of the γ_1^*-invariant which is always bounded from above by the latter one.

Note that a series of irreducible plane curves of degree d with r singularities of type A_k, k arbitrarily large, satisfying

$$r \cdot k^2 = \sum_{i=1}^{r} \tau^*(A_k)^2 = 9d^2 + \text{ terms of lower order}$$

constructed by Shustin (cf. [**Shu97**]) shows that asymptotically we cannot expect to do essentially better in general. For a survey on other known results on $\Sigma = \mathbb{P}^2_{\mathbb{C}}$ we refer to [**GLS00**] and [**GLS01**], and for results on Severi varieties on other surfaces see [**Tan80, GrK89, GLS98a, FlM01, Fla01**].

In this section we introduce the basic concepts and notations used throughout the paper, and we state several important known facts. Section 2 contains the main results and Section 3 their proofs.

1.1. General Assumptions and Notations

Throughout this article Σ will denote a smooth projective surface over $\mathbb{C}$.

We will denote by $\mathrm{Div}(\Sigma)$ the group of divisors on Σ and by K_Σ its canonical divisor. If D is any divisor on Σ, $\mathcal{O}_\Sigma(D)$ shall be the corresponding invertible sheaf and we will sometimes write $H^\nu(X,D)$ instead of $H^\nu\big(X,\mathcal{O}_X(D)\big)$. A *curve* $C \subset \Sigma$ will be an effective (non-zero) divisor, that is a one-dimensional locally principal scheme, not necessarily reduced; however, an *irreducible curve* shall be reduced by definition. $|D|_l$ denotes the system of curves linearly equivalent to D. We will use the notation $\mathrm{Pic}(\Sigma)$ for the *Picard group* of Σ, that is $\mathrm{Div}(\Sigma)$

modulo linear equivalence (denoted by $\sim_l$), and $\mathrm{NS}(\Sigma)$ for the *Néron–Severi group*, that is $\mathrm{Div}(\Sigma)$ modulo algebraic equivalence (denoted by $\sim_a$). Given a reduced curve $C \subset \Sigma$ we will write $g(C)$ for its *geometric genus*.

Given any closed subscheme X of a scheme Y, we denote by $\mathcal{J}_X = \mathcal{J}_{X/Y}$ the *ideal sheaf* of X in $\mathcal{O}_Y$. If X is zero-dimensional we denote by $\deg(X) = \sum_{z\in Y} \dim_{\mathbb{C}}(\mathcal{O}_{Y,z}/\mathcal{J}_{X/Y,z})$ its *degree*. If $X \subset \Sigma$ is a zero-dimensional scheme on Σ and $D \in \mathrm{Div}(\Sigma)$, we denote by $\left|\mathcal{J}_{X/\Sigma}(D)\right|_l$ the linear system of curves C in $|D|_l$ with $X \subset C$.

Given two curves C and D in Σ and a point $z \in \Sigma$, and let $f, g \in \mathcal{O}_{\Sigma,z}$ be local equations at z of C and D respectively, then we will denote by $i(C, D; z) = i(f, g) = \dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/\langle f, g\rangle)$ the intersection multiplicity of C and D at z.

1.2. Singularity Types

The germ $(C, z) \subset (\Sigma, z)$ of a reduced curve $C \subset \Sigma$ at a point $z \in \Sigma$ is called a *plane curve singularity*, and two plane curve singularities (C, z) and (C', z') are said to be *topologically* (respectively *analytically equivalent*) if there is a homeomorphism (respectively an analytical isomorphism) $\Phi : (\Sigma, z) \to (\Sigma, z')$ such that $\Phi(C) = C'$. We call an equivalence class with respect to these equivalence relations a *topological* (respectively *analytical*) *singularity type*.

When dealing with numerical conditions for T-smoothness some topological (respectively analytical) invariants of the singularities play an important role. We gather some results on them here for the convenience of the reader.

Let (C, z) be the germ at z of a reduced curve $C \subset \Sigma$ and let $f \in R = \mathcal{O}_{\Sigma,z}$ be a representative of (C, z) in local coordinates x and y. For the analytical type of the singularity the *Tjurina ideal*

$$I^{ea}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle$$

plays a very important role, as does the *equisingularity ideal*

$$I^{es}(f) = \left\{ g \in R \;\middle|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \right\} \supseteq I^{ea}(f)$$

for the topological type. They give rise to the following invariants of the topological (respectively analytical) singularity type $\mathcal{S}$ of (C, z).

(a) Analytical Invariants:
 (1) $\tau(\mathcal{S}) = \dim_{\mathbb{C}}\left(R/I^{ea}(f)\right)$ is the *Tjurina number*, i. e. the dimension of the base space of the semiuniversal deformation of (C, z).
 (2) $\tau_{ci}(\mathcal{S}) = \max\left\{ \dim_{\mathbb{C}}(R/I) \;\middle|\; I^{ea}(f) \subseteq I \text{ a complete intersection}\right\}$.
 (3) $\gamma^{ea}_{\alpha}(\mathcal{S}) = \max\left\{\gamma_\alpha(f; I) \;\middle|\; I^{ea}(f) \subseteq I \text{ a complete intersection}\right\}$.

(b) Topological Invariants:

(1) $\tau^{es}(\mathcal{S}) = \dim_{\mathbb{C}}\big(R/I^{es}(f)\big)$ is the codimension of the μ-constant stratum in the semiuniversal deformation of (C, z).

(2) $\tau_{ci}^{es}(\mathcal{S}) = \max\big\{\dim_{\mathbb{C}}(R/I) \;\big|\; I^{es}(C,z) \subseteq I \text{ a complete intersection}\big\}$.

(3) $\gamma_\alpha^{es}(\mathcal{S}) = \max\big\{\gamma_\alpha(f;I) \;\big|\; I^{es}(C,z) \subseteq I \text{ a complete intersection}\big\}$.

Here, for an ideal I containing $I^{ea}(f)$ and a rational number $0 \leq \alpha \leq 1$ we define

$$\gamma_\alpha(f;I) = \max\left\{(1+\alpha)^2\cdot\dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f;I,g) \;\big|\; g\in I, i(f,g)\leq 2\cdot\dim_{\mathbb{C}}(R/I)\right\},$$

where for $g \in I$

$$\lambda_\alpha(f;I,g) = \frac{\big(\alpha\cdot i(f,g)-(1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g)-\dim_{\mathbb{C}}(R/I)}.$$

Note that by Lemma 1.1 $i(f,g) > \dim_{\mathbb{C}}(R/I)$ for all $g \in I$ and $\gamma_\alpha(f,g)$ is thus a well-defined positive rational number.

Throughout this article we will frequently treat topological and analytical singularities at the same time. Whenever we do so, we will write $\tau^*(\mathcal{S})$ for $\tau^{es}(\mathcal{S})$ respectively for $\tau(\mathcal{S})$, and analogously we use the notation $\tau^*_{ci}(\mathcal{S})$ and $\gamma^*_\alpha(\mathcal{S})$. Analogously we will write $X^*(C)$ for the zero-dimensional schemes $X^{es}(C)$ respectively for $X^{ea}(C)$ introduced in Subsection 1.3.

One easily sees the following relations:

$$(1+\alpha)^2\cdot\tau^*_{ci}(\mathcal{S}) \leq \gamma^*_\alpha(\mathcal{S}) \leq \big(\tau^*_{ci}(\mathcal{S})+\alpha\big)^2 \leq \big(\tau^*(\mathcal{S})+\alpha\big)^2. \tag{1.3}$$

In [**KeL05**] the γ^*_α-invariant has been calculated for the simple singularities,

$\mathcal{S}$		$\gamma_\alpha^{ea}(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$
A_k,	$k \geq 1$	$(k+\alpha)^2$
D_k,	$4 \leq k \leq 4+\sqrt{2}\cdot(2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
D_k,	$k \geq 4+\sqrt{2}\cdot(2+\alpha)$	$(k-2+\alpha)^2$
E_k,	$k = 6,7,8$	$\frac{(k+2\alpha)^2}{2}$

and for the topological singularity type M_m of an ordinary m-fold point

$$\gamma_\alpha^{es}(M_m) = 2\cdot(m-1+\alpha)^2.$$

Moreover, upper and lower bounds for the γ_0^{es}-invariant and for the γ_1^{es}-invariant of a topological singularity type given by a convenient semi-quasihomogeneous power series can be found there. They also show that

$$\tau_{ci}^{es}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \geq 3 \text{ odd}, \\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even}, \\ 1, & \text{if } m = 2. \end{cases}$$

These results show in particular that the upper bound for $\gamma_\alpha^*(\mathcal{S})$ in (1.3) may be attained, while it may as well be far from the actual value.

The proof of the following lemma can be found in [**Shu97**] Lemma 4.1.

Lemma 1.1
Let (C,z) be a reduced plane curve singularity given by $f \in \mathcal{O}_{\Sigma,z}$ and let $I \subseteq \mathfrak{m}_{\Sigma,z} \subset \mathcal{O}_{\Sigma,z}$ be an ideal containing the Tjurina ideal $I^{ea}(C,z)$. Then for any $g \in I$ we have

$$\dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/I) < \dim_{\mathbb{C}}\big(\mathcal{O}_{\Sigma,z}/(f,g)\big) = i(f,g).$$

1.3. Singularity Schemes

For a reduced curve $C \subset \Sigma$ we recall the definition of the zero-dimensional schemes $X^{es}(C)$ and $X^{ea}(C)$ from [**GLS00**]. They are defined by the ideal sheaves $\mathcal{J}_{X^{es}(C)/\Sigma}$ and $\mathcal{J}_{X^{ea}(C)/\Sigma}$ respectively, given by the stalks $\mathcal{J}_{X^{es}(C)/\Sigma,z} = I^{es}(f)$ and $\mathcal{J}_{X^{ea}(C)/\Sigma,z} = I^{ea}(f)$ respectively, where $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of C at z. We call $X^{es}(C)$ the *equisingularity scheme* of C and $X^{ea}(C)$ the *equianalytical singularity scheme* of C.

1.4. Equisingular Families

Given a divisor $D \in \mathrm{Div}(\Sigma)$ and topological or analytical singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$, we denote by $V = V_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ the locally closed subspace of $|D|_l$ of reduced curves in the linear system $|D|_l$ having precisely r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$. By $V^{irr} = V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ we denote the open subset of V of irreducible curves. If a type $\mathcal{S}$ occurs $k > 1$ times, we rather write $k\mathcal{S}$ than $\mathcal{S}, \overset{k}{\ldots}, \mathcal{S}$. We call these families of curves *equisingular families of curves*.

We say that V is *T-smooth* at $C \in V$ if the germ (V, C) is smooth of the (expected) dimension $\dim |D|_l - \deg\big(X^*(C)\big)$. By [**Los98**] Proposition 2.1 (see also [**GrK89**], [**GrL96**], [**GLS00**]) T-smoothness of V at C follows from the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)$, since the tangent space of V at C may be identified with $H^0\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)/H^0(\Sigma, \mathcal{O}_\Sigma)$.

2. The Main Results

In this section we give sufficient conditions for the T-smoothness of equisingular families of curves on certain surfaces with Picard number one, including the projective plane, general surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and general K3-surfaces –, on general products of curves, and on geometrically ruled surfaces. Since we do not have any general relation between the γ-invariants used in the conditions for our smoothness results and the invariants used in **[KeT02]** for the existence results, we are in general only able to say that the families produced here are T-smooth if they are non-empty. However, if you consider singularity types where the γ-invariant is known explicitely (see [**KeL05**]), you can check the conditions for non-emptyness in [**KeT02**], and you will find that they are in general fulfilled as well.

2.1. Surfaces with Picard Number One

Theorem 2.1
Let Σ be a surface such that $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$ with L ample, let $D = d \cdot L \in \mathrm{Div}(\Sigma)$, let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and let $K_\Sigma = \kappa \cdot L$. Suppose that $d \geq \max\{\kappa+1, -\kappa\}$ and

$$\sum_{i=1}^{r} \gamma_\alpha^*(\mathcal{S}_i) < \alpha \cdot (D - K_\Sigma)^2 = \alpha \cdot (d-\kappa)^2 \cdot L^2 \quad \textit{with } \alpha = \tfrac{1}{\max\{1, 1+\kappa\}}. \tag{2.1}$$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

Corollary 2.2
Let $d \geq 3$, $H \subset \mathbb{P}^2_{\mathbb{C}}$ be a line, and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma_1^*(\mathcal{S}_i) < (d+3)^2. \tag{2.2}$$

Then either $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth. □

As soon as for one of the singularities we have $\gamma_1^*(\mathcal{S}_i) > 4 \cdot \tau_{ci}^*(\mathcal{S}_i)$, e. g. simple singularities or ordinary multiple points which are not simple double points, then the strict inequality in (2.2) can be replaced by "$\leq$", which then is the same sufficient condition as in **[GLS01]** Theorem 1 (see also (1.2)).

In particular, $V^{irr}_{|dH|}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2 \cdot m_i^2 \leq (d+3)^2.$$

Moreover, this condition has the right assymptotics, as the examples in [**GLS01**] show. For further results in the plane case see [**Wah74a, GrK89, Lue87a, Lue87b, Shu87, Vas90, Shu91, Shu94, GrL96, Shu96, Shu97, GLS98a, Los98, GLS00, GLS01**].

A smooth complete intersection surface with Picard number one satisfies the assumptions of Theorem 2.1. Thus by the Noether–Lefschetz Theorem (cf. [**GrH85**]) the result applies in particular to general surfaces in $\mathbb{P}^3_{\mathbb{C}}$. Moreover, if in Theorem 2.1 we have $\kappa > 0$, i. e. $\alpha < 1$, then the strict inequality in Condition (2.1) may be replaced by "$\leq$", since in (3.9) the second inequality is strict, as is the second inequality in (3.10).

Corollary 2.3
Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 5$, let $H \subset \Sigma$ be a hyperplane section, and suppose that the Picard number of Σ is one. Let $d \geq n-3$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma^*_{\frac{1}{n-3}}(\mathcal{S}_i) \leq \frac{n}{n-3} \cdot (d-n+4)^2.$$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

In particular, $V^{irr}_{|dH|}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2 \cdot \left(m_i - \frac{n-4}{n-3} \right)^2 \leq \frac{n}{n-3} \cdot (d-n+4)^2,$$

which is better than the conditions derived from [**GLS97**]. The condition

$$r \leq \frac{n \cdot (n-3)}{(n-2)^2} \cdot (d-n+4)^2,$$

which gives the T-smoothness of $V_{|dH|}(rA_1)$ is weaker than the condition provided in [**ChS97**], but for $n = 5$ it reads $r \leq \frac{10}{9} \cdot (d-1)^2$ and comes still close to the sharp bound $\frac{5}{4} \cdot (d-1)^2$ provided there for odd d.

A general K3-surface has also Picard number one..

Corollary 2.4
Let Σ be a smooth K3-surface with $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$, L ample, and set $n = L^2$. Let $d \geq 1$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma^*_1(\mathcal{S}_i) < d^2 n.$$

Then either $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

The best previously known condition for T-smoothness on K3-surfaces

$$\sum_{i=1}^{r} \left(\tau^*_{ci}(\mathcal{S}_i) + 1 \right)^2 < d^2 n$$

is thus completely replaced.

2.2. Products of Curves

If $\Sigma = C_1 \times C_2$ is the product of two smooth projective curves, then for a general choice of C_1 and C_2 the Néron–Severi group will be generated by two fibres of the canonical projections, by abuse of notation also denoted by C_1 and C_2. If both curves are elliptic, then "general" just means that the two curves are non-isogenous. (Cf. **[Kei01]** Appendix G.)

Theorem 2.5
Let C_1 and C_2 be two smooth projective curves of genera g_1 and g_2 with $g_1 \geq g_2$, such that for $\Sigma = C_1 \times C_2$ the Néron–Severi group is $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.

Let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_1 + bC_2$ with $a \geq \max\{2-2g_2, 2g_2-1\}$ and $b \geq \max\{2-2g_1, 2g_1-1\}$, let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \ < \ \gamma \cdot (D - K_\Sigma)^2, \tag{2.3}$$

where the constant γ may be read off the following table with $A = \frac{a-2g_2+2}{b-2g_1+2}$

g_1	g_2	γ
$0, 1$	$0, 1$	$\frac{1}{4}$
≥ 2	$0, 1$	$\min\left\{\frac{1}{4g_1}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$
≥ 2	≥ 2	$\min\left\{\frac{1}{4g_1+4g_2-4}, \frac{A}{4\cdot(g_2-1)}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$

Then either $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

In particular, on a product of non-isogenous elliptic curves for nodal curves we reproduce the previous sufficient condition

$$r < \frac{ab}{2},$$

for T-smoothness of $V_{|aC_1+bC_2|}^{irr}(rA_1)$ from **[GLS97]**, while the previous general condition

$$\frac{\left(m_i^2 + 2m_i + 5\right)^2}{32} < ab$$

for T-smoothness of $V_{|aC_1+bC_2|}^{irr}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, has been replaced by

$$\sum_{i=1}^{r} 4 \cdot (m_i - 1)^2 < ab,$$

which is better from $m_i = 7$ on.

Note that the coneficient γ in Theorem 2.5 depends on the ratio of a and b unless both g_1 and g_2 are at most one. This means that in general an asymptotical behaviour can only be examined if the ratio remains unchanged.

2.3. Geometrically Ruled Surfaces

Let $\pi : \Sigma = \mathbb{P}_{\mathbb{C}}(\mathcal{E}) \to C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [**Har77**] V.2.8.1). The Néron–Severi group of Σ is $\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z}$ with intersection matrix $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where $F \cong \mathbb{P}^1_{\mathbb{C}}$ is a fibre of π, C_0 a section of π with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, $g = g(C)$ the genus of C, $\mathfrak{e} = \Lambda^2\mathcal{E}$ and $e = -\deg(\mathfrak{e}) \geq -g$. For the canonical divisor we have $K_\Sigma \sim_a -2C_0 + (2g - 2 - e) \cdot F$.

Theorem 2.6
Let $\pi : \Sigma \to C$ be a geometrically ruled surface with $g = g(C)$. Let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_0 + bF$ with $b > \max\left\{2g - 2 + \frac{ae}{2}, 2 - 2g + \frac{ae}{2}, ae\right\}$ and $a > 2$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \; < \; \gamma \cdot (D - K_\Sigma)^2, \tag{2.4}$$

where with $A = \frac{a+2}{b+2-2g-\frac{ae}{2}}$ the constant γ satisfies

$$\gamma = \begin{cases} \frac{1}{4}, & \text{if } g \in \{0, 1\}, \\ \min\left\{\frac{1}{4g}, \frac{1}{4 \cdot (g-1) \cdot A}\right\}, & \text{if } g \geq 2. \end{cases}$$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

The results of [**GLS97**] only applied to eight Hirzebruch surfaces and a few classes of fibrations over elliptic curves, while our results apply to all geometrically ruled surfaces. Moreover, the results are in general better, e. g. for the Hirzebruch surface $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ already the previous sufficient condition for T-smoothness of families of curves with r cusps and $b = 3a$ the condition

$$9r < 2a^2 + 8a$$

has been replaced by the slightly better condition

$$8r < 3a^2 + 8a + 4.$$

For ordinary multiple points the difference will become more significant. Even for families of nodal curves the new conditions would always be slightly better, but for those families T-smoothness is guaranteed anyway by [**Tan80**].

Note that, as for products of curves, the conefficient γ in Theorem 2.6 depends on the ratio of a and b unless g is at most one.

3. The Proofs

The following Lemma is the technical key to the above results. Using the method of Bogomolov unstable vector bundles, it gives us a "small" curve which passes through a "large" part of $X^*(C)$, provided that

$h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big) \neq 0$. We will then show that its existence contradicts (2.1), (2.3), or (2.4) respectively.

Lemma 3.1

Let Σ a smooth projective surface, and let $D \in \operatorname{Div}(\Sigma)$ and $X \subset \Sigma$ be a zero-dimensional scheme satisfying

(0) $D - K_\Sigma$ *is big and nef, and* $D + K_\Sigma$ *is nef,*

(1) $\exists\, C \in |D|_l$ *irreducible* : $X \subseteq X^*(C)$*,*

(2) $h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) > 0$*, and*

(3) $4 \cdot \deg(X_0) < (D - K_\Sigma)^2$ *for all local complete intersection schemes* $X_0 \subseteq X$*.*

Then there exists a curve $\Delta \subset \Sigma$ and a zero-dimensional local complete intersection scheme $X_0 \subseteq X \cap \Delta$ such that with the notation $\operatorname{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$ and[1] *$\varepsilon_i = \min\{\deg(X_i), i(C, \Delta; z_i) - \deg(X_i)\} \geq 1$ we have*

(a) $D.\Delta \geq \deg(X_0) + \sum_{i=1}^s \varepsilon_i$*,*

(b) $\deg\big(X_0\big) \geq \big(D - K_\Sigma - \Delta\big).\Delta$*,*

(c) $\big(D - K_\Sigma - 2 \cdot \Delta\big)^2 > 0$*, and*

(d) $\big(D - K_\Sigma - 2 \cdot \Delta\big).H > 0$ *for all* $H \in \operatorname{Div}(\Sigma)$ *ample.*

Moreover, it follows

$$0 \leq \tfrac{1}{4} \cdot (D - K_\Sigma)^2 - \deg\big(X_0\big) \leq \big(\tfrac{1}{2} \cdot (D - K_\Sigma) - \Delta\big)^2. \tag{3.1}$$

Proof: Choose $X_0 \subseteq X$ minimal such that still $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$. By Assumption (0) the divisor $D - K_\Sigma$ is big and nef, and thus $h^1\big(\Sigma, \mathcal{O}_\Sigma(D)\big) = 0$ by the Kawamata–Viehweg Vanishing Theorem. Hence X_0 cannot be empty.

Due to the Grothendieck-Serre duality we have

$$0 \neq H^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) \cong \operatorname{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D - K_\Sigma), \mathcal{O}_\Sigma\big).$$

That is, there is an extension

$$0 \to \mathcal{O}_\Sigma \to E \to \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma) \to 0. \tag{3.2}$$

The minimality of X_0 implies that E is locally free and X_0 is a local complete intersection scheme (cf. [**Laz97**] Proposition 3.9). Moreover, we have

$$c_1(E) = D - K_\Sigma \quad \text{and} \quad c_2(E) = \deg(X_0). \tag{3.3}$$

By Assumption (3) and (3.3) we have

$$c_1(E)^2 - 4 \cdot c_2(E) = (D - K_\Sigma)^2 - 4 \cdot \deg(X_0) > 0,$$

[1] Since $X_0 \subseteq X^*(C) \subseteq X^{ea}(C)$, Lemma 1.1 applies to the local ideals of X_0, that is for the points $z \in \operatorname{supp}(X_0)$ we have $i(C, \Delta; z) \geq \deg(X_0, z) + 1$.

and thus E is Bogomolov unstable (cf. **[Laz97]** Theorem 4.2). This, however, implies that there exists a divisor $\Delta_0 \in \mathrm{Div}(\Sigma)$ and a zero-dimensional scheme $Z \subset \Sigma$ such that

$$0 \to \mathcal{O}_\Sigma(\Delta_0) \to E \to \mathcal{J}_{Z/\Sigma}(D - K_\Sigma - \Delta_0) \to 0 \tag{3.4}$$

is exact, and such that

$$(2\Delta_0 - D + K_\Sigma)^2 \geq c_1(E)^2 - 4 \cdot c_2(E) > 0 \tag{3.5}$$

and

$$(2\Delta_0 - D + K_\Sigma).H > 0 \quad \text{for all ample} \quad H \in \mathrm{Div}(\Sigma). \tag{3.6}$$

Tensoring (3.4) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to the following exact sequence

$$0 \to \mathcal{O}_\Sigma \to E(-\Delta_0) \to \mathcal{J}_{Z/\Sigma}(D - K_\Sigma - 2\Delta_0) \to 0, \tag{3.7}$$

and we deduce $h^0\big(\Sigma, E(-\Delta_0)\big) \neq 0$.

Now tensoring (3.2) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to

$$0 \to \mathcal{O}_\Sigma(-\Delta_0) \to E(-\Delta_0) \to \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0) \to 0. \tag{3.8}$$

Let H be some ample divisor. By (3.6) and since $D - K_\Sigma$ is nef by (0):

$$-\Delta_0.H < -\tfrac{1}{2} \cdot (D - K_\Sigma).H \leq 0.$$

Hence $-\Delta_0$ cannot be effective, that is $H^0\big(\Sigma, \mathcal{O}_\Sigma(-\Delta_0)\big) = 0$. But the long exact cohomology sequence of (3.8) then implies

$$0 \neq H^0\big(\Sigma, E(-\Delta_0)\big) \hookrightarrow H^0\left(\Sigma, \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\right).$$

In particular we may choose a curve

$$\Delta \in \big|\mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\big|_l.$$

Thus (c) and (d) follow from (3.5) and (3.6). It remains to show (a) and (b).

We note that $C \in |D|_l$ is irreducible and that Δ cannot contain C as an irreducible component: otherwise applying (3.6) with some ample divisor H we would get the following contradiction, since $D + K_\Sigma$ is nef by (0),

$$0 \leq (\Delta - C).H < -\tfrac{1}{2} \cdot (D + K_\Sigma).H \leq 0.$$

Since $X_0 \subset C \cap \Delta$ the Theorem of Bézout implies (a):

$$D.\Delta = C.\Delta = \sum_{z \in C \cap \Delta} i(C, \Delta; z) \geq \sum_{i=1}^{s} \big(\deg(X_i) + \varepsilon_i\big) = \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i.$$

Finally, by (3.3) and (3.4) we get (b):

$$\deg(X_0) = c_2(E) = \Delta_0.(D - K_\Sigma - \Delta_0) + \deg(Z) \geq (D - K_\Sigma - \Delta).\Delta.$$

Equation (3.1) is just a reformulation of (b). □

Using this result we can now prove the main theorems.

Proof of Theorem 2.1: Let $C \in V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \operatorname{supp}(X_0)$ we have

$$4 \cdot \deg(X_z) \leq \frac{4}{(1+\alpha)^2} \cdot \gamma_\alpha^*(C,z) \leq \frac{1}{\alpha} \cdot \gamma_\alpha^*(C,z) \tag{3.9}$$

Lemma 3.1 applies and there is curve $\Delta \in |\delta \cdot L|_l$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1). That is, fixing the notation $l = \sqrt{L^2}$, $\operatorname{supp}(X_0) = \{z_1,\ldots,z_s\}$, $X_i = X_{0,z_i}$ and $\varepsilon_i = \min\{\deg(X_i), i(C,\Delta;z_i) - \deg(X_i)\} \geq 1$, we have

(a) $d \cdot \delta \cdot l^2 \geq \deg(X_0) + \sum_{i=1}^s \varepsilon_i$,

(b) $\deg(X_0) \geq (d-\kappa-\delta)\cdot\delta\cdot l^2$,

and

$$\delta \cdot l \leq \tfrac{(d-\kappa)\cdot l}{2} - \sqrt{\tfrac{(d-\kappa)^2\cdot l^2}{4} - \deg(X_0)} = \frac{2\cdot\deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2\cdot l^2 - 4\cdot\deg(X_0)}}.$$

But then together with (a) and (b) we deduce

$$\sum_{i=1}^s \varepsilon_i \leq \delta\cdot(\delta+\kappa)\cdot l^2 \leq \frac{1}{\alpha}\cdot\left(\frac{2\cdot\deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2\cdot l^2 - 4\cdot\deg(X_0)}}\right)^2. \tag{3.10}$$

Applying the Cauchy inequality this leads to

$$\sum_{i=1}^s \frac{\deg(X_i)^2}{\varepsilon_i} \geq \frac{\deg(X_0)^2}{\sum_{i=1}^s \varepsilon_i} \geq \frac{\alpha\cdot(d-\kappa)^2\cdot l^2}{4}\cdot\left(1+\sqrt{1-\tfrac{4\cdot\deg(X_0)}{(d-\kappa)^2\cdot l^2}}\right)^2.$$

Setting

$$\beta = \frac{\sum_{i=1}^s \frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot(d-\kappa)^2\cdot l^2},\qquad \gamma = \frac{\sum_{i=1}^s \frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot\deg(X_0)},$$

we thus have

$$\beta \geq \frac{1}{4}\cdot\left(1+\sqrt{1-\tfrac{4\beta}{\gamma}}\right)^2,$$

and hence, $\beta \geq \left(\frac{\gamma}{\gamma+1}\right)^2$. But then, applying the Cauchy inequality once more, we find

$$\alpha\cdot(d-\kappa)^2\cdot l^2 = \frac{\alpha\cdot\gamma}{\beta}\cdot\deg(X_0) \leq \alpha\cdot\left(\gamma+2+\frac{1}{\gamma}\right)\cdot\deg(X_0)$$

$$\leq \sum_{i=1}^s\left(\frac{\deg(X_i)^2}{\varepsilon_i} + 2\alpha\deg(X_i) + \alpha^2\varepsilon_i\right) \leq \sum_{i=1}^r \gamma_\alpha^*(\mathcal{S}_i),$$

in contradiction to Equation (2.1). □

Proof of Theorem 2.5: Let $C \in V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma,\mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0\subseteq X^*(C)$ any local complete intersection scheme and $z\in\operatorname{supp}(X)$ we have

$$\deg(X_z)\leq\gamma_0^*(C,z),$$

and since $\gamma\leq\frac{1}{4}$, Lemma 3.1 applies and there is curve $\Delta\sim_a\alpha\cdot C_1+\beta\cdot C_2$ and a local complete intersection scheme $X_0\subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1). That is, fixing the notation $\operatorname{supp}(X_0)=\{z_1,\ldots,z_s\}$, $X_i=X_{0,z_i}$ and $\varepsilon_i=\min\{\deg(X_i),i(C,\Delta;z_i)-\deg(X_i)\}\geq 1$, we have

(a) $a\beta+b\alpha\geq\deg(X_0)+\sum_{i=1}^s\varepsilon_i$,

(b) $\deg(X_0)\geq(a-2g_2+2-\alpha)\cdot\beta+(b-2g_1+2-\beta)\cdot\alpha$, and

(c) $0\leq\alpha\leq\frac{a-2g_2+2}{2}$ and $0\leq\beta\leq\frac{b-2g_1+2}{2}$.

The last inequalities follow from (d) in Lemma 3.1 replacing the ample divisor H by the nef divisors C_2 respectively C_1.

From (b) and (c) we deduce

$$\deg(X_0)\geq\frac{a-2g_2+2}{2}\cdot\beta+\frac{b-2g_1+2}{2}\cdot\alpha,$$

and thus

$$\deg(X_0)^2\geq 4\cdot\frac{a-2g_2+2}{2}\cdot\frac{b-2g_1+2}{2}\cdot\alpha\cdot\beta=\frac{(D-K_\Sigma)^2}{2}\cdot\alpha\cdot\beta. \tag{3.11}$$

Considering now (a) and (b) we get

$$0<\sum_{i=1}^s\varepsilon_i\leq\Delta.(\Delta+K_\Sigma)=2\alpha\beta+(2g_1-2)\cdot\alpha+(2g_2-2)\cdot\beta\leq\frac{\alpha\beta}{2\gamma},$$

where the last inequality holds only if $\alpha\neq 0\neq\beta$. In particular, we see $\alpha\neq 0$ if $g_2\leq 1$ and $\beta\neq 0$ if $g_1\leq 1$. But this together with (3.11) gives

$$\sum_{i=1}^s\varepsilon_i\leq\frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

If $\alpha=0$, then from (a) and (b) we deduce again

$$0<\sum_{i=1}^s\varepsilon_i\leq(2g_2-2)\cdot\beta\leq\frac{4\cdot(g_1-1)}{A}\cdot\frac{\deg(X_0)^2}{(D-K_\Sigma)^2}\leq\frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2},$$

and similarly, if $\beta=0$,

$$0<\sum_{i=1}^s\varepsilon_i\leq(2g_1-2)\cdot\alpha\leq 4\cdot(g_1-1)\cdot A\cdot\frac{\deg(X_0)^2}{(D-K_\Sigma)^2}\leq\frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma\cdot(D-K_\Sigma)^2\leq\frac{\deg(X_0)^2}{\sum_{i=1}^s\varepsilon_i}\leq\sum_{i=1}^s\frac{\deg(X_i)^2}{\varepsilon_i}\leq\sum_{i=1}^r\gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (2.3). □

Proof of Theorem 2.6: Let $C \in V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \operatorname{supp}(X)$ we have

$$\deg(X_z) \leq \gamma_0^*(C, z),$$

and since $\gamma \leq \frac{1}{4}$, Lemma 3.1 applies and there is curve $\Delta \sim_a \alpha \cdot C_0 + \beta \cdot F$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1).

Remember that the Néron–Severi group of Σ is generated by a section C_0 of π and a fibre F with intersection pairing given by $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $K_\Sigma \sim_a -2C_0 + (2g - 2 - e) \cdot F$. Note that

$$\alpha \geq 0 \quad \text{ and } \quad \beta' := \beta - \tfrac{e}{2}\alpha \geq 0.$$

If we set $b' = b - \frac{ae}{2}$, $\kappa_1 = a + 2$ and $\kappa_2 = b + 2 - 2g - \frac{ae}{2} = b' + 2 - 2g$, we get

$$(D - K_\Sigma)^2 = -e \cdot (a+2)^2 + 2 \cdot (a+2) \cdot (b + 2 + e - 2g) = 2 \cdot \kappa_1 \cdot \kappa_2. \tag{3.12}$$

Now fixing the notation $\operatorname{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$, and finally $\varepsilon_i = \min\{\deg(X_i), i(C, \Delta; z_i) - \deg(X_i)\} \geq 1$, the conditions on Δ and $\deg(X_0)$ take the form

(a) $a\beta' + b'\alpha \geq \deg(X_0) + \sum_{i=1}^s \varepsilon_i$,

(b) $\deg(X_0) \geq \kappa_1 \cdot \beta' + \kappa_2 \cdot \alpha - 2\alpha\beta'$, and

(c) $0 \leq \alpha \leq \frac{\kappa_1}{2}$ and $0 \leq \beta' \leq \frac{\kappa_2}{2}$.

The last inequalities follow from (d) in Lemma 3.1 replacing the ample divisor H by the nef divisors F respectively $C_0 + \frac{e}{2} \cdot F$.

From (b) and (c) we deduce

$$\deg(X_0) \geq \frac{\kappa_1}{2} \cdot \beta' + \frac{\kappa_2}{2} \cdot \alpha,$$

and thus, taking (3.12) into account,

$$\deg(X_0)^2 \geq 4 \cdot \frac{\kappa_1}{2} \cdot \frac{\kappa_2}{2} \cdot \alpha \cdot \beta' = \frac{(D - K_\Sigma)^2}{2} \cdot \alpha \cdot \beta'. \tag{3.13}$$

Considering now (a) and (b) we get

$$0 < \sum_{i=1}^s \varepsilon_i \leq \Delta.(\Delta + K_\Sigma) = 2\alpha\beta' + (2g-2) \cdot \alpha - 2\beta' \leq \frac{\alpha\beta'}{2\gamma},$$

where the last inequality holds if $\beta' \neq 0$. We see, in particular, that $\beta' \neq 0$ if $g \leq 1$. But this together with (3.13) gives for $\beta' \neq 0$

$$\sum_{i=1}^{s} \varepsilon_i \leq \frac{\deg(X_0)^2}{\gamma \cdot (D - K_\Sigma)^2}.$$

If $\beta' = 0$, then we deduce from (a) and (b)

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g-2) \cdot \alpha \leq 4 \cdot (g-1) \cdot A \cdot \frac{\deg(X_0)^2}{(D - K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma \cdot (D - K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma \cdot (D - K_\Sigma)^2 \leq \frac{\deg(X_0)^2}{\sum_{i=1}^{s} \varepsilon_i} \leq \sum_{i=1}^{s} \frac{\deg(X_i)^2}{\varepsilon_i} \leq \sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (2.4). $\square$

PAPER II

A new Invariant for Plane Curve Singularities

Abstract: In [**GLS01**] the authors gave a general sufficient numerical condition for the T-smoothness (smoothness and expected dimension) of equisingular families of plane curves. This condition involves a new invariant γ^* for plane curve singularities, and it is conjectured to be asymptotically proper. In [**Kei05b**], similar sufficient numerical conditions are obtained for the T-smoothness of equisingular families on various classes surfaces. These conditions involve a series of invariants γ^*_α, $0 \leq \alpha \leq 1$, with $\gamma^*_1 = \gamma^*$. In the present paper we compute (respectively give bounds for) these invariants for semiquasihomogeneous singularities.

This paper is published as [**KeL05**] Thomas Keilen and Christoph Lossen, *A new invariant for plane curve singularities*, Rend. Semin. Mat. Torino **63** (2005), no. 1, 15–42.

When studying numerical conditions for the T-smoothness of equisingular families of curves, new invariants of plane curve singularities $V(f) \subset (\mathbb{C}^2, 0)$ turn up. These invariants are defined as the maximum of a function depending on the codimension of complete intersection ideals containing the Tjurina ideal, respectively the equisingularity ideal, of f, and on the intersection multiplicity of f with elements of the complete intersection ideals. In Section 1 we will define these invariants, and we will calculate them for several classes of singularities, the main results being Proposition 1.11, Proposition 1.12 and Proposition 1.13. It is the upper bound in Lemma 1.8 which ensures that the conditions for T-smoothness with these new conditions (see [**GLS00**], [**GLS01**], [**Kei05b**]) improve than the previously known ones (see [**GLS97**]). In the remaining sections we introduce some notation and we gather some necessary, though mainly well-known technical results used in the proofs of Section 1.

We should like to point out that the definition of the invariant γ^*_1 below is a modification of the invariant "γ^*" defined in [**GLS01**], and it is always bound from above by the latter. Moreover, the latter can be replaced by it in the conditions of [**GLS01**] Proposition 2.2.

Notation

Throughout this paper $R = \mathbb{C}\{x, y\}$ will be the ring of convergent power series in the variables x and y, and $\mathfrak{m} = \langle x, y \rangle \lhd R$ will be its maximal ideal.

1. The γ_α^*-Invariants

For the definition of the γ_α^*-invariants the Tjurina ideal, respectively the equisingularity ideal in the sense of [**Wah74b**], play an essential role. For the convenience of the reader we recall their definitions.

Definition 1.1
Let $f \in \mathfrak{m}$ be a reduced power series. The *Tjurina ideal* of f is defined as

$$I^{ea}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle,$$

and the *equisingularity ideal* of f is defined as

$$I^{es}(f) = \left\{ g \in R \;\middle|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \right\} \supseteq I^{ea}(f).$$

Their codimensions

$$\tau(f) = \dim_{\mathbb{C}} R/I^{ea}(f),$$

respectively

$$\tau^{es}(f) = \dim_{\mathbb{C}} R/I^{es}(f),$$

are analytical, respectively topological, invariants of the singularity type defined by f. Note that $\tau^{es}(f)$ is the codimension of the μ-constant stratum in the equisingular deformation of the plane curve singularity defined by f. It can be computed in terms of multiplicities of the strict transform of f at essential infinitely near points in the resolution tree of $\big(V(f), 0\big)$ (cf. [**Shu91**]).

Definition 1.2
Let $f \in \mathfrak{m}$ be a reduced power series, and let $0 \leq \alpha \leq 1$ be a rational number. If I is a zero-dimensional ideal in R with $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ and $g \in I$, we define

$$\lambda_\alpha(f; I, g) := \frac{\big(\alpha \cdot i(f,g) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)},$$

and

$$\gamma_\alpha(f; I) := \max\left\{ (1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\middle|\; g \in I, i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I) \right\},$$

where $i(f,g)$ denotes the intersection multiplicity of f and g. Note that, by Lemma 1.3, $i(f,g) > \dim_{\mathbb{C}}(R/I)$ for all $g \in I$. Thus $\gamma_\alpha(f; I)$ is a well-defined positive rational number.
We then set

$$\gamma_\alpha^{ea}(f) := \max\left\{ 0,\ \gamma_\alpha(f; I) \;\middle|\; I \supseteq I^{ea}(f) \text{ is a complete intersection ideal} \right\}$$

and

$$\gamma_\alpha^{es}(f) := \max\left\{ 0,\ \gamma_\alpha(f; I) \;\middle|\; I \supseteq I^{es}(f) \text{ is a complete intersection ideal} \right\}$$

Note that if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $I^{ea}(f) = I^{es}(f) = R$ and there is no zero-dimensional complete intersection ideal containing any of those two, hence $\gamma_\alpha^{ea}(f) = \gamma_\alpha^{es}(f) = 0$.

Lemma 1.3
Let $f \in \mathfrak{m}^2$ be reduced, and let I be an ideal such that $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$.
Then, for any $g \in I$, we have

$$\dim_{\mathbb{C}}(R/I) < \dim_{\mathbb{C}}\big(R/\langle f, g\rangle\big) = i(f, g).$$

Proof: Cf. **[Shu97]** Lemma 4.1; the idea is mainly to show that not both derivatives of f can belong to $\langle f, g\rangle$. $\square$

Up to embedded isomorphism the Tjurina ideal only depends on the analytical type of the singularity. More precisely, if $f \in R$ any power series, $u \in R$ a unit and $\phi : R \to R$ an isomorphism, then $I^{ea}(u \cdot f \circ \phi) = \{g \circ \phi \mid g \in I^{ea}(f)\}$. Thus the following definition makes sense.

Definition 1.4
Let $\mathcal{S}$ be an analytical, respectively topological, singularity type, and let $f \in R$ be a representative of $\mathcal{S}$. We then define

$$\gamma_\alpha^{ea}(\mathcal{S}) := \gamma_\alpha^{ea}(f),$$

respectively

$$\gamma_\alpha^{es}(\mathcal{S}) := \max\{\gamma_\alpha^{es}(g) \mid g \text{ is a representative of } \mathcal{S}\}.$$

Since $i(f, g) > \dim_{\mathbb{C}}(R/I)$ in the above situation, we deduce the following lemma.

Lemma 1.5
Let $f \in \mathfrak{m}^2$ be reduced, $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ be a zero-dimensional ideal, and $0 \leq \alpha < \beta \leq 1$, then $\gamma_\alpha(f; I) < \gamma_\beta(f; I)$.

In particular, for any analytical, respectively topological, singularity type

$$\gamma_\alpha^{ea}(\mathcal{S}) < \gamma_\beta^{ea}(\mathcal{S}) \qquad \textit{respectively} \qquad \gamma_\alpha^{es}(\mathcal{S}) < \gamma_\beta^{es}(\mathcal{S}).$$

For reasons of comparison let us also recall the definition of τ_{ci}^{ea}, τ_{ci}^{es}, κ and δ.

Definition 1.6
For $f \in R$ we define

$$\tau_{ci}^{ea}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{ea}(f) \text{ a complete intersection}\},$$

and

$$\tau_{ci}^{es}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{es}(f) \text{ a complete intersection}\}.$$

Again, for analytically equivalent singularities the values coincide, so that for an analytical singularity type $\mathcal{S}$, choosing some representative $f \in R$, we may define

$$\tau_{ci}^{ea}(\mathcal{S}) := \tau_{ci}(f).$$

For a topological singularity type we set

$$\tau_{ci}^{es}(\mathcal{S}) := \max\{\tau_{ci}^{es}(g) \mid g \text{ a representative of } \mathcal{S}\}.$$

Note that obviously

$$\tau_{ci}^{ea}(\mathcal{S}) \leq \tau(\mathcal{S}) \quad \text{and} \quad \tau_{ci}^{es}(\mathcal{S}) \leq \tau^{es}(\mathcal{S}),$$

where $\tau(\mathcal{S})$ is the Tjurina number of $\mathcal{S}$ and $\tau^{es}(\mathcal{S})$ is as defined in Definition 1.1.

Definition 1.7
For $f \in R$ and $\mathcal{O} = R/\langle f\rangle$, we define the δ-invariant

$$\delta(f) = \dim_{\mathbb{C}} \widetilde{\mathcal{O}}/\mathcal{O}$$

where $\mathcal{O} \subset \widetilde{\mathcal{O}}$ is the normalisation of $\mathcal{O}$, and the κ-invariant

$$\kappa(f) = i\left(f, \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial x}\right),$$

where $(\alpha : \beta) \in \mathbb{P}^1_{\mathbb{C}}$ is generic.

δ and κ are topological (thus also analytical) invariants of the singularity defined by f so that for the topological, respectively analytical, singularity type $\mathcal{S}$ given by f we can set

$$\delta(\mathcal{S}) = \delta(f) \quad \text{and} \quad \kappa(\mathcal{S}) = \kappa(f).$$

Throughout this article we will sometimes treat topological and analytical singularities at the same time. Whenever we do so, we will write $I^*(f)$ for $I^{ea}(f)$ respectively for $I^{ea}(f)$, and analogously we will use the notation γ_α^*, τ_{ci}^* and τ^*.

The following lemma is again obvious from the definition of $\gamma_\alpha(f; I)$, once we take into account that $\kappa(f) = i(f, g)$ for a generic element $g \in I^{ea}(f)$ of f and that for a fixed value of $d = \dim_{\mathbb{C}}(R/I)$ the function $i \mapsto \frac{(\alpha i+(1-\alpha)\cdot d)^2}{i-d}$ takes its maximum on $[d+1, 2d]$ for the minimal possible value $i = d+1$.

Lemma 1.8
Let $f \in \mathfrak{m}^2$ be reduced, and let I be an ideal in R such that $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$.

Then

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \gamma_\alpha(f; I) \leq \big(\dim_{\mathbb{C}}(R/I) + \alpha\big)^2.$$

Moreover, if $\kappa(f) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$, then

$$\gamma_\alpha(f; I) \geq \frac{\big(\alpha \cdot \kappa(f) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{\kappa(f) - \dim_{\mathbb{C}}(R/I)}.$$

In particular, for any analytical, respectively topological, singularity type $\mathcal{S}$

$$(1+\alpha)^2 \cdot \tau_{ci}^*(\mathcal{S}) \leq \gamma_\alpha^*(\mathcal{S}) \leq \big(\tau_{ci}^*(\mathcal{S}) + \alpha\big)^2,$$

and if $\kappa(\mathcal{S}) \leq 2 \cdot \tau_{ci}^*(\mathcal{S})$*, then*

$$\gamma_\alpha^*(\mathcal{S}) \geq \frac{\big(\alpha \cdot \kappa(\mathcal{S}) + (1-\alpha)\cdot \tau_{ci}^*(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau_{ci}^*(\mathcal{S})}.$$

In order to make the conditions for T-smoothness in **[Kei05b]** as sharp as possible, it is useful to know under which circumstances the term $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ involved in the definition of $\gamma_\alpha^*(f)$ is actually exceeded.

Lemma 1.9
If $\mathcal{S}$ is a topological or analytical singularity type such that $\kappa(\mathcal{S}) < 2 \cdot \tau_{ci}^*(\mathcal{S})$, then

$$(1+\alpha)^2 \cdot \tau_{ci}^*(\mathcal{S}) < \gamma_\alpha^*(\mathcal{S}).$$

This is in particular the case, if $\mathcal{S} \neq A_1$ and $\tau_{ci}^*(\mathcal{S}) = \tau^*(\mathcal{S})$, i. e. if the Tjurina ideal, respectively the equisingularity ideal, of some representative is a complete intersection.

Proof: Lemma 1.8 gives

$$\gamma_\alpha^*(\mathcal{S}) \geq \frac{\big(\alpha \cdot \kappa(\mathcal{S}) + (1-\alpha)\cdot \tau_{ci}^*(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau_{ci}^*(\mathcal{S})}.$$

If we consider the right-hand side as a function in $\kappa(\mathcal{S})$, it is strictly decreasing on the interval $[0, 2\cdot\tau_{ci}^*(\mathcal{S})]$ and takes its minimum thus at $2\cdot\tau_{ci}^*(\mathcal{S})$. By the assumption on $\kappa(\mathcal{S})$ we, therefore, get

$$\gamma_\alpha^*(\mathcal{S}) > (1+\alpha)^2 \cdot \tau_{ci}^*(\mathcal{S}).$$

Suppose now that $\tau_{ci}^*(\mathcal{S}) = \tau^*(\mathcal{S})$ and $\mathcal{S} \neq A_1$. By Lemma 1.10 we know $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S}) \leq \tau(\mathcal{S})$. On the other hand we have $\kappa(\mathcal{S}) \leq 2\cdot\delta(\mathcal{S})$ (see **[GLS05]**). Therefore, $\kappa(\mathcal{S}) < 2\cdot\tau_{ci}^*(\mathcal{S})$. □

Lemma 1.10
If $\mathcal{S} \neq A_1$ is any analytical or topological singularity type, then $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S})$.

Proof: If (C,z) is a representative of $\mathcal{S}$ and if $\mathcal{T}^*(C,z)$ is the essential subtree of the complete embedded resolution tree of (C,z), then

$$\delta(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C)\cdot(\mathrm{mult}_p(C)-1)}{2}$$

and

$$\tau^{es}(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C)\cdot(\mathrm{mult}_p(C)+1)}{2} - \#\text{ free points in } \mathcal{T}^*(C,z) - 1,$$

where $\mathrm{mult}_p(C)$ denotes the multiplicity of the strict transform of C at p (see **[GLS05]**). Setting $\varepsilon_p = 0$ if p is satellite, $\varepsilon_p = 1$ if $p \neq z$ is free, and $\varepsilon_z = 2$, then

$\mathrm{mult}_p(C) \geq \varepsilon_p$ and therefore

$$\tau^{es}(\mathcal{S}) = \delta(\mathcal{S}) + \sum_{p \in \mathcal{T}^*(C,z)} \big(\mathrm{mult}_p(C) - \varepsilon_p\big) \geq \delta(\mathcal{S}).$$

Moreover, we have equality if and only if $\mathrm{mult}_z(C) = 2$, $\mathrm{mult}_p(C) = 1$ for all $p \neq z$ and there is no satellite point, but this implies that $\mathcal{S} = A_1$. $\square$

For some classes of singularities we can calculate the γ_α^*-invariant concretely, and for some others we can at least give an upper bound, which in general is much better than the one derived from Lemma 1.8. We restrict our attention to singularities having a convenient semi-quasihomogeneous representative $f \in R$ (see Definition 4.1). Throughout the following proofs we will frequently make use of monomial orderings, see Section 2.

Proposition 1.11 ((Simple Singularities))
Let α be a rational number with $0 \leq \alpha \leq 1$. Then we obtain the following values for $\gamma_\alpha^{es}(\mathcal{S}) = \gamma_\alpha^{ea}(\mathcal{S})$, where $\mathcal{S}$ is a simple singularity type.

$\mathcal{S}$		$\gamma_\alpha^{ea}(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$
A_k,	$k \geq 1$	$(k+\alpha)^2$
D_k,	$4 \leq k \leq 4 + \sqrt{2} \cdot (2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
D_k,	$k \geq 4 + \sqrt{2} \cdot (2+\alpha)$	$(k-2+\alpha)^2$
E_k,	$k = 6, 7, 8$	$\frac{(k+2\alpha)^2}{2}$

Proof: Let $\mathcal{S}_k$ be one of the simple singularity types A_k, D_k or E_k, and let $f \in R$ be a representative of $\mathcal{S}_k$. Note that the Tjurina ideal $I^{ea}(f)$ and the equisingularity ideal $I^{es}(f)$ coincide, and hence so do the γ_α^*-invariants, i. e.

$$\gamma_\alpha^{ea}(\mathcal{S}_k) = \gamma_\alpha^{es}(\mathcal{S}_k).$$

Moreover, in the considered cases the Tjurina ideal is indeed a complete intersection ideal with $\dim_{\mathbb{C}}\big(R/I^{ea}(f)\big) = k$, so that in particular the given values are upper bounds for $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ for any complete intersection ideal I containing the Tjurina ideal. By Lemma 1.8 we know

$$\frac{(\alpha \cdot \kappa(\mathcal{S}_k) + (1-\alpha) \cdot k)^2}{\kappa(\mathcal{S}_k) - k} \leq \gamma_\alpha(\mathcal{S}_k) \leq (k+\alpha)^2.$$

Note that $\kappa(A_k) = k+1$, $\kappa(D_k) = k+2$ and $\kappa(E_k) = k+2$, which in particular gives the result for $\mathcal{S}_k = A_k$. Moreover, it shows that for $\mathcal{S}_k = D_k$ or $\mathcal{S}_k = E_k$ we have

$$\gamma_\alpha(\mathcal{S}_k) \geq \frac{(k+2\alpha)^2}{2}.$$

If we fix a complete intersection ideal I with $I^{ea}(f) \subseteq I$, then

$$\lambda_\alpha(f;I,g) = \frac{\big(\alpha\cdot i(f,g) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g)-\dim_{\mathbb{C}}(R/I)},$$

with $g \in I$ such that $i(f,g) \leq 2\cdot\dim_{\mathbb{C}}(R/I)$, considered as a function in $i(f,g)$ is maximal, when $i(f,g)$ is minimal. If $i(f,g)-\dim_{\mathbb{C}}(R/I) \geq 2$, then

$$\lambda_\alpha(f;I,g) \leq \frac{(k+2\alpha)^2}{2}.$$

It therefore remains to consider the case where

$$i(f,g) - \dim_{\mathbb{C}}(R/I) = 1 \tag{1.1}$$

for some I and some $g \in I$, and to maximise the possible $\dim_{\mathbb{C}}(R/I)$.

We claim that for $\mathcal{S}_k = D_k$ with $f = x^2y - y^{k-1}$ as representative, $\dim_{\mathbb{C}}(R/I) \leq k-2$, and thus $I = \langle x, y^{k-2}\rangle$ and $g = x$ are suitable with

$$\lambda_\alpha(f;I,x) = (k-2+\alpha)^2,$$

which is greater than $\frac{(k+2\alpha)^2}{2}$ if and only if $k \geq 4+\sqrt{2}\cdot(2+\alpha)$. Suppose, therefore, $\dim_{\mathbb{C}}(R/I) = k-1$. Then $y^{k-1}, x^3 \in I^{ea}(f) = \langle xy, x^2-(k-1)\cdot y^{k-2}\rangle \subset I$, the leading ideal $L_{<_{ls}}\big(I^{ea}(f)\big) = \langle x^3, xy, y^{k-2}\rangle \subset L_{<_{ls}}(I)$, and since by Proposition 2.3 $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_{<_{ls}}(I)\big)$, either $L_{<_{ls}}(I) = \langle x^3, xy, y^{k-3}\rangle$ or $L_{<_{ls}}(I) = \langle x^2, xy, y^{k-2}\rangle$. In the first case there is a power series $g \in I$ such that $g \equiv y^{k-3} + ax + bx^2 \pmod{I}$, and hence $I \ni yg \equiv y^{k-2} \pmod{I}$, i. e. $y^{k-2} \in I$. But then $x^2 \in I$ and $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case, similarly, there is a $g \in I$ such that $g \equiv x^2 \pmod{I}$, and hence $x^2 \in I$ which in turn implies that $y^{k-2} \in I$. Thus $I = \langle x^2, xy, y^{k-2}\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ which by Remark 3.7 contradicts the fact that I is a complete intersection.

If $\mathcal{S}_k = E_6$, then $f = x^3 - y^4$ is a representative and $I^{ea}(f) = \langle x^2, y^3\rangle$. Suppose that $\dim_{\mathbb{C}}(R/I) = k-1 = 5$, then $L_{<_{ds}}(I) = \langle x^2, y^3, xy^2\rangle$ and $H^0_{R/I} = H^0_{R/L_{<_{ds}}(I)}$, in contradiction to Lemma 3.6, since $H^0_{R/L_{<_{ds}}(I)}(2) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(3) = 0$. Thus $\dim_{\mathbb{C}}(R/I) \leq 4$ and $\lambda_\alpha(f;I,g) \leq (4+\alpha)^2 \leq \frac{(6+2\alpha)^2}{2}$.

If $\mathcal{S}_k = E_7$, then $f = x^3 - xy^3$ is a representative and $I^{ea}(f) = \langle 3x^2-y^3, xy^2\rangle \ni x^3, y^5$. If $\dim_{\mathbb{C}}(R/I) \leq 4$, then $\lambda_\alpha(f;I,g) \leq (4+\alpha)^2 \leq \frac{(7+2\alpha)^2}{2}$, and we are done. It thus remains to exclude the cases where $\dim_{\mathbb{C}}(R/I) \in \{5,6\}$. For this we note first that if there is a $g \in I$ such that $L_{<_{ls}}(g) = y^2$, then

$$g \equiv y^2 + ax + bx^2 + cxy + dx^2y \pmod{I}, \tag{1.2}$$

and therefore $y^2g \equiv y^4 \pmod{I}$, which implies $y^4 \in I$ and hence $x^2y \in I$. Analogously, if there is a $g \in I$ such that $L_{<_{ls}}(g) = x^2y$, then $g \equiv x^2y \pmod{I}$ and again $x^2y, y^4 \in I$. Suppose now that $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^2, x^3\rangle$ or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2y, x^3\rangle$. In both cases we thus have $x^2y, y^4 \in I$. However, in the first case then $x^2y \in L_{<_{ls}}(I)$, in contradiction to the assumption. While in the second case we find $I = \langle xy^2, x^2y, 3x^2 - y^3\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts the fact that I is a complete intersection by Lemma 3.7. Suppose, therefore, that $\dim_{\mathbb{C}}(R/I) = 5$. Then $L_{<_{ls}}(I) = \langle y^2, x^2y, x^3\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy, x^3\rangle$. In the first case, we know already that $y^4, x^2y \in I$. Looking once more on (1.2) we consider the cases $a = 0$ and $a \neq 0$. If $a = 0$, then $yg \equiv y^3 \pmod{I}$, and thus $y^3 \in I$, which in turn implies $x^2 \in I$. Similarly, if $a \neq 0$, then $xg \equiv ax^2 \pmod{I}$ implies $x^2 \in I$. But then also $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case there is a $g \in I$ such that $g \equiv x^2 + ax^2y \pmod{I}$, and thus $yg \equiv x^2y \in I$. But then also $x^2 \in I$ and $y^3 \in I$, so that $I = \langle y^3, xy^2, x^2\rangle$. However, $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts again the fact that I is a complete intersection. Finally in the third case there is a $g \in I$ with $g \equiv xy + ax^2 + bx^2y \pmod{I}$, and thus $xg \equiv x^2y \pmod{I}$ implies $x^2y \in I$ and then $xy + ax^2 \in I$. Therefore, $I = \langle xy + ax^2, 3x^2 - y^3\rangle$, and for for $h \in I$ and for generic $b, c \in \mathbb{C}$ we have $i(f,h) \geq i(x,h) + i\big(x^2 - y^3, b\cdot(xy + ax^2) + c\cdot(3x^2 - y^3)\big) \geq 3 + 5 = 8$, in contradiction to (1.1).

Finally, if $\mathcal{S}_k = E_8$ with representative $f = x^3 - y^5$ and $I^{ea}(f) = \langle x^2, y^4\rangle$, we get for $\dim_{\mathbb{C}}(R/I) \leq 5$ that $\lambda_\alpha(f; I, g) \leq (5+\alpha)^2 \leq \frac{(8+2\alpha)^2}{2}$. It therefore remains to exclude the cases $\dim_{\mathbb{C}}(R/I) \in \{6, 7\}$. If $\dim_{\mathbb{C}}(R/I) = 7$ then $L_{<_{ds}}(I) = \langle x^2, y^4, xy^3\rangle$. But then $H^0_{R/L_{<_{ds}}(I)}(3) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(4) = 0$ are in contradiction to Lemma 3.6. And if $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^3, x^2\rangle$ or $L_{<_{ls}}(I) = \langle y^4, xy^2, x^2\rangle$. In the first case there is some $g \in I$ such that $g \equiv y^3 + ax + bxy + cxy^2 + dxy^3 \pmod{I}$, and thus $xg \equiv xy^3 \pmod{I}$ and $xy^3 \in I$. But then $yg \equiv axy + bxy^2 \pmod{I}$ and hence $axy + bxy^2 \in I$. Since neither $xy \in L_{<_{ls}}(I)$ nor $xy^2 \in L_{<_{ls}}(I)$, we must have $a = 0 = b$. Therefore, $g \equiv y^3 + cxy^2 \pmod{I}$ and $I = \langle x^2, y^3 + cxy^2\rangle$, which for $h \in I$ and $a, b \in \mathbb{C}$ generic gives $i(f,g) \geq i\big(x^3 - y^4, ax^2 + b\cdot(y^3 + cxy^2)\big) \geq 8$, in contradiction to (1.1). In the second case, there is $g \in I$ such that $g \equiv xy^2 + axy^3 \pmod{I}$, therefore $yg \equiv xy^3 \pmod{I}$ and $xy^3 \in I$. But then $xy^2 \in I$ and $I = \langle y^4, xy^2, x^2\rangle$. This, however, is not a complete intersection, since $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$, in contradiction to the assumption.

This finishes the proof. □

Proposition 1.12 ((Ordinary Multiple Points))
Let α be a rational number with $0 \leq \alpha \leq 1$, and let M_k denote the topological singularity type of an ordinary k-fold point with $k \geq 3$. Then

$$\gamma_\alpha^{es}(M_k) = 2\cdot(k - 1 + \alpha)^2.$$

In particular

$$\gamma_\alpha^{es}(M_k) > (1+\alpha)^2 \cdot \tau_{ci}^{es}(M_k).$$

Proof: Note that for any representative f of M_k we have

$$I^{es}(f) = I^{ea}(f) + \mathfrak{m}^k = \left\langle \frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y} \right\rangle + \mathfrak{m}^k,$$

where f_k is the homogeneous part of degree k of f, so that we may assume f to be homogeneous of degree k.

If I is a complete intersection ideal with $\mathfrak{m}^k \subset I^{es}(f) \subseteq I$, then by Lemma 3.10

$$\dim_{\mathbb{C}}(R/I) \leq \big(k - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

We note moreover that for any $g \in I$

$$i(f,g) \geq \mathrm{mult}(f) \cdot \mathrm{mult}(g) \geq k \cdot \mathrm{mult}(I),$$

and that for a fixed I we may attain an upper bound for $\lambda_\alpha(f;I,g)$ by replacing $i(f,g)$ by a lower bound for $i(f,g)$.

Hence, if $\mathrm{mult}(I) \geq 2$, we have

$$\lambda_\alpha(f;I,g) \leq \frac{\big(k - (1-\alpha)\cdot(\mathrm{mult}(I)-1)\big)^2 \cdot \mathrm{mult}(I)^2}{\mathrm{mult}(I) \cdot \big(\mathrm{mult}(I)-1\big)} \leq 2 \cdot (k-1+\alpha)^2, \quad \textbf{(1.3)}$$

while $\dim_{\mathbb{C}}(R/I) \leq k-1$ for $\mathrm{mult}(I) = 1$ and the above inequality (1.3) is still satisfied. To see $\dim_{\mathbb{C}}(R/I) \leq k-1$ for $\mathrm{mult}(I) = 1$ note that the ideal I contains an element g of order 1 with $g_1 = ax + by$ as homogeneous part of degree 1 and the partial derivatives of f; applying a linear change of coordinates we may assume $g_1 = x$ and $f = \prod_{i=1}^k (x - a_i y)$ with pairwise different a_i, and we may consider the negative degree lexicographical monomial ordering $>$ giving preference to y; if some $a_i = 0$, then $L_>\big(\frac{\partial f}{\partial x}\big) = y^{k-1}$, while otherwise $L_>\big(\frac{\partial f}{\partial y}\big) = y^{k-1}$, so that in any case $\langle x, y^{k-1}\rangle \subseteq L_>(I)$, and by Proposition 2.3 therefore $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_>(I)\big) \leq \dim_{\mathbb{C}}(R/\langle x, y^{k-1}\rangle) = k-1$.

Equation (1.3) together with Lemma 3.10 shows

$$\gamma_\alpha^{es}(M_k) \leq 2 \cdot (k-1+\alpha)^2.$$

On the other hand, considering the representative $f = x^k - y^k$, we have

$$I^{es}(f) = \langle x^{k-1}, y^{k-1}, x^a y^b \mid a+b=k \rangle,$$

and $I = \langle y^{k-1}, x^2 \rangle$ is a complete intersection ideal containing $I^{es}(f)$. Moreover, $i\big(f,x^2\big) = 2k$, $\dim_{\mathbb{C}}(R/I) = 2\cdot(k-1)$, thus

$$\gamma_\alpha^{es}(M_k) \geq \frac{\big(\alpha \cdot i(f,x^2) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i\big(f,x^2\big) - \dim_{\mathbb{C}}(R/I)} = 2 \cdot (k-1+\alpha)^2.$$

The “in particular” part then follows right away from Corollary 3.11. $\square$

Since a convenient semi-quasihomogeneous power series of multiplicity 2 defines an A_k-singularity and one with a homogeneous leading term defines an ordinary multiple point, the following proposition together with the previous two gives upper bounds for all singularities defined by a convenient semi-quasihomogeneous representative.

Proposition 1.13 ((Semiquasihomogeneous Singularities))
Let $\mathcal{S}_{p,q}$ be a singularity type with a convenient semi-quasihomogeneous representative $f \in R$, $q > p \geq 3$.

Then $\gamma_\alpha^{es}(\mathcal{S}_{p,q}) \geq \frac{\left(q-(1-\alpha)\cdot\left\lfloor\frac{q}{p}\right\rfloor\right)^2}{\left\lfloor\frac{q}{p}\right\rfloor} \geq \frac{q\cdot(p-1+\alpha)^2}{p}$ and we obtain the following upper bound for $\gamma_\alpha^{es}(f)$:

p,q	$\gamma_\alpha^{es}(f)$
$q \geq 39$	$\leq 3\cdot(q-2+\alpha)^2$
$\frac{q}{p} \in (1,2)$	$\leq 3\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [2,4)$	$\leq 2\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [4,\infty)$	$\leq (q-1+\alpha)^2$

Proof: To see the claimed lower bound for $\gamma_\alpha^{es}(\mathcal{S}_{p,q})$ recall that (see **[GLS05]**)

$$I^{es}(f) = \left\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, x^\alpha y^\beta \;\middle|\; \alpha p + \beta q \geq pq \right\rangle. \tag{1.4}$$

In particular, $I^{es}(f) \subseteq \left\langle y, x^{q-\lfloor\frac{q}{p}\rfloor}\right\rangle$, $\dim_{\mathbb{C}}(R/I) = q - \left\lfloor\frac{q}{p}\right\rfloor$ and $i(f,y) = q$, which implies the claim.

Let now I be a complete intersection ideal with $I^{es}(f) \subseteq I$. Applying Lemma 3.10 and $\mathrm{d}(I) \leq q$, we first of all note that

$$(1+\alpha)^2\cdot\dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2\cdot(q+1)^2}{4} \leq 2\cdot(q-1+\alpha)^2.$$

Moreover, if $\frac{q}{p} \geq 3$, then

$$(1+\alpha)^2\cdot\dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2\cdot\left(q^2+4q+3\right)}{6} \leq (q-1+\alpha)^2.$$

since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}}\left(R/I^{es}(f)\right) \leq \frac{(p+1)\cdot(q+1)}{2}$ by (1.4).

It therefore suffices to show

$$\lambda_\alpha(f;I,g) \leq \begin{cases} 3\cdot(q-2+\alpha)^2, & \text{if } q\geq 39,\\ 3\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p}\in(1,2),\\ 2\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p}\in[2,4),\\ (q-1+\alpha)^2, & \text{if } \frac{q}{p}\in[4,\infty),\end{cases} \tag{1.5}$$

where $g\in I$ with $i(f,g)\leq 2\cdot\dim_{\mathbb{C}}(R/I)$. Recall that

$$\lambda_\alpha(f;I,g) = \frac{\big(\alpha\cdot i(f,g)+(1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g)-\dim_{\mathbb{C}}(R/I)}.$$

Fixing I and considering $\lambda_\alpha(f;I,g)$ as a function in $i(f,g)$, where due to (1.12) the latter takes values between $\dim_{\mathbb{C}}(R/I)+1$ and $2\cdot\dim_{\mathbb{C}}(R/I)$, we note that the function is monotonously decreasing. In order to calculate an upper bound for $\lambda_\alpha(f;I,g)$ we may therefore replace $i(f,g)$ by some lower bound, which still exceeds $\dim_{\mathbb{C}}(R/I)+1$. Having done this we may then replace $\dim_{\mathbb{C}}(R/I)$ by an upper bound in order to find an upper bound for $\lambda(f;I,g)$.

Note that for $q\geq 39$ we have

$$\frac{54}{19}\cdot(q-1+\alpha)^2\leq 3\cdot(q-2+\alpha)^2. \tag{1.6}$$

Fix I and g, and let $L_{(p,q)}(g)=x^Ay^B$ be the leading term of g w. r. t. the weighted ordering $<_{(p,q)}$ (see Definition 2.1). By Remark 4.2 we know

$$i(f,g)\geq Ap+Bq. \tag{1.7}$$

Working with this lower bound for $i(f,g)$ we reduce the problem to find suitable upper bounds for $\dim_{\mathbb{C}}(R/I)$. For this purpose we may assume that $L_{(p,q)}(g)$ is minimal, and thus, in particular, $B\leq \operatorname{mult}(I)$.

If $A=0$, in view of Remark 3.8 we therefore have

$$B=\operatorname{mult}(I)\leq\frac{\mathrm{d}(I)+1}{2}\leq\frac{q+1}{2},$$

and thus by Lemma 3.10 then

$$\dim_{\mathbb{C}}(R/I)\leq B\cdot(q-B+1). \tag{1.8}$$

Moreover, for $A=0$ Lemma 4.4 applies with $h=g$ and we get

$$\dim_{\mathbb{C}}(R/I)\leq B\cdot q-1-\sum_{i=1}^{B-1}\left\lfloor\tfrac{qi}{p}\right\rfloor\leq B\cdot q-1-\left\lfloor\frac{q}{p}\right\rfloor\cdot\frac{B\cdot(B-1)}{2}. \tag{1.9}$$

Since $x^\alpha y^\beta\in I$ for $\alpha p+\beta q\geq pq$, we may assume $Ap+Bq\leq pq$. But then, since $\dim_{\mathbb{C}}(R/I)\leq\dim_{\mathbb{C}} R/\big\langle\frac{\partial f}{\partial y},g,x^\alpha y^\beta\mid \alpha p+\beta q\geq pq\big\rangle$, we may apply Lemma 4.5

with $h = \frac{\partial f}{\partial y}$ and $C = p-1$. This gives

$$\dim_{\mathbb{C}}(R/I) \leq Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \min\left\{A, \left\lceil \tfrac{q}{p} \right\rceil\right\}, \tag{1.10}$$

and if $B = 0$ we get in addition

$$\dim_{\mathbb{C}}(R/I) \leq A \cdot (p-1). \tag{1.11}$$

Finally note that by Lemma 1.3

$$i(f,g) > \dim_{\mathbb{C}}(R/I). \tag{1.12}$$

Let us now use the inequalities (1.6)-(1.12) to show (1.5). For this we have to consider several cases for possible values of A and B.

Case 1: $A = 0$, $B \geq 1$.

If $B = 1$, then by (1.9) and (1.12) we have $\lambda_\alpha(f; I, g) \leq (q-1+\alpha)^2$.

We may thus assume that $B \geq 2$. By (1.7) and (1.8)

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{B \cdot (B-1)} \leq 2 \cdot (q-1+\alpha)^2.$$

If, moreover, $\frac{q}{p} \geq 3$, then we may apply (1.9) to find

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{\left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \leq (q-1+\alpha)^2.$$

Taking (1.6) into account, this proves (1.5) in the case $A = 0$ and $B \geq 1$.

Case 2: $A = 1$, $B \geq 1$.

From (1.10) we deduce

$$\dim_{\mathbb{C}}(R/I) \leq B \cdot (q-1) + (p-1) - \left\lfloor \tfrac{q}{p} \right\rfloor \cdot \tfrac{B\cdot(B-1)}{2}.$$

Since $\frac{p-1+\alpha}{q-1+\alpha} \leq \frac{p}{q}$ we thus get

$$\begin{aligned}
\lambda_\alpha(f; I, g) &\leq \frac{\left(B + \frac{p-1+\alpha}{q-1+\alpha}\right)^2}{B + \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \cdot (q-1+\alpha)^2 \\
&\leq \begin{cases}
\frac{(B+\frac{1}{3})^2}{\frac{3B^2}{2} - \frac{B}{2} + 1} \cdot (q-1+\alpha)^2 & \leq (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} \geq 3, \\
\frac{(B+\frac{1}{2})^2}{B^2+1} \cdot (q-1+\alpha)^2 & \leq \frac{5}{4} \cdot (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} \geq 2, \\
2 \cdot \frac{(B+1)^2}{B^2+B+2} \cdot (q-1+\alpha)^2 & \leq \frac{16}{7} \cdot (q-1+\alpha)^2, \quad \text{if } \frac{q}{p} > 1.
\end{cases}
\end{aligned}$$

Once more we are done, since $\frac{16}{7} \leq \frac{54}{19}$.

Case 3: $A \geq 2$, $B \geq 1$.

Note that $\lfloor r\rfloor \geq r-1$ for any rational number r, and set $s=\frac{q}{p}$, then by (1.10)

$$\dim_{\mathbb{C}}(R/I) \leq Ap+Bq-(A-1)\cdot(B-1)-\frac{A\cdot(A-1)}{2s}-\frac{s\cdot B\cdot(B-1)}{2}-1-\min\left\{A,\lceil s\rceil\right\}.$$

This amounts to

$$\begin{aligned}
&\lambda_\alpha(f;I,g) \leq \\
&\frac{\Big(Ap+Bq-(1-\alpha)\cdot\big((A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+1+\min\{A,\lceil s\rceil\}\big)\Big)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}\\
&\qquad\leq \frac{\big(A\cdot(p-1+\alpha)+B\cdot(q-1+\alpha)\big)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3} \leq \varphi(A,B)\cdot(q-1+\alpha)^2,
\end{aligned}$$

where

$$\varphi(A,B)=\frac{\left(\frac{A}{s}+B\right)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}.$$

For the last inequality we just note again that $\frac{p-1+\alpha}{q-1+\alpha}\leq\frac{p}{q}=\frac{1}{s}$, while for the second inequality a number of different cases has to be considered. We postpone this for a moment.

In order to show (1.5) in the case $A\geq 2$ and $B\geq 1$ it now suffices to show

$$\varphi(A,B)\leq\begin{cases}\frac{54}{19}, & \text{if } s\geq 1,\\ 2, & \text{if } s\geq 2,\\ 1, & \text{if } s\geq 4.\end{cases} \tag{1.13}$$

Elementary calculus shows that for $B\geq 1$ fixed the function $[2,\infty)\to\mathbb{R}: A\mapsto\varphi(A,B)$ takes its maximum at

$$A=\max\left\{2,\frac{16-3B}{2+\frac{1}{s}}\right\}.$$

If $B\leq 3$, then the maximum is attained at $A=\frac{16-3B}{2+\frac{1}{s}}$, and

$$\varphi(A,B)\leq\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)=\frac{8sB-8B+64}{4s^2B-4s^2-4sB+28s-1}.$$

Again elementary calculus shows that the function $B\mapsto\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)$ is monotonously decreasing on $[1,3]$ and, therefore,

$$\varphi(A,B)\leq\varphi\left(\frac{13}{2+\frac{1}{s}},1\right)=\frac{8s+56}{24s-1}=:\psi_1(s).$$

Since also the function ψ_1 is monotonously decreasing on $[1,\infty)$ and $\psi_1(1)=\frac{64}{23}\leq\frac{54}{19}$, $\psi_1(2)=\frac{72}{47}\leq 2$ and $\psi_1(4)=\frac{88}{95}\leq 1$ Equation (1.13) follows in this case.

As soon as $B\geq 4$ the maximum for $\varphi(A,B)$ is attained for $A=2$ and

$$\varphi(A,B)\leq\varphi(2,B)=\frac{2\cdot(sB+2)^2}{s^3B^2-s^3B+2s^2B+4s^2+2s}.$$

Once more elementary calculus shows that the function $B \mapsto \varphi(2,B)$ is monotonously decreasing on $[4,\infty)$. Thus

$$\varphi(A,B) \leq \varphi(2,4) = \frac{4\cdot(1+2s)^2}{6s^3+6s^2+s} =: \psi_2(s).$$

Applying elementary calculus again, we find that the function ψ_2 is monotonously decreasing on $[1,\infty)$, so that we are done since $\psi_2(1) = \frac{36}{13} \leq \frac{54}{19}$, $\psi_2(2) = \frac{50}{37} \leq 2$ and $\psi_2(4) = \frac{81}{121} \leq 1$.

Let us now come back to proving the missing inequality above. We have to show

$$A+B \leq (A-1)\cdot(B-1) + \frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 1 + \min\big\{A, \lceil s\rceil\big\},$$

or equivalently

$$\frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\big\{A,\lceil s\rceil\big\} + AB - 2A - 2B \geq 0.$$

If $B \geq 2$, then $AB \geq 2A$ and $\frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\big\{A,\lceil s\rceil\big\} \geq 2B$, so we are done. It remains to consider the case $B=1$, and we have to show

$$A^2 - A - 2sA + 2s\cdot\min\big\{A,\lceil s\rceil\big\} \geq 0.$$

If $A \leq \lceil s\rceil$ or $A=2$ this is obvious. We may thus suppose that $A > \lceil s\rceil$ and $A \geq 3$. Since $\frac{A^2}{3} \geq A$ it remains to show

$$\frac{2A^2}{3} - 2sA + 2s\cdot\lceil s\rceil \geq 0.$$

For this

$$\frac{2A^2}{3} - 2sA + 2s\cdot\lceil s\rceil \geq \begin{cases} \frac{2A^2}{3} - 2sA & \geq 0, \quad \text{if } A \geq 3s,\\ \frac{2A^2}{3} - \frac{4sA}{3} & \geq 0, \quad \text{if } 2s \leq A \leq 3s,\\ \frac{2A^2}{3} - sA & \geq 0, \quad \text{if } \frac{3s}{2} \leq A \leq 2s,\\ \frac{2A^2}{3} - \frac{2sA}{3} & \geq 0, \quad \text{if } \lceil s\rceil \leq A \leq \frac{3s}{2}. \end{cases}$$

Case 4: $A \geq 1$, $B = 0$.

Applying (1.10) and (1.11) we get

$$\lambda_\alpha(f;I,g) \leq \begin{cases} \frac{A^2\cdot(p-1+\alpha)^2}{A} \leq \begin{Bmatrix} \frac{A}{s^2}\cdot(q-1+\alpha)^2 \\ A\cdot(q-2+\alpha)^2 \end{Bmatrix} & \text{for any } A, \quad \text{and}\\ \frac{A^2\cdot(p-1+\alpha)^2}{\sum_{i=1}^{A-1}\lfloor\frac{pi}{q}\rfloor + \min\{A,\lceil\frac{q}{p}\rceil\}} \leq \varphi_{\nu,s}(A)\cdot(q-1+\alpha)^2, & \text{if } A \geq 3, \end{cases}$$

where

$$\varphi_{\nu,s}(A) = \frac{\frac{A^2}{s^2}}{\frac{A\cdot(A-1)}{2s} - (A-1) + \nu} = \frac{2A^2}{sA^2 - (2s^2+s)\cdot A + 2\cdot(\nu+1)\cdot s^2}$$

with $\nu = 2$ for $s \in (1,2]$ and $\nu = 3$ for $s \in (2,\infty)$.

In particular, due to the first two inequalities we may thus assume that

$$A > \begin{cases} 3, & \text{if } q \geq 39, \\ 3s^2, & \text{if } s \in (1,2), \\ 2s^2, & \text{if } s \in [2,4), \\ s^2, & \text{if } s \in [4,\infty). \end{cases}$$

Note that $\varphi_{3,s}(A) \leq 1$ for $s \geq 4$, since

$$A \geq s^2 = \frac{9s^2}{16} + \frac{7s^2}{16} \geq \frac{s\cdot(1+2s)}{2\cdot(s-2)} + \frac{s}{s-2}\cdot\sqrt{s^2-3s+\tfrac{33}{4}}.$$

This gives (1.5) for $s \geq 4$.

If now $s \in (2,4)$, then $\varphi_{3,s}$ is monotonously decreasing on $\big[2s^2,\infty\big)$, as is $s \mapsto \varphi_{3,s}\big(2s^2\big)$ on $[2,4)$, and thus

$$\varphi_{3,s}(A) \leq \varphi_{3,s}\big(2s^2\big) = \frac{4s^2}{2s^3-2s^2-s+4} \leq \frac{8}{5} \leq 2,$$

while for $s = 2$ the function $\varphi_{2,2}$ is monotonously decreasing on $[8,\infty)$ and thus $\varphi_{2,2}(A) \leq \frac{16}{9} \leq 2$. This finishes the case $s \in [2,4)$.

Let's now consider the case $s \in (1,2)$ and $q \geq 39$ parallel. Applying elementary calculus, we find that $\varphi_{2,s}$ takes its maximum on $[3,\infty)$ at $A = \frac{12s}{1+2s}$ and is monotonously decreasing on $\big[\frac{12s}{1+2s},\infty\big)$. Moreover, the function $s \mapsto \varphi_{2,s}\big(\frac{12s}{1+2s}\big)$ is monotonously decreasing on $(1,2)$. If $s \geq \frac{7}{6}$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,s}\big(\tfrac{12s}{1+2s}\big) \leq \varphi_{2,\frac{7}{6}}\big(\tfrac{21}{5}\big) = \frac{54}{19}.$$

Due to (1.6) it thus remains to consider the case $s \in \big(1,\frac{7}{6}\big)$ and $A > 3$. If $A \geq 8$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,1}(8) = \frac{64}{23} \leq \frac{54}{19},$$

since the function $s \mapsto \varphi_{2,s}(8)$ is monotonously decreasing on $[1,2)$.

So, we are finally stuck with the case $A \in \{4,5,6,7\}$ and $1 \leq \frac{q}{p} = s \leq \frac{7}{6}$. We want to apply Lemma 3.10. For this we note first that by Lemma 4.6 in our situation $\mathrm{d}(I) \leq p+1$ and $A = \mathrm{mult}(I) \leq \frac{p+2}{2}$. But then

$$\dim_{\mathbb{C}}(R/I) \leq A\cdot(p-A+2)$$

and thus,

$$\lambda_\alpha(f;I,g) \leq \frac{A^2\cdot\big(p-(1-\alpha)\cdot(A-2)\big)^2}{A\cdot(A-2)} \leq \frac{A}{(A-2)}\cdot(q-2+\alpha)^2 \leq 2\cdot(q-2+\alpha)^2.$$

This finishes the proof. $\square$

Remark 1.14
In the proof of the previous proposition we achieved for almost all cases $\lambda_\alpha(f;I,g) \leq \frac{54}{19}\cdot(q-1+\alpha)^2$, apart from the single case $L_{<_{(p,q)}}(g) = x^3$. The following example shows that indeed in this case we cannot, in general, expect any better coefficient than 3. More precisely, the example shows that the bound

$$3\cdot(q-2+\alpha)^2$$

is sharp for the family of singularities given by $x^q - y^{q-1}$, $q \geq 39$. A closer investigation should allow to lower the bound on q, but we cannot get this for all $q \geq 4$, as the example of E_6 and E_8 show.

Moreover, we give series of examples for which the bound $(q-1+\alpha)^2$ is sharp, respectively for which $2\cdot(q-1+\alpha)^2$ is a lower bound.

Example 1.15
Throughout these examples $q > p \geq 3$ are integers.

(a) Let $f = x^q - y^{q-1}$, then $\gamma_\alpha^{es}(f) \geq 3\cdot(q-2+\alpha)^2$. In particular, for $q \geq 39$,

$$\gamma_\alpha^{es}(f) = 3\cdot(q-2+\alpha)^2.$$

For this we note that $I = \langle x^3, y^{q-2}\rangle$ is a complete intersection ideal in R with $I^{es}(f) = \big\langle x^{q-1}, y^{q-2}, x^\alpha y^\beta \;\big|\; \alpha\cdot(q-1)+\beta q \geq q\cdot(q-1)\big\rangle \subseteq I$, since $2\cdot(q-1)+(q-3)\cdot q = q^2-q-2 < q\cdot(q-1)$ and thus $x^2y^{q-3} \notin I^{es}(f)$. This also shows that the monomial x^iy^j with $0 \leq i \leq 2$ and $0 \leq j \leq q-3$ form a $\mathbb{C}$-basis of R/I, so that $\dim_{\mathbb{C}}(R/I) = 3q-6$. Since $i\big(f,x^3\big) = 3q-3$, the claim follows.

(b) Let $\frac{q}{p} < 2$ and $f = x^q - y^p$, then

$$\gamma_\alpha^{es}(f) \geq 2\cdot(q-1+\alpha)^2.$$

By the assumption on p and q we have $(q-2)\cdot p+q < pq$ and hence $x^{q-2}y \notin I^{es}(f)$. Thus $I^{es}(f) = \big\langle x^{q-1}, y^{p-1}, x^\alpha y^\beta \;\big|\; \alpha p+\beta q \geq pq\big\rangle \subseteq I = \langle y^2, x^{q-1}\rangle$, and we are done since $\dim_{\mathbb{C}}(R/I) = 2q-2$ and $i\big(f,y^2\big) = 2q$.

(c) Let $f \in R$ be convenient, semi-quasihomogeneous of $\operatorname{ord}_{(p,q)}(f) = pq$, and suppose that in f no monomial x^ky, $k \leq q-2$, occurs (e. g. $f = x^q - y^p$), then $\gamma_\alpha^{es}(f) \geq (q-1+\alpha)^2$. In particular, if $\frac{q}{p} \geq 4$, then

$$\gamma_\alpha^{es}(f) = (q-1+\alpha)^2.$$

By the assumption, $I^{es}(f) \subseteq I = \langle x^{q-1}, y\rangle$, since $\frac{\partial f}{\partial x} \equiv x^{q-1}\cdot u(x) \pmod y$ for a unit u and $\frac{\partial f}{\partial y} \equiv 0 \;\big(\mathrm{mod}\ \langle y, x^{q-1}\rangle\big)$. Hence we are done since $\dim_{\mathbb{C}}(R/I) = q-1$ and $i(f,y) = q$.

(d) Let $f = y^3 - 3x^8y + 3x^{12}$, then f does not satisfy the assumptions of (c), but still $\gamma_\alpha^{es}(f) = (11+\alpha)^2 = (q-1+\alpha)^2$.

For this note that $I = \langle y - x^4, x^{11}\rangle$ contains $I^{es}(f)$, $\dim_{\mathbb{C}}(R/I) = 11$ and $i\big(f, y - x^4\big) = 12$.

(e) Let $f = 7y^3+15x^7-21x^5y$, then f is semi-quasihomogeneous with weights $(p, q) = (3, 7)$ and convenient, but $\gamma_0^{es}(f) \leq 25 < 36 = (q-1)^2$. This shows that $(q-1)^2$ is not a general lower bound for $\gamma_0^{es}(\mathcal{S}_{p,q})$.
We note first that $I^{es}(f) = \langle x^7, y^2 - x^5, x^6 - x^4y\rangle$ is not a complete intersection and $\dim_{\mathbb{C}}\big(R/I^{es}(f)\big) = 11$. Let now I be a complete intersection ideal with $I^{es}(f) \subset I$ and let $h \in I$ such that $L_{<_{(3,7)}}(h) = x^Ay^B$ is minimal, in particular, $\operatorname{ord}_{(3,7)}(h) = 3A + 7B$ is minimal. Then $\dim_{\mathbb{C}}(R/I) \leq 10$ and $i(f, g) \geq 3A + 7B$ for all $g \in I$.
If, therefore, $3A + 7B \geq 14$, then
$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 25.$$
We may thus assume that $3A + 7B \leq 13$, in particular $B < 2$. If $B = 0$, and hence $A \leq 4$, then by Lemma 4.5 $\dim_{\mathbb{C}}(R/I) \leq 2A$, so that
$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 4A \leq 16.$$
Similarly, if $B = 1$ and $A = 2$, then by the same Lemma $\dim_{\mathbb{C}}(R/I) \leq 9$ and $i(f, g) \geq 13$, so that
$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{81}{4}.$$
So it remains to consider the case $B = 1$ and $A \in \{0, 1\}$. That is $h = x^Ay+h'$ with $\operatorname{ord}_{(3,7)}(h') \geq 9+3A$. Consider the ideal $J = \big\langle x^\alpha y^\beta \bigm| 3\alpha+7\beta \geq 21\big\rangle \subseteq I$. Then $x^{4-A}\cdot h \equiv x^4y \pmod J$, and thus $x^6-x^4y \equiv x^6 \pmod{\langle h\rangle+J}$, i. e. $\langle h, x^6 - x^4y\rangle + J = \langle h, x^6\rangle + J$. Moreover, $x^6 \not\in \langle h\rangle + J$, so that $\dim_{\mathbb{C}}\big(R/\langle g, x^6-x^4y\rangle+J\big) \leq 6+A$. If we can show that $\langle g, x^6-x^4y\rangle+J \subsetneqq I$, then
$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{(5+A)^2}{3A+7-5-A} \leq \frac{25}{2}.$$
We are therefore done, once we know that $y^2 - x^5 \not\in \langle g, x^6\rangle + J$. Suppose there was a g such that $gh = y^2 - x^5 \pmod{\langle x^6\rangle + J}$. Then $y^2 = L_{<_{(3,7)}}(g)\cdot L_{<_{(3,7)}}(h)$, which in particular means $A = 0$ and $L_{<_{(3,7)}}(h) = L_{<_{(3,7)}}(g) = y$. But then the coefficients of 1, x and x^2 in h and g must be zero, so that x^5 cannot occur with a non-zero coefficient in the product. This gives the desired contradiction.

2. Local Monomial Orderings

Throughout the proofs of the auxilary statements in Section 4 we make use of some results from computer algebra concerning properties of local monomial orderings. In this section we recall the relevant definitions and results.

Definition 2.1
A *monomial ordering* is a total ordering $<$ on the set of monomials $\{x^\alpha y^\beta \mid \alpha, \beta \geq 0\}$ such that for all $\alpha,\beta,\gamma,\delta,\mu,\nu \geq 0$

$$x^\alpha y^\beta < x^\gamma y^\delta \quad \Longrightarrow \quad x^{\alpha+\mu} y^{\beta+\nu} < x^{\gamma+\mu} y^{\delta+\nu}.$$

A monomial ordering $<$ is called *local* if $1 > x^\alpha y^\beta$ for all $(\alpha,\beta) \neq (0,0)$, and it is a local *degree ordering* if

$$\alpha+\beta > \gamma+\delta \quad \Longrightarrow \quad x^\alpha y^\beta < x^\gamma y^\delta.$$

Finally, if $<$ is any local monomial ordering, then we define the *leading monomial* $L_<(f)$ with respect to $<$ of a non-zero power series $f \in R$ to be the maximal monomial $x^\alpha y^\beta$ such that the coefficient of $x^\alpha y^\beta$ in f does not vanish. For $f = 0$, we set $L_<(f) := 0$.
If $I \trianglelefteq R$ is an ideal in R, then $L_<(I) = \langle L_<(f) \mid f \in I \rangle$ is called its *leading ideal*.

We will give now some examples of local monomial orderings which are used in the proofs.

Example 2.2
Let $\alpha,\beta,\gamma,\delta \geq 0$ be integers.

(a) The *negative lexicographical ordering* $<_{ls}$ is defined by the relation

$$x^\alpha y^\beta <_{ls} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha > \gamma \text{ or } (\alpha = \gamma \text{ and } \beta > \delta).$$

(b) The *negative degree reverse lexicographical ordering* $<_{ds}$ is defined by the relation

$$x^\alpha y^\beta <_{ds} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha+\beta > \gamma+\delta \text{ or } (\alpha+\beta = \gamma+\delta \text{ and } \beta > \delta).$$

(c) If positive integers p and q are given, then we define the *local weighted degree ordering* $<_{(p,q)}$ with weights (p,q) by the relation

$$\begin{aligned} x^\alpha y^\beta <_{(p,q)} x^\gamma y^\delta \quad :\Longleftrightarrow \quad & \alpha p + \beta q > \gamma p + \delta q \text{ or} \\ & (\alpha p + \beta q = \gamma p + \delta q \text{ and } \beta < \delta). \end{aligned}$$

We note that $<_{ds}$ is a local degree ordering, while $<_{ls}$ is not and $<_{(p,q)}$ is if and only if $p = q$.

Let us finally recall some useful properties of local orderings (see e. g. **[GrP02]** Corollary 7.5.6 and Proposition 5.5.7).

Proposition 2.3
Let $<$ be any local monomial ordering, and let I be a zero-dimensional ideal in R.

(a) *The monomials of $R/L_<(I)$ form a $\mathbb{C}$-basis of R/I. In particular*

$$\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big).$$

(b) *If $<$ is a degree ordering, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide (see Definition 3.1, and see also Remark 3.3).*

3. The Hilbert Samuel Function

A useful tool in the study of the degree of zero-dimensional schemes and their subschemes is the Hilbert Samuel function of the structure sheaf, that is of the corresponding Artinian ring.

Definition 3.1
Let $I \lhd R$ be a zero-dimensional ideal.

(a) The function

$$H^1_{R/I} : \mathbb{Z} \to \mathbb{Z} : d \mapsto \begin{cases} \dim_{\mathbb{C}}\big(R/(I+\mathfrak{m}^{d+1})\big), & d \geq 0, \\ 0, & d < 0, \end{cases}$$

is called the *Hilbert Samuel function* of R/I.

(b) We define the *slope* of the Hilbert Samuel function of R/I to be the function

$$H^0_{R/I} : \mathbb{N} \to \mathbb{N} : d \mapsto H^1_{R/I}(d) - H^1_{R/I}(d-1).$$

Thus

$$H^0_{R/I}(d) = \dim_{\mathbb{C}}\big(\mathfrak{m}^d/((I \cap \mathfrak{m}^d) + \mathfrak{m}^{d+1})\big),$$

is just the number $d+1$ of linearly independent monomials of degree d in $\mathfrak{m}^d$, minus the number of linearly independent monomials of degree d in $\big(I \cap \mathfrak{m}^d\big) + \mathfrak{m}^{d+1}$.
Note that if $\overline{\mathfrak{m}} = \mathfrak{m}/I$ denotes the maximal ideal of R/I and $\mathrm{Gr}_{\mathfrak{m}}(R/I) = \bigoplus_{d \geq 0} \overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}$ the associated graded ring, then

$$H^0_{R/I}(d) = \dim_{\mathbb{C}}\big(\overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}\big)$$

is just the dimension of the graded piece of degree d of $\mathrm{Gr}_{\mathfrak{m}}(R/I)$.

(c) Finally, we define the *multiplicity* of I to be

$$\mathrm{mult}(I) := \min\big\{\,\mathrm{mult}(f) \;\big|\; 0 \neq f \in I\big\},$$

and the *degree bound* of I as

$$\mathrm{d}(I) := \min\big\{d \in \mathbb{N} \;\big|\; \mathfrak{m}^d \subseteq I\big\}.$$

Let us gather some straight forward properties of the slope of the Hilbert Samuel function.

Lemma 3.2
Let $J \subseteq I \lhd R$ be zero-dimensional ideals.

(a) $H^0_{R/I}(d) = d+1$ *for all* $0 \leq d < \mathrm{mult}(I)$.

(b) $H^0_{R/I}(d) \leq H^0_{R/I}(d-1)$ *for all* $d \geq \operatorname{mult}(I)$.

(c) $H^0_{R/I}(d) \leq \operatorname{mult}(I)$.

(d) $H^0_{R/I}(d) = 0$ *for all* $d \geq \operatorname{d}(I)$ *and* $H^0_{R/I} \neq 0$ *for all* $d < \operatorname{d}(I)$. *In particular*

$$\dim_{\mathbb{C}}(R/I) = \sum_{d=0}^{\operatorname{d}(I)-1} H^0_{R/I}(d).$$

(e) $H^0_{R/I}(d) \leq H^0_{R/J}(d)$ *for all* $d \in \mathbb{N}$.

(f) $\operatorname{d}(I)$ *and* $\operatorname{mult}(I)$ *are completely determined by* $H^0_{R/I}$.

Proof: For (a) we note that $I \subseteq \mathfrak{m}^d$ for all $d \leq \operatorname{mult}(I)$ and thus $H^0_{R/I}(d) = \dim_{\mathbb{C}}\left(\mathfrak{m}^d/\mathfrak{m}^{d+1}\right) = d+1$ for all $0 \leq d < \operatorname{mult}(I)$.

By definition we see that $H^0_{R/I}(d)$ is just the number of linearly independent monomials of degree d in $\mathfrak{m}^d$, which is $d+1$, minus the number of linearly independent monomials, say $m_1, \ldots, m_r$, of degree d in $\left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$. We note that then the set

$$\{xm_1, \ldots, xm_r, ym_1, \ldots, ym_r\} \subseteq \mathfrak{m} \cdot \left((I \cap \mathfrak{m}^d) + \mathfrak{m}^{d+1}\right) \subseteq \left(I \cap \mathfrak{m}^{d+1}\right) + \mathfrak{m}^{d+2}$$

contains at least $r+1$ linearly independent monomials of degree $d+1$, once r was non-zero. However, for $d = \operatorname{mult}(I)$ and $g = g_d + h.o.t \in I$ with homogeneous part $g_d \neq 0$ of degree d, we have $g_d \in \left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$, that is, $d = \operatorname{mult}(I)$ is the smallest integer d for which there is a monomial of degree d in $\left(I \cap \mathfrak{m}^d\right) + \mathfrak{m}^{d+1}$. Thus for $d \geq \operatorname{mult}(I) - 1$

$$H^0_{R/I}(d+1) \leq (d+2) - (r+1) = d+1-r = H^0_{R/I}(d),$$

which proves (b), while (c) is an immediate consequence of (a) and (b).

If $d \geq \operatorname{d}(I)$, then $H^1_{R/I}(d) = \dim_{\mathbb{C}}(R/I)$ is independent of d, and hence $H^0_{R/I}(d) = 0$ for all $d \geq \operatorname{d}(I)$. In particular,

$$\sum_{i=0}^{\operatorname{d}(I)-1} H^0_{R/I}(d) = H^1_{R/I}(\operatorname{d}(I)-1) - H^1_{R/I}(-1) = \dim_{\mathbb{C}}(R/I).$$

Moreover, $\mathfrak{m}^{\operatorname{d}(I)-1} + I \neq I = I + \mathfrak{m}^{\operatorname{d}(I)}$, so that $H^0_{R/I}\big(\operatorname{d}(I)-1\big) \neq 0$, and by (b) then $H^0_{R/I}(d) \neq 0$ for all $d < \operatorname{d}(I)$. This proves (d), and (e) and (f) are obvious. □

Remark 3.3

Let $<$ be a local degree ordering on R, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide by Proposition 2.3, and hence we have as well

$$H^0_{R/I} = H^0_{R/L_<(I)}, \quad \operatorname{d}(I) = \operatorname{d}\left(L_<(I)\right), \quad \text{and} \quad \operatorname{mult}(I) = \operatorname{mult}\left(L_<(I)\right),$$

since by the previous lemma the multiplicity and the degree bound only depend on the slope of the Hilbert Samuel function.

Remark 3.4
The slope of the Hilbert Samuel function of R/I gives rise to a histogram as the graph of the function $H^0_{R/I}$. By the Lemma 3.2 we know that up to $\mathrm{mult}(I)-1$ the histogram is just a staircase with steps of height one, and from $\mathrm{mult}(I)-1$ on it can only go down, which it eventually will do until it reaches the value zero for $d=\mathrm{d}(I)$. This means that we get a histogram of form shown in Figure 1.

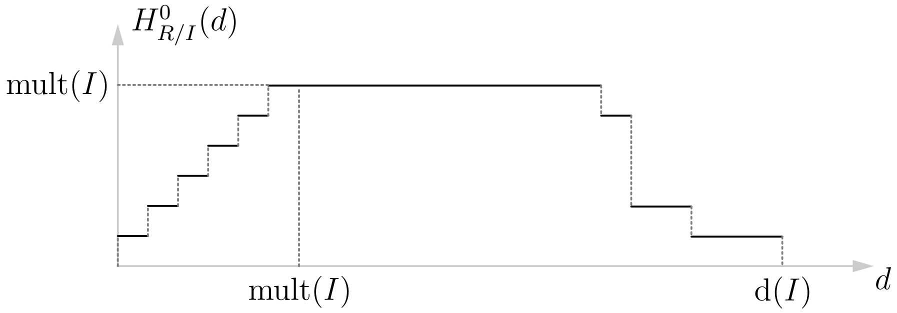

FIGURE 1. The histogram of $H^0_{R/I}$ for a general ideal I.

Note also, that by Lemma 3.2 (a) the area of the histogram is just $\dim_{\mathbb{C}}(R/I)$!

Example 3.5
In order to understand the slope of the Hilbert Samuel function better, let us consider some examples.

(a) Let $f=x^2-y^{k+1}$, $k\geq 1$, and let $I=I^{ea}(f)=\langle x,y^k\rangle$ the equisingularity ideal of an A_k-singularity. Then $\mathrm{d}(I)=k$, $\mathrm{mult}(I)=1$ and $\dim_{\mathbb{C}}(R/I)=k$.

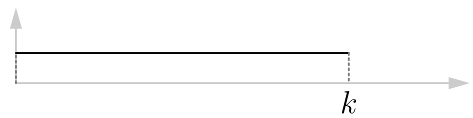

FIGURE 2. The histogram of $H^0_{R/I}$ for an A_k-singularity

(b) Let $f=x^2y-y^{k-1}$, $k\geq 4$, and let $I=I^{ea}(f)=\langle xy,x^2-(k-1)\cdot y^{k-2}\rangle$ the equisingularity ideal of a D_k-singularity. Then $x^3,xy,y^{k-1}\in I$, and thus $\mathfrak{m}^{k-1}\subset I$, which gives $\mathrm{d}(I)=k-1$, $\mathrm{mult}(I)=2$ and $\dim_{\mathbb{C}}(R/I)=k$, which shows that the bound in Lemma 3.10 need not be obtained.

(c) Let $f=x^3-y^4$ and let $I=I^{ea}(f)=\langle x^2,y^3\rangle$ the equisingularity ideal of an E_6-singularity. Then $\mathrm{d}(I)=4$, $\mathrm{mult}(I)=2$ and $\dim_{\mathbb{C}}(R/I)=6$.

Let $f=x^3-xy^3$ and let $I=I^{ea}(f)=\langle 3x^2-y^3,xy^2\rangle$ the equisingularity ideal of an E_7-singularity. Then $x^3,xy^2,y^5\in I$, and thus $\mathfrak{m}^5\subset I$, which gives $\mathrm{d}(I)=5$, $\mathrm{mult}(I)=2$ and $\dim_{\mathbb{C}}(R/I)=7$.

FIGURE 3. The histogram of $H^0_{R/I}$ for a D_k-singularity

Let $f = x^3 - y^5$ and let $I = I^{ea}(f) = \langle x^2, y^4\rangle$ the equisingularity ideal of an E_8-singularity. Then $\mathrm{d}(I) = 6$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 8$.

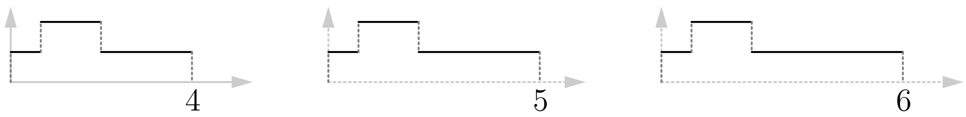

FIGURE 4. The histogram of $H^0_{R/I}$ for E_6, E_7 and E_8.

(d) Let $I = \langle x^3, x^2y, y^3\rangle$, then $\mathrm{d}(I) = 4$, $\mathrm{mult}(I) = 3$ and $\dim_{\mathbb{C}}(R/I) = 7$.

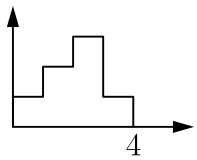

FIGURE 5. The histogram of $H^0_{R/I}$ for $I = \langle x^3, x^2y, y^3\rangle$.

The following result providing a lower bound for the minimal number of generators of a zero-dimensional ideal in R is due to A. Iarrobino.

Lemma 3.6

Let $I \lhd R$ be a zero-dimensional ideal. Then I cannot be generated by less than $1 + \sup\left\{H^0_{R/I}(d-1) - H^0_{R/I}(d) \;\middle|\; d \geq \mathrm{mult}(I)\right\}$ elements.

In particular, if I is a complete intersection ideal then for $d \geq \mathrm{mult}(I)$

$$H^0_{R/I}(d-1) - 1 \leq H^0_{R/I}(d) \leq H^0_{R/I}(d-1).$$

Proof: See **[Iar77]** Theorem 4.3 or **[Bri77]** Proposition III.2.1. □

Moreover, by the Lemma of Nakayama and Proposition 2.3 we can compute the minimal number of generators for a zero-dimensional ideal exactly.

Lemma 3.7

Let $I \lhd R$ be zero-dimensional ideal and let $<$ denote any local ordering on R. Then the minimal number of generators of I is

$$\dim_{\mathbb{C}}(I/\mathfrak{m}I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big) - \dim_{\mathbb{C}}\big(R/L_<(\mathfrak{m}I)\big).$$

Remark 3.8
If we apply Lemma 3.6 to a zero-dimensional complete intersection ideal $I \lhd R$, i. e. a zero-dimensional ideal generated by two elements, then we know that the histogram of $H^0_{R/I}$ will be as shown in Figure 6; that is, up to the value

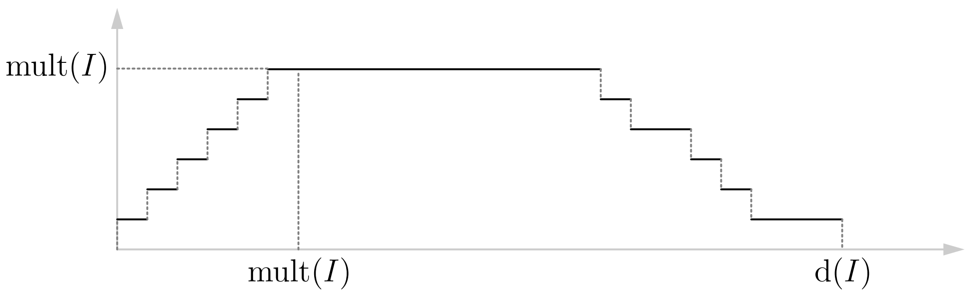

FIGURE 6. The histogram of $H^0_{R/I}$ for a complete intersection.

$d = \mathrm{mult}(I)$ the histogram of $H^0_{R/I}$ is an ascending staircase with steps of height and length one, then it remains constant for a while, and finally it is a descending staircase again with steps of height one, but a possibly longer length. In particular we see that

$$\mathrm{mult}(I) \leq \begin{cases} \frac{\mathrm{d}(I)+1}{2}, & \text{if } \mathrm{d}(I) \text{ is odd,} \\ \frac{\mathrm{d}(I)}{2}, & \text{if } \mathrm{d}(I) \text{ is even.} \end{cases} \tag{3.1}$$

Example 3.9
Let $I = \mathfrak{m}^k$ for $k \geq 1$. Then $\mathrm{d}(I) = \mathrm{mult}(I) = k$ and $\dim_{\mathbb{C}}(R/I) = \binom{k+1}{2}$.

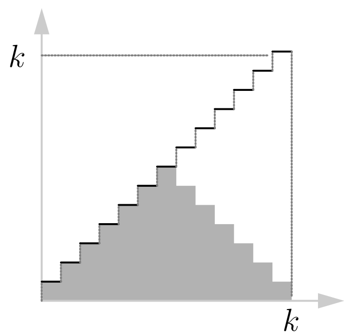

FIGURE 7. The histogram of $H^0_{R/\mathfrak{m}^k}$. The shaded region is the maximal possible value of $\dim_{\mathbb{C}}(R/I)$ for a complete intersection ideal I containing $\mathfrak{m}^k$.

Lemma 3.10
Let $I \lhd R$ be a zero-dimensional complete intersection ideal, then

$$\dim_{\mathbb{C}}(R/I) \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

In particular

$$\dim_{\mathbb{C}}(R/I) \leq \begin{cases} \frac{(\mathrm{d}(I)+1)^2}{4}, & \textit{if } \mathrm{d}(I) \textit{ odd}, \\ \frac{\mathrm{d}(I)^2+2\,\mathrm{d}(I)}{4}, & \textit{if } \mathrm{d}(I) \textit{ even}. \end{cases}$$

Proof: By Remark 3.4 we have to find an upper bound for the area A of the histogram of $H^0_{R/I}$. This area would be maximal, if in the descending part the steps had all length one, i. e. if the histogram was as shown in Figure 8. Since

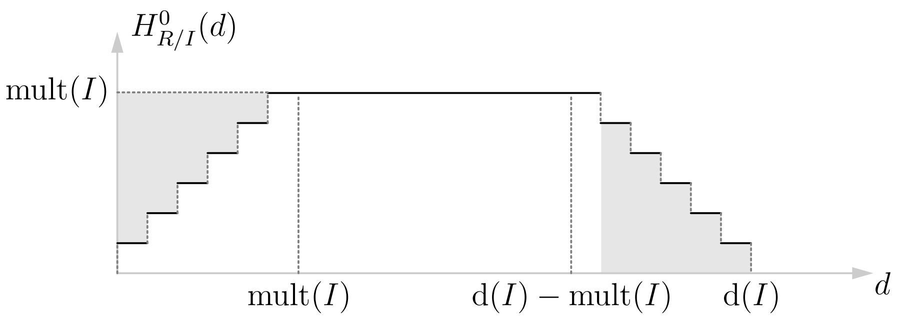

FIGURE 8. Maximal possible area.

the two shaded regions have the same area, we get

$$A \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

Consider now the function

$$\varphi : \left[\mathrm{mult}(I), \tfrac{\mathrm{d}(I)+1}{2}\right] \longrightarrow \mathbb{R} : x \mapsto \big(\mathrm{d}(I) - x + 1\big) \cdot x,$$

then this function is monotonously increasing, which finishes the proof in view of Equation (3.1). □

Corollary 3.11

For an ordinary m-fold point M_m we have

$$\tau^{es}_{ci}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \textit{if } m \geq 3 \textit{ odd}, \\ \frac{m^2+2m}{4}, & \textit{if } m \geq 4 \textit{ even}, \\ 1, & \textit{if } m = 2. \end{cases}$$

Proof: Let f be a representative of M_m. Then

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle + \mathfrak{m}^m,$$

and as in the proof of Proposition 1.12 we may assume that f is a homogeneous of degree m.

In particular, if $m = 2$, then $I^{es}(f) = \mathfrak{m}$ is a complete intersection and $\tau^{es}_{ci}(M_2) = 1$. We may therefore assume that $m \geq 3$.

For any complete intersection ideal I with $\mathfrak{m}^m \subset I^{es}(f) \subseteq I$ we automatically have $\mathrm{d}(I) \leq m$, and by Lemma 3.10

$$\tau_{ci}^{es}(f) \leq \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \text{ odd}, \\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even}. \end{cases}$$

Consider now the representative $f = x^m - y^m$. If $m = 2k$ is even, then the ideal $I = \langle x^k, y^{k+1} \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau_{ci}^{es}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 + k = \frac{m^2 + 2m}{4}.$$

Similarly, if $m = 2k - 1$ is odd, then the ideal $I = \langle x^k, y^k \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau_{ci}^{es}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 = \frac{m^2 + 2m + 1}{4}.$$

□

Remark 3.12
Let $I \lhd R$ be any zero-dimensional ideal, not necessarily a complete intersection, then still

$$\dim_{\mathbb{C}}(R/I) \leq \left(\mathrm{d}(I) - \frac{\mathrm{mult}(I) - 1}{2} \right) \cdot \mathrm{mult}(I).$$

Proof: The proof is the same as for the complete intersection ideal, just that we cannot ensure that the histogram goes down to zero at $\mathrm{d}(I)$ with steps of size one. The dimension is thus bounded by the region of the histogram in Figure 9. □

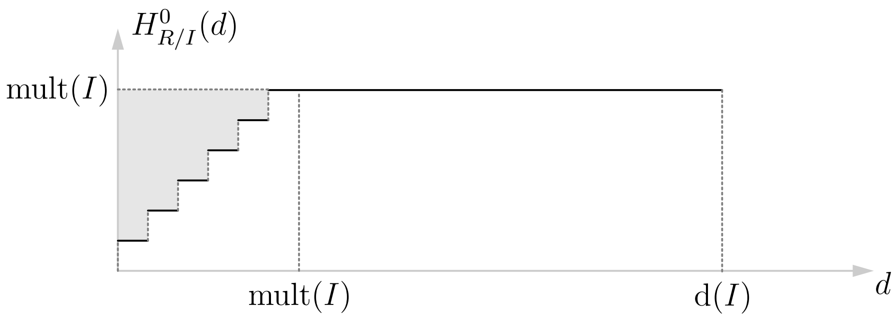

FIGURE 9. Maximal possible area.

4. Semi-Quasihomogeneous Singularities

Definition 4.1
A non-zero polynomial of the form $f = \sum_{\alpha \cdot p + \beta \cdot q = d} a_{\alpha,\beta} x^\alpha y^\beta$ is called *quasihomogeneous* of (p, q)-degree d. Thus the Newton polygon of a quasihomogeneous polynomial has just one side of slope $-\frac{p}{q}$.

A quasihomogeneous polynomial is said to be *non-degenerate* if it is reduced, that is if it has no multiple factors, and it is said to be *convenient* if $\frac{d}{p}, \frac{d}{q} \in \mathbb{Z}$ and $a_{\frac{d}{p},0}$ and $a_{0,\frac{d}{q}}$ are non-zero, that is if the Newton polygon meets the x-axis and the y-axis.

If $f = f_0 + f_1$ with f_0 quasihomogeneous of (p,q)-degree d and for any monomial $x^\alpha y^\beta$ occurring in f_1 with a non-zero coefficient we have $\alpha \cdot p + \beta \cdot q > d$, we say that f is of (p,q)-*order* d, and we call f_0 the (p,q)-*leading form* of f and denote it by $\operatorname{lead}_{(p,q)}(f)$. We denote the (p,q)-order of f by $\operatorname{ord}_{(p,q)}(f)$.

A power series $f \in R$ is said to be *semi-quasihomogeneous* with respect to the weights (p,q) if the (p,q)-leading form is non-degenerate.

Remark 4.2
Let $f \in R$ with $\deg_{(p,q)}(f) = pq$ and let f_0 denote its (p,q)-leading form.

(a) If $\gcd(p,q) = r$, then f_0 has r factors of the form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$, $i = 1, \ldots, r$. If, moreover, f_0 is non-degenerate, then these will all be irreducible and pairwise different, i. e. not scalar multiples of each other.

(b) If f is irreducible, then f_0 has only one irreducible factor, possibly of higher multiplicity.

(c) If f_0 is non-degenerate, then f has $r = \gcd(p,q)$ branches $f_1, \ldots, f_r$, which are all semi-quasihomogeneous with irreducible (p,q)-leading form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ for pairwise distinct points $(a_i : b_i) \in \mathbb{P}^1_{\mathbb{C}}$, $i = 1, \ldots, r$. The characteristic exponents of f_i are $\frac{q}{r}$ and $\frac{p}{r}$ for all $i = 1, \ldots, r$, and thus f_i admits a parametrisation of the form

$$\big(x_i(t), y_i(t)\big) = \Big(\alpha_i t^{\frac{p}{r}} + h.o.t, \beta_i t^{\frac{q}{r}} + h.o.t\Big).$$

(d) If f_0 is non-degenerate, i. e. f is semi-quasihomogeneous, and $g \in R$, then

$$i(f,g) \geq \operatorname{ord}_{(p,q)}(g).$$

Proof:

(a) If $\alpha p + \beta q = pq$, then $p \mid \beta q$ and hence $p \mid \beta r$, so that $\beta \cdot \frac{r}{p}$ is a natural number. Similarly $\alpha \cdot \frac{r}{q}$ is a natural number. We may therefore consider the transformation

$$f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big) \in \mathbb{C}[x,y]_r$$

which is a homogeneous polynomial of degree r. Thus $f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big)$ factors in r linear factors $a_i x - b_i y$, $i = 1, \ldots, r$, so that f_0 factors as

$$f_0 = \prod_{i=1}^{r} \big(a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}\big). \tag{4.1}$$

Since $\gcd\left(\frac{p}{r}, \frac{q}{r}\right) = 1$, the factors $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ are irreducible once neither a_i nor b_i is zero.

If f_0 is non-degenerate, then the irreducible factors of f_0 are pairwise distinct. So, $a_i = 0$ implies $r = p$ and still $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = b_i y$ irreducible, while $b_i = 0$ similarly gives $r = q$ and $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = a_i x$ irreducible. Thus, in any case the factors in (4.1) are irreducible and, hence, pairwise distinct.

(b) With the notation from Lemma 4.3 and the factorisation of f_0 from (4.1) we get

$$g = \frac{\prod_{i=1}^r a_i u^{\frac{bq}{r}} v^{\frac{pq}{r^2}} - b_i u^{\frac{ap}{r}} v^{\frac{pq}{r^2}}}{u^{ap} v^{\frac{pq}{r}}} = \prod_{i=1}^r (a_i u - b_i).$$

By assumption f is irreducible, hence according to Lemma 4.3 g has at most one, possibly repeated, zero. But thus the factors of f_0 all coincide – up to scalar multiple.

(c) The first assertion is an immediate consequence from (a) and (b), while the "in particular" part follows by Puiseux expansion.

(d) Let g_0 be the (p,q)-leading form of g. Using the notation from (c) we have

$$\begin{aligned} i(f,g) &= \sum_{i=1}^r i(f_i,g) = \sum_{i=1}^r \operatorname{ord}\big(g(x_i(t),y_i(t))\big) \\ &= \sum_{i=1}^r \operatorname{ord}\Big(g_0\big(\alpha_i t^{\frac{p}{r}}, \beta_i t^{\frac{q}{r}}\big) + h.o.t\Big) \geq \sum_{i=1}^r \frac{\operatorname{ord}_{(p,q)}(g)}{r} = \operatorname{ord}_{(p,q)}(g). \end{aligned}$$

□

Lemma 4.3
Let $f \in R$ with $\operatorname{ord}_{(p,q)}(f) = pq$ and let f_0 denote its (p,q)-leading form. Let $r = \gcd(p,q)$ and $a, b \geq 0$ such that $qb - pa = r$. Finally set

$$g = \frac{f_0\big(u^b v^{\frac{p}{r}}, u^a v^{\frac{q}{r}}\big)}{u^{ap} v^{\frac{pq}{r}}} \in \mathbb{C}[u].$$

Then the number of different zeros of g is a lower bound for the number of branches of f.

Proof: See [**BrK86**] Remark on p. 480. □

The following investigations are crucial for the proof of Proposition 1.13.

Lemma 4.4
Let $f \in R$ be convenient semi-quasihomogeneous with leading form f_0 and $\operatorname{ord}_{(p,q)}(f) = pq$, let $I = \langle x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$, and let $h \in R$. Then

$$\dim_{\mathbb{C}} R/\big(\langle h\rangle + I^{es}(f)\big) < \dim_{\mathbb{C}} R/\big(\langle h\rangle + I\big).$$

In particular, if $L_{(p,q)}(h) = y^B$ with $B \leq p$, then

$$\dim_{\mathbb{C}} R/\langle h\rangle + I^{es}(f) \leq Bq - 1 - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor.$$

Proof: As

$$I^{es}(f) = \left\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y} \right\rangle + I,$$

it suffices to show that

$$I^{es}(f) \not\subseteq \langle h \rangle + I,$$

which is the same as showing that not both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ belong to $\langle h \rangle + I$.

Suppose the contrary, that is, there are $h_x, h_y \in R$ such that

$$\tfrac{\partial f}{\partial x} \equiv h_x \cdot h \;(\mathrm{mod}\; I) \quad \text{ and } \quad \tfrac{\partial f}{\partial y} \equiv h_y \cdot h \;(\mathrm{mod}\; I).$$

We note that

$$\mathrm{lead}_{(p,q)}\left(\tfrac{\partial f}{\partial x}\right) = \tfrac{\partial f_0}{\partial x} \quad \text{ and } \quad \mathrm{lead}_{(p,q)}\left(\tfrac{\partial f}{\partial y}\right) = \tfrac{\partial f_0}{\partial y},$$

and none of the monomials involved is contained in I. Therefore

$$\mathrm{lead}_{(p,q)}(h_x) \cdot \mathrm{lead}_{(p,q)}(h) = \tfrac{\partial f_0}{\partial x} \quad \text{ and } \quad \mathrm{lead}_{(p,q)}(h_y) \cdot \mathrm{lead}_{(p,q)}(h) = \tfrac{\partial f_0}{\partial y},$$

which in particular implies that $\frac{\partial f_0}{\partial x}$ and $\frac{\partial f_0}{\partial y}$ have a common factor. This, however, is then a multiple factor of the quasihomogeneous polynomial f_0, in contradiction to f being semi-quasihomogeneous.

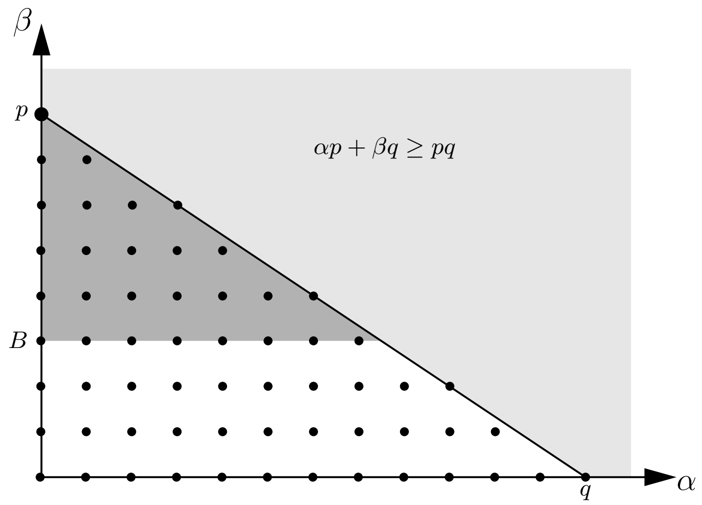

FIGURE 10. A Basis of $R/\langle h \rangle + I$.

For the "in particular" part, we note that by Proposition 2.3

$$\dim_{\mathbb{C}} R/\langle h \rangle + I = \dim_{\mathbb{C}} R/L_{<_{(p,q)}}\big(\langle h \rangle + I\big) \leq \dim_{\mathbb{C}} R/\big\langle y^B \big\rangle + I,$$

and the monomials $x^\alpha y^\beta$ with $\alpha p + \beta q < pq$ and $\beta < B$ form a $\mathbb{C}$-basis of the latter vector space (see also Figure 10). Hence,

$$\dim_{\mathbb{C}} R/\langle h \rangle + I \leq \sum_{i=0}^{B-1} \left\lceil q - \tfrac{qi}{p} \right\rceil = Bq - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor.$$

□

Lemma 4.5
Let $g, h \in R$ *such that* $L_{(p,q)}(g) = x^A y^B$ *and* $L_{(p,q)}(h) = y^C$*, and consider the ideals* $J = \langle x^A y^B, y^C, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$ *and* $J' = \langle g, h, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$*. Then*

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/J,$$

and if $Ap + Bq \leq pq$ *and* $B \leq C \leq p$*, then*

$$\dim_{\mathbb{C}} R/J = Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \sum_{i=C}^{p-1} \min \left\{ A, \left\lceil q - \tfrac{Cq}{p} \right\rceil \right\}.$$

Moreover, if $B = 0$*, then* $\dim_{\mathbb{C}} R/J \leq A \cdot C$*.*

Proof: By Proposition 2.3

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/L_{<_{(p,q)}}(J') \leq \dim_{\mathbb{C}} R/J.$$

Let $I = \langle x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$. Then the monomials $x^\alpha y^\beta$ with $(\alpha, \beta) \in \Lambda = \{(\alpha, \beta) \in \mathbb{N} \times \mathbb{N} \mid \alpha p + \beta q < pq\}$ form a basis of R/I. Moreover, the monomials $x^\alpha y^\beta$ with $(\alpha, \beta) \in \Lambda_1 \cup \Lambda_2$ are a basis of J/I, where

$$\Lambda_1 = \{(\alpha, \beta) \in \Lambda \mid \alpha \geq A \text{ and } \beta \geq B\}$$

and

$$\Lambda_2 = \{(\alpha, \beta) \in \Lambda \setminus \Lambda_1 \mid \beta \geq C\}.$$

(See also Figure 11.) This gives rise to the above values for $\dim_{\mathbb{C}} R/J$.

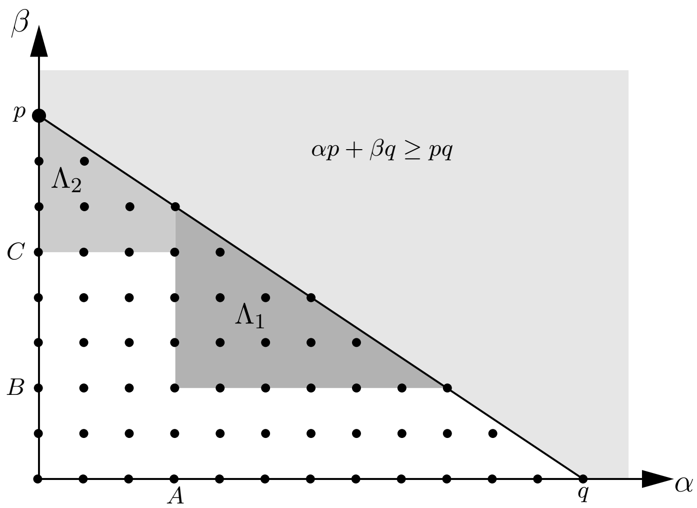

FIGURE 11. A Basis of R/J.

□

Lemma 4.6
Let $q > p$ *be such that* $\frac{q}{p} < \frac{d}{d-1}$ *for some integer* $d \geq 2$*, and let* $0 \leq A \leq d$*.*

(a) *If* $L_{(p,q)}(g) = x^A$*, then* $L_{<_{ds}}(g) = x^A$*.*

(b) $\mathfrak{m}^{p+1} \subseteq \langle x^A, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$.

(c) *If I is an ideal such that $g, h, x^\alpha y^\beta \in I$ for $\alpha p + \beta q \geq pq$ and where $L_{<_{(p,q)}}(g) = x^A$ and $L_{<_{(p,q)}}(h) = y^{p-1}$, then* $\mathrm{d}(I) \leq p+1$.

Moreover, if $L_{<_{(p,q)}}(g)$ is minimal among the leading monomials of elements in I w. r. t. $<_{(p,q)}$, then $\mathrm{mult}(I) = A$.

Proof: It suffices to consider the case $A = d$, since this implies the other cases. Note that by assumption $d \leq p$.

(a) Since x^d is less than any monomial of degree at least d with respect to $<_{ds}$, we have to show that in g no monomial of degree less than d can occur with a non-zero coefficient. x^d being the leading monomial of g with respect to $<_{(p,q)}$, it suffices to show that $\alpha + \beta < d$ implies $\alpha p + \beta q < dp$, or alternatively, since $\frac{q}{p} < \frac{d}{d-1}$,

$$\alpha + \beta \cdot \frac{d}{d-1} \leq d.$$

For $\alpha + \beta < d$ the left hand side of this inequality will be maximal for $\alpha = 0$ and $\beta = d-1$, and thus the inequality is satisfied.

(b) We only have to show that $x^\gamma y^{p+1-\gamma} \in \langle x^d, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$ for $\gamma = 3, \ldots, d-1$, since the remaining generators of $\mathfrak{m}^{p+1}$ definitely are. However, by assumption $\frac{q}{p} < \frac{d}{d-1} \leq \frac{\gamma}{\gamma-1}$, and thus $\gamma \cdot p + (p+1-\gamma) \cdot q \geq pq$.

(c) By the assumption on I we deduce form (a) and (b) that $\mathrm{d}\left(L_{<_{ds}}(I)\right) \leq p+1$. However, by Remark 3.3 $\mathrm{d}(I) = \mathrm{d}\left(L_{<_{ds}}(I)\right)$, which proves the first assertion.

Suppose now that $\mathrm{mult}(I) < A$, i. e. there is an $f \in I$ such that $\mathrm{mult}(f) \leq A-1$. The considerations for (a) show that then $L_{<_{(p,q)}}(f) < x^A$ in contradiction to the assumption.

$\square$

PAPER III

Reducible Families of Curves with Ordinary Multiple Points on Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

Abstract: In [**Kei03**], [**Kei05a**] and [**Kei05b**] we gave numerical conditions which ensure that an equisingular family is irreducible respectively T-smooth. Combining results from [**GLS01**] and an idea from [**ChC99**] we give in the present paper series of examples of families of irreducible curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ with only ordinary multiple points which are reducible and where at least one component does not have the expected dimension. The examples show that for families of curves with ordinary multiple points the conditions for T-smoothness in [**Kei05b**] have the right asymptotics.

This paper is published as [**Kei06**] Thomas Keilen, *Reducible families of curves with ordinary multiple points on surfaces in* $\mathbb{P}^3$, Comm. in Alg. **34** (2006), no. 5, 1921–1926.

Throughout this article Σ will denote a smooth projective surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 2$, and H will be a hyperplane section of Σ. For a positive integer m we denote by M_m the topological singularity type of an ordinary m-fold point, i. e. the singularity has m smooth branches with pairwise different tangents. And for positive integers d and r we denote by $V^{irr}_{|dH|}(rM_m)$ the family of irreducible curves in the linear system $|dH|$ with precisely r singular points all of which are ordinary m-fold points. $V^{irr}_{|dH|}(rM_m)$ is called T-smooth if it is smooth of the expected dimension

$$\operatorname{expdim}\left(V^{irr}_{|dH|}(rM_m)\right) = \dim|dH| - r \cdot \frac{m^2+m-4}{2}.$$

Theorem 1.1
For $m \geq 18n$ there is an integer $l_0 = l_0(m,\Sigma)$ such that for all $l \geq l_0$ the family $V^{irr}_{|dH|}(rM_m)$ with $d = 2lm + l$ and $r = 4l^2n$ has at least one T-smooth component and one component of higher dimension.

Moreover, the T-smooth component dominates $\operatorname{Sym}^r(\Sigma)$ *under the map*

$$V^{irr}_{|dH|}(rM_m) \longrightarrow \operatorname{Sym}^r(\Sigma) : C \mapsto \operatorname{Sing}(C)$$

sending a curve C to its singular locus, and the fundamental group $\pi_1(\Sigma \setminus C)$ of the complement of any curve $C \in V^{irr}_{|dH|}(rM_m)$ is abelian.

Before we prove the theorem let us compare the result with the conditions for T-smoothness in [**Kei05b**] and for irreducibility in [**Kei05a**].

Here we have given examples of non-T-smooth families $V_{|dH|}(rM_m)$ where

$$r \cdot m^2 \equiv n \cdot d^2,$$

if we neglect the terms of lower order in ml. If $n \geq 4$ and the Picard number of Σ is one, then according to [**Kei05b**] Corollary 2.3 respectively Corollary 2.4 – neglecting terms of lower order in m and d –

$$r \cdot m^2 < \frac{1}{2n-6} \cdot n \cdot d^2$$

would be a sufficient condition for T-smoothness. Similarly, if $n = 2$, then Σ is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and we may apply [**Kei05b**] Theorem 2.5 to find that

$$r \cdot m^2 < \frac{1}{8} \cdot n \cdot d^2$$

implies T-smoothness. Since the families fail to satisfy the conditions only by a constant factor we see that asymptotically in d, m and r the conditions for T-smoothness are proper.

For irreducibility the situation is not quite as good. The conditions in [**Kei05a**] Corollary 2.4 for irreducibility if $n \geq 4$ and the Picard number of Σ is one is roughly

$$r \cdot m^2 < \frac{24}{n^2 m^2} \cdot n \cdot d^2,$$

and similarly for $n = 2$ [**Kei05a**] Theorem 2.6 it is

$$r \cdot m^2 < \frac{1}{6m^2} \cdot n \cdot d^2.$$

Here the "constant" by which the families fail to satisfy the condition depends on the multiplicity m, so that with respect to m the asymptotics are not proper. However, we should like to point out that it does not depend on the number r of singular points which are imposed.

The families in Theorem 1.1 thus exhibit the same properties as the families of plane curves provided in [**GLS01**], which we use to construct the non-T-smooth component. The idea is to intersect a family of cones in $\mathbb{P}^3_{\mathbb{C}}$ over the plane curves provided by [**GLS01**] with Σ and to calculate the dimension of the resulting family. Under the conditions on m and l requested this family turns out to be of higher dimension than the expected one. The same idea was used by Chiantini and Ciliberto in [**ChC99**] in order to give examples of nodal families of curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ which are not of the expected dimension. We then combine an asymptotic h^1-vanishing result by Alexander and Hirschowitz [**AlH00**] with an existence statement from [**KeT02**] to show that there is also a T-smooth component, where actually the curves have their singularities in very general position.

Proof of Theorem 1.1: Fix a general plane P in $\mathbb{P}^3_{\mathbb{C}}$ and a general point p. By [**GLS01**] there is an integer $l_1 = l_1(m)$ such that for any $l \geq \max\{l_1, m\}$ the family of curves in P of degree $2lm + l$ with $4l^2$ ordinary m-fold points as only singularities has a component W of dimension

$$\begin{aligned}\dim(W) &\geq (m+1) \cdot \frac{(l+1)\cdot(l+2)}{2} + (2l+1)\cdot(2l+2) - 4\\ &= \frac{l^2m + 9l^2 + 3lm + 15l + 2m - 4}{2}.\end{aligned}$$

Let $\mathcal{W}$ be the family of cones with vertex p over curves C in W, then $\dim(\mathcal{W}) = \dim(W)$, since a cone is uniquely determined by the curve C and the vertex p. Moreover, any cone in $\mathcal{W}$ has precisely $4l^2$ lines of multiplicity m, so that when we intersect it with Σ we get in general an irreducible curve in Σ with $4l^2n$ ordinary m-fold points. In particular, $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ must have a component W' of dimension

$$\dim(W') \geq \dim(W) \geq \frac{l^2m + 9l^2 + 3lm + 15l + 2m - 4}{2}.$$

However, since the dimension of the linear system $|dH|$ is

$$\dim|dH| = \binom{d+3}{3} - \binom{d+3-n}{3} - 1,$$

and since

$$\tau^{es}(M_m) = \frac{m\cdot(m+1)}{2} - 2$$

is the expected number of conditions imposed by an ordinary m-fold point, the expected dimension of $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ is

$$\begin{aligned}\operatorname{expdim}\Big(V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)\Big) &= \dim|(2lm+l)H| - 4l^2n\tau^{es}(M_m)\\ &= \frac{17l^2n + \big(4n-n^2\big)\cdot l\cdot(2m+1)}{2} + \frac{n^3 - 6n^2 + 11n}{6}.\end{aligned}$$

Due to the conditions on m and l this number is strictly smaller than the dimension of W'. It remains to show that $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ also has a T-smooth component, after possibly enlarging l_1.

For $\underline{z} = (z_1, \ldots, z_r) \in \Sigma^r$ we denote by $X(m;\underline{z})$ the zero-dimensional scheme with ideal sheaf $\mathcal{J}_{X(m;\underline{z})}$ given by the stalks

$$\mathcal{J}_{X(m;\underline{z}),z} = \begin{cases} \mathfrak{m}^m_{\Sigma,z}, & \text{if } z \in \{z_1,\ldots,z_r\},\\ \mathcal{O}_{\Sigma,z}, & \text{else,}\end{cases}$$

where $\mathcal{O}_{\Sigma,z}$ denotes the local ring of Σ at z and $\mathfrak{m}_{\Sigma,z}$ is its maximal ideal.

By [**AlH00**] Theorem 1.1 there is an integer $l_2 = l_2(m,\Sigma)$ such that for $l \geq l_2$ and $\underline{z} \in \Sigma^r$ in very general position the canonical map

$$H^0\big(\Sigma, \mathcal{O}_\Sigma((2lm+l-1)H)\big) \longrightarrow H^0\big(\Sigma, \mathcal{O}_{X(m;\underline{z})}((2lm+l-1)H)\big)$$

has maximal rank. In particular, since $h^1\big(\Sigma,\mathcal{O}_\Sigma((2lm+l-1)H)\big)=0$ we have

$$h^1\big(\Sigma,\mathcal{J}_{X(m;\underline{z})}(2lm+l-1)H)\big)=0,$$

once $\deg\big(X(m;\underline{z})\big)\le h^0\big(\Sigma,\mathcal{O}_\Sigma((2lm+l-1)H)\big)$, which is equivalent to

$$\frac{4l^2n\cdot m\cdot(m+1)}{2}\le\binom{2lm+l+2}{3}-\binom{2lm+l+2-n}{3},$$

or alternatively

$$\frac{nl\cdot\big(l-(n-2)\cdot(2m+1)\big)}{2}+\frac{n^3-3n^2+2n}{6}\ge 0.$$

The latter inequality is fulfilled as soon as $l\ge(n-2)\cdot(2m+1)$. Moreover, under this hypothesis we have

$$(2lm+l)\cdot H^2-2g(H)=(2lm+l)\cdot n-(n-1)\cdot(n-2)\ge 2m,$$

where $g(H)$ denotes the geometric genus of H, and

$$(2lm+l)^2\cdot H^2>4l^2nm^2. \tag{1.1}$$

Thus **[KeT02]** Theorem 3.3 (see also **[Kei01]** Theorem 1.2) implies that the family $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ has a non-empty T-smooth component, more precisely it contains a curve in a T-smooth component with singularities in $z_1,\ldots,z_r$. In particular, since there is only a finite number of components and $\underline{z}$ is in very general position, some T-smooth component must dominate $\mathrm{Sym}^r(\Sigma)$. Actually, due to **[Los98]** Proposition 2.1 (e) and since $h^1(\Sigma,\mathcal{O}_\Sigma)=0$ every T-smooth component dominates $\mathrm{Sym}^r(\Sigma)$.

Thus the statement follows with

$$l_0(m,\Sigma):=\max\big\{l_1(m),l_2(m,\Sigma),(\deg(\Sigma)-2)\cdot(2m+1),m\big\}.$$

It just remains to show that the fundamental group of the complement of a curve $C\in V^{irr}_{|dH|}(rM_m)$ is abelian. Note first of all that by the Lefschetz Hyperplane Section Theorem Σ is simply connected. But then $\pi_1(\Sigma\setminus C)$ is abelian by **[Nor83]** Proposition 6.5 because of (1.1). $\square$

PAPER IV

Some Obstructed Equisingular Families of Curves on Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

Abstract: Very few examples of obstructed equsingular families of curves on surfaces other than $\mathbb{P}^2_{\mathbb{C}}$ are known. Combining results from [**Wes04**] and [**Hir92**] with an idea from [**ChC99**] we give in the present paper series of examples of families of irreducible curves with simple singularities on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ which are not T–smooth, i.e. do not have the expected dimension, (Section 1) and we compare this with conditions (showing the same asymptotics) which ensure the existence of a T–smooth component (Section 2).

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Below we are going to construct two series of equisingular families of curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. In both examples the families are obstructed in the sense that they do not have the expected dimension. However, while in the first example at least the existence of such curves was expected, the families in the second example were expected to be empty. It would be interesting to see if the equisingular families contain further components which are well–behaved. However, the families which we construct fail to satisfy the numerical conditions for the existence of such a component given in Section 2 by a factor of two. We do not know whether the families are reducible or not, or if they are smooth.

1. Examples of obstructed families

Throughout this section Σ will denote a smooth projective surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 2$, and H will be a hyperplane section of Σ. $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ will be a finite set of *simple* singularity types, that is the $\mathcal{S}_i$ are of type A_k (given by $x^2 - y^{k+1} = 0$, $k \geq 1$), D_k (given by $x^2y - y^{k-1} = 0$, $k \geq 4$), or E_k (given by $x^3 - y^4 = 0$, $x^3 - xy^3 = 0$, or $x^3 - y^5 = 0$ for $k = 6, 7, 8$ respectively). In general, for positive integers $r_1, \ldots, r_s$ and d we denote by $V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ the family of irreducible curves in the linear system $|dH|$ with precisely $r = r_1 + \ldots + r_s$ singular points, r_i of which are of the type $\mathcal{S}_i$, $i = 1, \ldots, s$, where $\mathcal{S}_i$ may be any analytic type of an isolated singularity. $V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ is called *T–smooth* or *not obstructed* if it is smooth of the expected dimension

$$\begin{aligned}\operatorname{expdim}\big(V^{irr}_{|dH|}(r_1\mathcal{S}_1,\ldots,r_s\mathcal{S}_s)\big) &= \dim|dH| - \sum_{i=1}^{s} r_i\cdot\tau(\mathcal{S}_i)\\ &= \frac{nd^2+(4n-n^2)d}{2}+\frac{n^3-6n^2+11n-6}{6}-\sum_{i=1}^{s} r_i\cdot\tau(\mathcal{S}_i),\end{aligned}$$

where $\tau(\mathcal{S})=\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\big\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},f\big\rangle$ is the Tjurina number of the singularity type $\mathcal{S}$ given by the local equation $f=0$. Note that $\tau(A_k)=\tau(D_k)=\tau(E_k)=k$.

In this note we give examples of such equisingular families of curves which are obstructed in the sense that they have dimension larger than the expected one. We use the idea by which Chiantini and Ciliberto in **[ChC99]** showed the existence of obstructed families of nodal curves.

Let us fix a plane P in $\mathbb{P}^3_{\mathbb{C}}$, a point p outside P, and a curve C of degree $d>1$ in P. If we intersect the cone $K_{C,p}$ over C with vertex p with Σ, this gives a curve $C'=K_{C,p}\cap\Sigma$ in $|dH|$ which is determined by the choice of C and p (see Lemma 3.1). In particular, if C varies in an N-dimensional family in P, then C' varies in an N-dimensional family on Σ, and if C is irreducible, then for a general choice of p the curve C' will be irreducible as well (see Lemma 3.2). Moreover, if C has a singular point q of (simple) singularity type $\mathcal{S}$ and Σ meets the line joining p and q transversally in n points, then C' will have a singularity of the same type in each of these points.

Example 1.1
Fix the set $S=\{\mathcal{S}_1,\ldots,\mathcal{S}_s\}$ and let $m=\max\{\tau(\mathcal{S})\mid\mathcal{S}\in S\}$. Suppose that $n>2m+4$ and $d>>n$, and let $r_1,\ldots,r_s\geq 0$ be such that

$$\frac{d^2+(4-n)d+2}{2}\quad\leq\quad\sum_{i=1}^{s} r_i\cdot\tau(\mathcal{S}_i)\quad\leq\quad\frac{d^2+(4-n)d+2}{2}+m-1.$$

Then

$$\sum_{i=1}^{s} r_i\cdot\tau(\mathcal{S}_i)\leq\frac{d^2+(4-n)d+2}{2}+m-1\leq\frac{d^2}{2}-m\cdot d-3.$$

Hence, by **[Wes04]** Remark 3.3.5 the family $V=V^{irr}_d(r_1\mathcal{S}_1,\ldots,r_s\mathcal{S}_s)$ of irreducible plane curves C of degree d with precisely $r=r_1+\ldots+r_s$ singular points, r_i of which are of type $\mathcal{S}_i$, is non-empty, and we may estimate its dimension:

$$\begin{aligned}\dim(V)\geq\operatorname{expdim}(V) &= \frac{d(d+3)}{2}-\sum_{i=1}^{s} r_i\cdot\tau(\mathcal{S}_i)\\ &\geq\frac{d(d+3)}{2}-\frac{d^2+(4-n)d+2}{2}-m+1=\frac{n-1}{2}\cdot d-m.\end{aligned}$$

By the above construction we see that hence the family of curves C' satisfies

$$\dim\left(V^{irr}_{|dH|}(nr_1\mathcal{S}_1,\ldots,nr_s\mathcal{S}_s)\right)\geq\frac{n-1}{2}\cdot d-m.$$

However, the expected dimension of this family is

$$\begin{aligned}\operatorname{expdim}&\left(V^{irr}_{|dH|}(nr_1\mathcal{S}_1,\ldots,nr_s\mathcal{S}_s)\right)\\&=\frac{nd^2+(4n-n^2)d}{2}+\frac{n^3-6n^2+11n-6}{6}-\sum_{i=1}^{s}n\cdot r_i\cdot\tau(\mathcal{S}_i)\\&\leq\frac{nd^2+(4n-n^2)d}{2}+\frac{n^3-6n^2+11n-6}{6}-n\cdot\left(\frac{d^2+(4-n)d+2}{2}\right)\\&=\frac{n^3-6n^2+5n-6}{6}.\end{aligned}$$

For $d>>n$, more precisely for

$$d>\frac{n^3-6n^2+5n-6+6m}{3n-3},$$

the expected dimension will be smaller than the actual dimension, which proves that the family is obstructed.

In particular, if $S=\{\mathcal{S}\}$, $\mathcal{S}\in\{A_k,D_k,E_k\}$, and

$$r=\left\lceil\frac{d^2+(4-n)d+2}{2k}\right\rceil,$$

then $V_{|dH|}(nr\mathcal{S})$ is obstructed, once $d>>n>3k+4$.

Note that in the previous example

$$\operatorname{expdim}\left(V_{|dH|}(nr_1\mathcal{S}_1,\ldots,nr_s\mathcal{S}_s)\right)\geq\frac{n^3-6n^2+5n-6}{6}-n\cdot(m-1)>0,$$

that is, the existence of curves in $|dH|$ with the given singularities was expected. This not so in the following example.

Example 1.2

Let k be an *even*, positive integer, $m\geq 1$, $d=2(k+1)^m$, and

$$r=\frac{3\cdot(k+1)\cdot\left((k+1)^{2m}-1\right)}{(k+1)^2-1}.$$

Hirano proved in **[Hir92]** the existence of an irreducible plane curve of degree d with precisely r singular points all of type A_k. Thus the above construction shows that

$$V^{irr}_{|dH|}(nrA_k)$$

is non-empty. However, the expected dimension is

$$\begin{aligned}\operatorname{expdim}\left(V^{irr}_{|dH|}(nrA_k)\right)&=\frac{nd^2+(4n-n^2)d}{2}+\frac{n^3-6n^2+11n-6}{6}-knr\\&=\left(2-\frac{3\cdot(k^2+k)}{k^2+2k}\right)\cdot(k+1)^{2m}+o\big((k+1)^m\big),\end{aligned}$$

which is negative for m sufficiently large, since

$$\frac{3\cdot(k^2+k)}{k^2+2k} > 2.$$

This shows that $V^{irr}_{|dH|}(nrA_k)$ is obstructed for sufficiently large k.

2. Some remarks on conditions for T–smoothness

Unless otherwise specified in this section Σ will be an arbitrary smooth projective surface, H a very ample divisor on Σ, and $\mathcal{S}_1,\dots,\mathcal{S}_s$ arbitrary (not necessarily different) topological or analytical singularity types. As in Section 1 we denote for $d \geq 0$ by $V^{irr}_{|dH|}(\mathcal{S}_1,\dots,\mathcal{S}_s)$ the equisingular family of irreducible curves in $|dH|$ with precisely s singular points of types $\mathcal{S}_1,\dots,\mathcal{S}_s$, and again the expected dimension is

$$\operatorname{expdim}\big(V^{irr}_{|dH|}(\mathcal{S}_1,\dots,\mathcal{S}_s)\big) = \dim|dH| - \sum_{i=1}^{s}\tau(\mathcal{S}_i).$$

$V^{irr}_{|dH|}(\mathcal{S}_1,\dots,\mathcal{S}_s)$ is called T–smooth if it is smooth of the expected dimension.

By **[Kei01]** Theorem 1.2 and 2.3 (which is a slight improvement of **[KeT02]** Theorem 3.3 and Theorem 4.3) there is a curve $C \in V^{irr}_{|dH|}(\mathcal{S}_1,\dots,\mathcal{S}_s)$ if

- $d\cdot H^2 - g(H) \geq m_i + m_j$, and
- $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big) = 0$ for $\underline{z}\in\Sigma^r$ very general,

where $\underline{m} = (m_1,\dots,m_s)$ with $m_i = e^*(\mathcal{S}_i)$, a certain invariant which only depends on $\mathcal{S}_i$. Moreover, $V^{irr}_{|dH|}(\mathcal{S}_1,\dots,\mathcal{S}_s)$ is T–smooth at this curve C (see e.g. **[Shu99]** Theorem 1). Finally, by **[AlH00]** Theorem 1.1 there is a number $d(m)$ depending only on $m = \max\{m_1,\dots,m_s\}$, such that for all $d \geq d(m)$ and for $\underline{z}\in\Sigma^r$ very general the map

$$H^0\big(\Sigma, \mathcal{O}_\Sigma((d-1)H)\big) \longrightarrow H^0\big(\Sigma, \mathcal{O}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big)$$

has maximal rank. In particular, if

$$\dim|(d-1)H| \geq \deg\big(X(\underline{m};\underline{z})\big) = \sum_{i=1}^{s}\frac{m_i\cdot(m_i+1)}{2},$$

then $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big) = 0$. This proves the following Proposition.

Proposition 2.1
Let $S = \{\mathcal{S}_1,\dots,\mathcal{S}_s\}$ be a finite set of pairwise different topological or analytical singularity types. Then there exists a number $d(S)$ such that for all $d \geq d(S)$ and $r_1,\dots,r_s \geq 0$ satisfying

$$\sum_{i=1}^{s} r_i\cdot\frac{e^*(\mathcal{S}_i)\cdot\big(e^*(\mathcal{S}_i)+1\big)}{2} < \dim|(d-1)H| \tag{2.1}$$

the equisingular family $V^{irr}_{|dH|}(r_1\mathcal{S}_1,\dots,r_s\mathcal{S}_s)$ has a non-empty T–smooth component.

In [**Shu03**] upper bounds for $e^*(\mathcal{S})$ are given. For a non-simple analytical singularity type we have

$$e^*(\mathcal{S}) = e^a(\mathcal{S}) \leq 3\sqrt{\mu(\mathcal{S})} - 2$$

where $\mu(\mathcal{S})$ is the Milnor number of $\mathcal{S}$, and for any topological singularity type

$$e^*(\mathcal{S}) = e^s(\mathcal{S}) \leq \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S})} - 1,$$

where $\delta(\mathcal{S})$ is the delta invariant of $\mathcal{S}$.

For *simple* singularity types there are the better bounds

$\mathcal{S}$	$e^*(\mathcal{S})$	$\mathcal{S}$	$e^*(\mathcal{S})$
A_1	2	D_4	3
A_2	3	D_5	4
$A_k, k = 3, \ldots, 7$	4	$D_k, k \leq 6, \ldots, 10$	5
$A_k, k = 8, \ldots, 10$	5	$D_k, k \leq 11, \ldots, 13$	6
$A_k, k \geq 1$	$\leq 2 \cdot \lfloor\sqrt{k+5}\rfloor$	$D_k, k \geq 1$	$\leq 2 \cdot \lfloor\sqrt{k+7}\rfloor + 1$
E_6	4	E_7	4
E_8	5		

In particular, if $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ is a finite set of *simple* singularities, then there is a $d(S)$ such that for all $d \geq d(S)$ and all $r_1, \ldots, r_s \geq 0$ satisfying

$$2 \cdot \sum_{i=1}^{s} r_i \cdot \Big(\tau(\mathcal{S}_i) + o\big(\sqrt{\tau(\mathcal{S}_i)}\big)\Big) \leq \dim |dH| \tag{2.2}$$

the family $V^{irr}_{|dH|}(nr_1\mathcal{S}_1, \ldots, nr_s\mathcal{S}_s)$ has a non-empty T–smooth component.

The families in Example 1.1 fail to satisfy this condition roughly by the factor 2. We thus cannot conclude that these families are reducible as we could in a similar situation in [**Kei06**].

However, if we compare Condition 2.1 respectively 2.2 to the conditions in [**GLS00**] or [**Kei05b**] which ensure that the equisingular family is T–smooth at *every* point, the latter basically involve the square of the Tjurina number and are therefore much more restrictive. This, of course, was to be expected.

3. Some remarks on cones

In this section we collect some basic properties on cones used for the construcion in Section 1, in particular the dimension counts.

For points $p_1, \ldots, p_r \in \mathbb{P}^3_{\mathbb{C}}$ we will denote by $\overline{p_1 \ldots p_r}$ the linear span in $\mathbb{P}^3_{\mathbb{C}}$ of $p_1, \ldots, p_r$, i.e. the smallest linear subspace containing $p_1, \ldots, p_r$.

Let $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane, $C \subset P$ a curve, and $p \in \mathbb{P}^3_{\mathbb{C}} \setminus P$ a point. Then we denote by

$$K_{C,p} = \bigcup_{q \in C} \overline{qp}$$

the cone over C with vertex p. Note that

$$K_{C,p} = \bigcup_{q \in K_{C,p}} \overline{qp}$$

and that

$$K_{C,p} \cap P = C.$$

We first show that C and p fix the cone uniquely except when C is a line.

Lemma 3.1
Let $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane, and $C \subseteq P$ be an irreducible curve which is not a line. Then for $p, p' \in \mathbb{P}^3_{\mathbb{C}}$ with $p \neq p'$ we have that $K_{C,p} \neq K_{C,p'}$.

Proof: Suppose there are points $p \neq p'$ such that $K_{C,p} = K_{C,p'}$. Choose a point $x \in C \setminus \overline{pp'}$ and let $E = \overline{xpp'}$. Then for any point $y \in \overline{xp} \subset K_{C,p} = K_{C,p'}$ we have

$$\overline{yp'} \subset K_{C,p'},$$

and thus $E = \bigcup_{y \in \overline{xp}} \overline{yp'} \subset K_{C,p'}$. This, however, implies that the line

$$l = E \cap P \subseteq K_{C,p'} \cap P = C$$

is contained in C, and since C is irreducible we would have $C = l$ in contradiction to our assumption that C is not a line. Hence, $K_{C,p} \neq K_{C,p'}$ for $p \neq p'$. $\square$

Finally we show that for a general p the cone $K_{C,p}$ intersects Σ in an irreducible curve.

Lemma 3.2
Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth projective surface, $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane such that $P \neq \Sigma$, and $C \subseteq P$ an irreducible curve which is not a line and not contained in Σ. Then for $p \in \mathbb{P}^3_{\mathbb{C}} \setminus P$ general $K_{C,p} \cap \Sigma$ is irreducible.

Proof: Consider the linear system $\mathcal{L}$ in $\mathbb{P}^3_{\mathbb{C}}$ which is given as the closure of

$$\left\{K_{C,p} \mid p \in \mathbb{P}^3_{\mathbb{C}} \setminus P\right\},$$

and set for $q \in \mathbb{P}^3_{\mathbb{C}} \setminus P$

$$\mathcal{L}_q = \{D \in \mathcal{L} \mid q \in D\}.$$

First we show that for $q' \in C$ and $q \notin P$

$$\bigcap_{p \in \overline{qq'}} K_{C,p} = C \cup \overline{qq'}. \tag{3.1}$$

Choose pairwise different point $p_1, \ldots, p_n \in \overline{qq'} \setminus \{q, q'\}$. Suppose that there is a $z \in \bigcap_{i=1}^n K_{C,p_i} \setminus (C \cup \overline{qq'})$. Since $z \in K_{C,p_i}$ there is a unique intersection point

$$x_i = \overline{zp_i} \cap C,$$

and these points $x_1, \ldots, x_n$ are pairwise different, since $z \notin \overline{qq'} = \overline{p_i p_j}$ for $i \neq j$. However,

$$x_i \in \overline{zp_i} \subset \overline{zp_ip_j} = \overline{zqq'}$$

and $x_i \in C \subset P$, so that

$$q', x_1, \ldots, x_n \in P \cap \overline{zqq'}$$

and $q', x_1, \ldots, x_n$ are pairwise different collinear points on C. Since C is irreducible but not a line, this implies $\deg(C) \geq n + 1$. In particular, if $n \geq \deg(C)$, then

$$\bigcap_{i=1}^{n} K_{C,p_i} = C \cup \overline{qq'},$$

which implies (3.1).

Note that by (3.1) for $q \in \mathbb{P}^3_{\mathbb{C}} \setminus P$

$$\bigcap_{D \in \mathcal{L}_q} D \subseteq \bigcap_{K_{C,p} \in \mathcal{L}_q} K_{C,p} = \bigcap_{q' \in C} \bigcap_{p \in \overline{qq'}} K_{C,p} = \bigcap_{q' \in C} \left(C \cup \overline{qq'}\right) = C \cup \{q\},$$

and thus

$$\bigcap_{D \in \mathcal{L}} D \subseteq \bigcap_{q \in \mathbb{P}^3_{\mathbb{C}} \setminus P} \bigcap_{D \in \mathcal{L}_q} D = C. \tag{3.2}$$

Consider now the linear systems

$$\mathcal{L}_\Sigma = \{D \cap \Sigma \mid D \in \mathcal{L}\} \quad \text{and} \quad \mathcal{L}_{q,\Sigma} = \{D \cap \Sigma \mid D \in \mathcal{L}_q\} = \{D \in \mathcal{L}_\Sigma \mid q \in D\}.$$

Suppose that $\mathcal{L}_\Sigma$ does not contain any irreducible curve. By (3.2) and since $C \not\subset \Sigma$ the linear system $\mathcal{L}_\Sigma$ has no fixed component. Thus by Bertini's Theorem $\mathcal{L}_\Sigma$ must be composed with a pencil $\mathcal{B}$, and since for a general point $q \in \Sigma$ the pencil $\mathcal{B}$ contains only one element, say $\widetilde{C}$, through q, the linear system $\mathcal{L}_{q,\Sigma}$ has a fixed component $\widetilde{C}$. But then

$$\widetilde{C} \subseteq \bigcap_{D \in \mathcal{L}_q} D \cap \Sigma = C \cap \Sigma.$$

However, $C \cap \Sigma$ is zero-dimensional, while $\widetilde{C}$ has dimension one.

This proves that $\mathcal{L}_\Sigma$ contains an irreducible element, and thus its general element is irreducible. In particular, for $p \in \mathbb{P}^3_{\mathbb{C}} \setminus P$ general $K_{C,p} \cap \Sigma$ is irreducible. $\square$

Part B

PAPER V

A Note on Equimultiple Deformations

Abstract: While the tangent space to an equisingular family of curves can be discribed by the sections of a twisted ideal sheaf, this is no longer true if we only prescribe the multiplicity which a singular point should have. However, it is still possible to compute the dimension of the tangent space with the aid of the equimulitplicity ideal. In this note we consider families $\mathcal{L}_m = \{(C, p) \in |L| \times S \mid \operatorname{mult}_p(C) = m\}$ for some linear system $|L|$ on a smooth projective surface S and a fixed positive integer m, and we compute the dimension of the tangent space to $\mathcal{L}_m$ at a point (C, p) depending on whether p is a unitangential singular point of C or not. We deduce that the expected dimension of $\mathcal{L}_m$ at (C, p) in any case is just $\dim |L| - \frac{m\cdot(m+1)}{2} + 2$. The result is used in the study of triple-point defective surfaces in **[ChM07a]** and **[ChM07b]**.

The paper is based on considerations about the Hilbert scheme of curves in a projective surface (see e.g. [**Mum66**], Lecture 22) and about local equimultiple deformations of plane curves (see [**Wah74b**]).

Definition 1.1
Let T be a complex space. An *embedded family of curves in S with section* over T is a commutative diagram of morphisms

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d, "\varphi"'] & T \times S \arrow[dl] \\
T \arrow[u, bend left=60, "\sigma"]
\end{tikzcd}$$

where $\operatorname{codim}_{T\times S}(\mathcal{C}) = 1$, φ is flat and proper, and σ is a section, i.e. $\varphi \circ \sigma = \mathrm{id}_T$. Thus we have a morphism $\mathcal{O}_T \to \varphi_*\mathcal{O}_\mathcal{C} = \varphi_*\big(\mathcal{O}_{T\times S}/\mathcal{J}_\mathcal{C}\big)$ such that $\varphi_*\mathcal{O}_\mathcal{C}$ is a flat $\mathcal{O}_T$-module.

The family is said to be *equimultiple* of multiplicity m *along the section* σ if the ideal sheaf $\mathcal{J}_\mathcal{C}$ of $\mathcal{C}$ in $\mathcal{O}_{T\times S}$ satisfies

$$\mathcal{J}_\mathcal{C} \subseteq \mathcal{J}_{\sigma(T)}^m \quad \text{and} \quad \mathcal{J}_\mathcal{C} \not\subseteq \mathcal{J}_{\sigma(T)}^{m+1},$$

where $\mathcal{J}_{\sigma(T)}$ is the ideal sheaf of $\sigma(T)$ in $\mathcal{O}_{T\times S}$.

Remark 1.2
Note that the above notion commutes with base change, i.e. if we have an

equimultiple embedded family of curves in S over T as above and if $\alpha : T' \to T$ is a morphism, then the fibre product diagram

$$\begin{tikzcd}
T'\times S \arrow[rrr] \arrow[ddr] & & & T\times S \arrow[ddl] \\
& \mathcal{C}' \arrow[ul, hook] \arrow[r] \arrow[d, "\varphi'"] & \mathcal{C} \arrow[ur, hook] \arrow[d, "\varphi"'] & \\
& T' \arrow[r, "\alpha"'] \arrow[u, bend right=60, "\sigma'"'] & T \arrow[u, bend left=60, "\sigma"] &
\end{tikzcd}$$

gives rise to an embedded equimultiple family of curves over T' of the same multiplicity, since locally it is defined via the tensor product.

Example 1.3
Let us denote by $T_\varepsilon = \operatorname{Spec}(\mathbb{C}[\varepsilon])$ with $\varepsilon^2 = 0$. Then a family of curves in S over T_ε is just a Cartier divisor of $T_\varepsilon \times S$, that is, it is given on a suitable open covering $S = \bigcup_{\lambda\in\Lambda} U_\lambda$ by equations

$$f_\lambda + \varepsilon \cdot g_\lambda \in \mathbb{C}[\varepsilon] \otimes_{\mathbb{C}} \Gamma(U_\lambda, \mathcal{O}_S) = \Gamma(U_\lambda, \mathcal{O}_{T\times S}),$$

which glue together to give a global section $\left\{\frac{g_\lambda}{f_\lambda}\right\}_{\lambda\in\Lambda}$ in $H^0\big(C, \mathcal{O}_C(C)\big)$, where C is the curve defined locally by the f_λ (see e.g. [**Mum66**], Lecture 22).

A section of the family through p is locally in p given as $(x, y) \mapsto (x_a, y_b) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$ for some $a, b \in \mathbb{C}\{x, y\} = \mathcal{O}_{S,p}$.

Example 1.4
Let H be a connected component of the Hilbert scheme Hilb_S of curves in S, then H comes with a universal family

$$\pi : \mathcal{H} \longrightarrow H : (C, p) \mapsto C. \tag{1.1}$$

Let us now fix a positive integer m and set

$$\mathcal{H}_m = \{(C, p) \in H \times S \mid C \in H, \operatorname{mult}_p(C) = m\}.$$

Then $\mathcal{H}_m$ is a locally closed subvariety of $H \times S$, and (1.1) induces via base change a flat and proper family $\mathcal{F}_m = \{(C_p, q) \in \mathcal{H}_m \times S \mid C_p = (C, p) \in \mathcal{H}_m, q \in C\}$ which has a distinguished section σ

$$\begin{tikzcd}
\mathcal{F}_m \arrow[r, hook] \arrow[d] & \mathcal{H}_m \times S \arrow[dl] \\
\mathcal{H}_m \arrow[u, bend left=60, "\sigma"] &
\end{tikzcd} \tag{1.2}$$

sending $C_p = (C, p)$ to $(C_p, p) \in \mathcal{F}_m$. Moreover, this family is equimultiple along σ of multiplicity m by construction.

Example 1.5
Similarly, if $|L|$ is a linear system on S, then it induces a universal family

$$\pi : \mathcal{L} = \{(C, p) \in |L| \times S \mid p \in C\} \longrightarrow |L| : (C, p) \mapsto C. \tag{1.3}$$

If we now fix a positive integer m and set

$$\mathcal{L}_m = \{(C,p) \in |L| \times S \mid C \in |L|, \mathrm{mult}_p(C) = m\}.$$

Then $\mathcal{L}_m$ is a locally closed subvariety of $|L| \times S$, and (1.3) induces via base change a flat and proper family $\mathcal{G}_m = \{(C_p, q) \in \mathcal{L}_m \times S \mid C_p = (C,p) \in \mathcal{L}_m, q \in C\}$ which has a distinguished section σ

$$\begin{tikzcd}
\mathcal{G}_m \arrow[r, hook] \arrow[d] & \mathcal{L}_m \times S \arrow[dl] \\
\mathcal{L}_m \arrow[u, bend left=60, "\sigma"] &
\end{tikzcd} \tag{1.4}$$

sending $C_p = (C,p)$ to $(C_p, p) \in \mathcal{G}_m$. Moreover, this family is equimultiple along σ of multiplicity m by construction.

We may interpret $\mathcal{L}_m$ as the family of curves in $|L|$ with m-fold points together with a section which distinguishes the m-fold point. This is important if the m-fold point is not isolated or if it splits in a neighbourhood into several simpler m-fold points.

Of course, since (1.3) can be viewed as a subfamily of (1.1) we may view (1.4) in the same way as a subfamily of (1.2).

Definition 1.6
Let $t_0 \in T$ be a pointed complex space, $C \subset S$ a curve, and $p \in C$ a point of multiplicity m. Then an *embedded (equimultiple) deformation of* C *in* S *over* $t_0 \in T$ *with section* σ *through* p is a commutative diagram of morphisms

$$\begin{tikzcd}
& S \arrow[rr, hook] \arrow[ddl] & & T \times S \arrow[ddl] \\
p \in C \arrow[ur, hook] \arrow[rr, hook] \arrow[d] & & \mathcal{C} \arrow[ur, hook] \arrow[d, "\varphi"'] & \\
t_0 \arrow[u, bend left=60, "\sigma"] \arrow[rr, mapsto] & & T \arrow[u, bend left=60, "\sigma"] &
\end{tikzcd}$$

where the right hand part of the diagram is an embedded (equimultiple) family of curves in S over T with section σ. Sometimes we will simply write (φ, σ) to denote a deformation as above.

Given two deformations, say (φ, σ) and (φ', σ'), of C over $t_0 \in T$ as above, a morphism of these deformations is a morphism $\psi : \mathcal{C}' \to \mathcal{C}$ which makes the obvious diagram commute:

$$\begin{tikzcd}
T\times S \arrow[rrrr, equal] \arrow[ddddr] & & & & T\times S \arrow[ddddl] \\
& \mathcal{C}' \arrow[ul, hook] \arrow[rr, "\psi"] \arrow[ddd, "\varphi'"'] & & \mathcal{C} \arrow[ur, hook] \arrow[ddd, "\varphi"] & \\
& & C \arrow[ul, hook] \arrow[ur, hook] \arrow[d] & & \\
& & t_0 \arrow[dl] \arrow[dr] & & \\
& T \arrow[uuu, bend right, "\sigma'"'] \arrow[rr, equal] & & T \arrow[uuu, bend left, "\sigma"] &
\end{tikzcd}$$

This gives rise to the deformation functor

$$\underline{\mathrm{Def}}^{sec,em}_{p\in C/S} : (\text{pointed complex spaces}) \to (\text{sets})$$

of embedded equimultiple deformations of C with section through p from the category of pointed complex spaces into the category of sets, where for a pointed complex space $t_0 \in T$

$$\begin{aligned}\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(t_0 \in T) = \{&\text{isomorphism classes of embedded equimultiple}\\ &\text{deformations } (\varphi,\sigma) \text{ of } C \text{ in } S \text{ over } t_0 \in T\\ &\text{with section through } p\}.\end{aligned}$$

Moreover, forgetting the section we have a natural transformation

$$\underline{\mathrm{Def}}^{sec,em}_{p\in C/\Sigma} \longrightarrow \underline{\mathrm{Def}}_{C/\Sigma}, \tag{1.5}$$

where the latter is the deformation functor

$$\underline{\mathrm{Def}}_{C/\Sigma} : (\text{pointed complex spaces}) \to (\text{sets})$$

of embedded deformations of C in S given by

$$\begin{aligned}\underline{\mathrm{Def}}_{C/S}(t_0 \in T) = \{&\text{isomorphism classes of embedded deformations}\\ &\text{of } C \text{ in } S \text{ over } t_0 \in T\ \}.\end{aligned}$$

Example 1.7
According to Example 1.3 a deformation of C in S over T_ε along a section through p is given by

- local equations $f + \varepsilon \cdot g$ such that f is a local equation for C and the $\frac{g}{f}$ glue to a global section of $\mathcal{O}_C(C)$,
- together with a section which in local coordinates in p is given as $\sigma : (x, y) \mapsto (x_a, y_b) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$ for some $a, b \in \mathbb{C}\{x, y\}$.

If we forget the section it is well known (see e.g. [**Mum66**], Lecture 22) that two such deformations are isomorphic if and only if they induce the same global section of $\mathcal{O}_C(C)$ and this one-to-one correspondence is functorial so that we have an isomorphism of vector spaces

$$\underline{\mathrm{Def}}_{C/S}(T_\varepsilon) \xrightarrow{\cong} H^0\big(C, \mathcal{O}_C(C)\big).$$

Considering the natural transformation from (1.5) we may now ask what the image of $\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon)$ in $H^0\big(C, \mathcal{O}_C(C)\big)$ is. These are, of course, the sections which allow a section σ through p along which the deformation is equimultiple, and according to Lemma 1.8 we thus have an epimorphism

$$\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon) \twoheadrightarrow H^0\big(C, \mathcal{J}_{Z/C}(C)\big),$$

where $\mathcal{J}_{Z/C}$ is the restriction to C of the ideal sheaf $\mathcal{J}_Z$ on S given by

$$\mathcal{J}_{Z,q} = \begin{cases} \mathcal{O}_{S,q}, & \text{if } q \neq p, \\ \big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \big\rangle + \langle x,y\rangle^m, & \text{if } q = p, \end{cases} \tag{1.6}$$

here f is a local equation for C in local coordinates x and y in p.

It remains the question what the dimension of the kernel of this map is, that is, how many different sections such an isomorphism class of embedded deformations of C in S over T_ε through p can admit.

J. Wahl showed in [**Wah74b**], Proposition 1.9, that locally the equimultiple deformation admits a unique section if and only if C in p is not unitangential. If C is unitangential we may assume that locally in p it is given by $f = y^m + h.o.t.$. If we have an embedded deformation of C in S which along some section is equimultiple of multiplicity m, then locally it looks like

$$f + \varepsilon \cdot \Big(a \cdot \tfrac{\partial f}{\partial x} + b \cdot \tfrac{\partial f}{\partial y} + h \Big)$$

with $h \in \langle x,y\rangle^m$. However, since $\frac{\partial f}{\partial x} \in \langle x,y\rangle^m$ the deformation is equimultiple along the sections $(x,y) \mapsto (x+\varepsilon\cdot(c+a), y+\varepsilon\cdot b)$ for all $c \in \mathbb{C}$. Thus in this case the kernel turns out to be one-dimensional, i.e. there is a one-dimensional vector space $\mathcal{K}$ such that the following sequence is exact:

$$0 \to \mathcal{K} \to \underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon) \to H^0\big(C, \mathcal{J}_{Z/C}(C)\big) \to 0. \tag{1.7}$$

Lemma 1.8

Let $f+\varepsilon\cdot g$ be a first-order infinitesimal deformation of $f \in \mathbb{C}\{x,y\}$, $m = \mathrm{ord}(f)$, $a, b \in \mathbb{C}\{x,y\}$, and $x_a = x + \varepsilon \cdot a$, $y_b = y + \varepsilon \cdot b$.

Then $f + \varepsilon \cdot g$ is equimultiple along the section $(x,y) \mapsto (x_a, y_b)$ if and only if

$$g - a \cdot \frac{\partial f}{\partial x} - b \cdot \frac{\partial f}{\partial y} \in \langle x,y\rangle^m.$$

In particular, $f + \varepsilon \cdot g$ is equimultiple along some section if and only if

$$g \in \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle + \langle x, y\rangle^m.$$

Proof: If $a, b \in \mathbb{C}\{x, y\}$ and $h \in \langle x, y\rangle^m$ then by Taylor expansion and since $\varepsilon^2 = 0$ we have

$$f + \varepsilon \cdot \left(a \cdot \tfrac{\partial f}{\partial x} + b \cdot \tfrac{\partial f}{\partial y} + h\right) = f(x_a, y_b) + \varepsilon \cdot h(x_a, y_b),$$

where $f(x_a, y_b), h(x_a, y_b) \in \langle x_a, y_b\rangle^m$, i.e. the infinitesimal deformation $f + \varepsilon \cdot \big(a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h\big)$ is equimultiple along $(x, y) \mapsto (x_a, y_b)$.

Conversely, if $f + \varepsilon \cdot g$ is equimultiple along $(x, y) \mapsto (x_a, y_b)$ then

$$f(x, y) + \varepsilon \cdot g(x, y) = F(x_a, y_b) + \varepsilon \cdot G(x_a, y_b)$$

with $F(x_a, y_b), G(x_a, y_b) \in \langle x_a, y_b\rangle^m$. Again, by Taylor expansion and since $\varepsilon^2 = 0$ we have

$$f(x, y) = f(x_a, y_b) - \varepsilon \cdot \left(a \cdot \tfrac{\partial f}{\partial x}(x_a, y_b) + b \cdot \tfrac{\partial f}{\partial y}(x_a, y_b)\right)$$

and

$$\varepsilon \cdot g(x, y) = \varepsilon \cdot g(x_a, y_b).$$

Thus

$$F(x_a, y_b) = f(x_a, y_b)$$

and

$$\langle x_a, y_b\rangle^m \ni G(x_a, y_b) = g(x_a, y_b) - a \cdot \tfrac{\partial f}{\partial x}(x_a, y_b) - b \cdot \tfrac{\partial f}{\partial y}(x_a, y_b).$$

□

Example 1.9
If we fix a curve $C \subset S$ and a point $p \in C$ such that $\mathrm{mult}_p(C) = m$, i.e. if using the notation of Example 1.4 we fix a point $C_p = (C, p) \in \mathcal{H}_m$, then the diagram

$$\begin{tikzcd}
S \arrow[r, "\cong"] & \{C_p\} \times S \arrow[rr, hook] & & \mathcal{H}_m \times S \arrow[ddl] \\
C \arrow[u, hook] \arrow[d] \arrow[r, "\cong"] & \{(C_p, q) \mid q \in C\} \arrow[u, hook] \arrow[d] \arrow[r, hook] & \mathcal{F}_m \arrow[ur, hook] \arrow[d] & \\
t_0 \arrow[r, equal] & C_p \arrow[r, mapsto] & \mathcal{H}_m \arrow[u, bend left, "\sigma"] &
\end{tikzcd} \tag{1.8}$$

is an embedded equimultiple deformation of C in S along the section σ through p. Moreover, any embedded equimultiple deformation of C in S with section through p as a family is up to isomorphism induced via (1.1) in a unique way and thus factors obviously uniquely through (1.8). This means that every equimultiple deformation of C in S through p is induced up to isomorphism in a unique way from (1.8).

We now want to examine the tangent space to $\mathcal{H}_m$ at a point $C_p = (C, p)$, which is just

$$T_{C_p}(\mathcal{H}_m) = \operatorname{Hom}_{loc-K-Alg}\big(\mathcal{O}_{\mathcal{H}_m,C_p}, \mathbb{C}[\varepsilon]\big) = \operatorname{Hom}\big(T_\varepsilon, (\mathcal{H}_m, C_p)\big),$$

where $(\mathcal{H}_m, C_p)$ denotes the germ of $\mathcal{H}_m$ at C_p. However, a morphism

$$\psi : T_\varepsilon \longrightarrow (\mathcal{H}_m, C_p)$$

gives rise to a commutative fibre product diagram

$$\begin{tikzcd} T_\varepsilon \times_{\mathcal{H}_m} \mathcal{F}_m \arrow[r] \arrow[d, "\varphi'"] & \mathcal{F}_m \arrow[d] \\ T_\varepsilon \arrow[r, "\psi"] \arrow[u, bend left, "\sigma'"] & \mathcal{H}_m \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

sending the closed point of T_ε to C. Thus $(\varphi', \sigma') \in \underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon)$ is an *embedded equimultiple deformation of* C *in* S *with section through* p. The universality of (1.8) then implies that up to isomorphism each one is of this form for a unique φ', and this construction is functorial. We thus have

$$T_{C_p}(\mathcal{H}_m) \cong \underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon),$$

and hence (1.7) gives the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow T_{C_p}(\mathcal{H}_m) \longrightarrow H^0\big(C, \mathcal{J}_{Z/C}(C)\big) \longrightarrow 0.$$

In particular,

$$\dim_{\mathbb{C}}\big(T_{C_p}(\mathcal{H}_m)\big) = \begin{cases} \dim_{\mathbb{C}} H^0\big(C, \mathcal{J}_{Z/C}(C)\big) - 2, & \text{if } C \text{ is unitangential}, \\ \dim_{\mathbb{C}} H^0\big(C, \mathcal{J}_{Z/C}(C)\big) - 1, & \text{else}. \end{cases}$$

Example 1.10

If we do the same constructions replacing in (1.8) the family (1.2) by (1.4) we get for the tangent space to $\mathcal{L}_m$ at $C_p = (C, p)$ the diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & T_{C_p}(\mathcal{H}_m) \arrow[r] & H^0\big(C, \mathcal{J}_{Z/C}(C)\big) \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & T_{C_p}(\mathcal{L}_m) \arrow[r] \arrow[u, hook] & H^0\big(S, \mathcal{J}_Z(C)\big)/H^0(\mathcal{O}_S) \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

In order to see this consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

induced from the structure sequence of C. This sequence shows that the tangent space to $|L|$ at C considered as a subspace of the tangent space $H^0\big(C, \mathcal{O}_C(C)\big)$ of H at C is just $H^0\big(S, \mathcal{O}_S(C)\big)/H^0(S, \mathcal{O}_S)$ – that is, a global section of $\mathcal{O}_C(C)$ gives rise to an embedded deformation of C in S which is actually a deformation in the linear system $|L|$ if and only if it comes from a global section of $\mathcal{O}_S(C)$, and the constant sections induce the trivial deformations. This construction carries over to the families (1.2) and (1.4).

In particular we get the following proposition.

Proposition 1.11
Using the notation from above let C be a curve in the linear system $|L|$ on S and suppose that $p \in C$ such that $\operatorname{mult}_p(C) = m$.

Then the tangent space of $\mathcal{L}_m$ at $C_p = (C, p)$ satisfies

$$\dim_{\mathbb{C}}\big(T_{C_p}(\mathcal{L}_m)\big) = \begin{cases} \dim_{\mathbb{C}} H^0\big(S, \mathcal{J}_Z(C)\big) - 2, & \textit{if } C \textit{ is unitangential}, \\ \dim_{\mathbb{C}} H^0\big(S, \mathcal{J}_Z(C)\big) - 1, & \textit{else}. \end{cases}$$

Moreover, the expected dimension of $T_{C_p}(\mathcal{L}_m)$ and thus of $\mathcal{L}_m$ at C_p is just

$$\operatorname{expdim}_{C_p}(\mathcal{L}_m) = \operatorname{expdim}_{\mathbb{C}}\big(T_{C_p}(\mathcal{L}_m)\big) = \dim|L| - \frac{(m+1)\cdot m}{2} + 2.$$

For the last statement on the expected dimension just consider the exact sequence

$$0 \to H^0\big(S, \mathcal{J}_Z(C)\big) \to H^0\big(S, \mathcal{O}_S(L)\big) \to H^0(S, \mathcal{O}_Z)$$

and note that the dimension of $H^0\big(S, \mathcal{J}_Z(C)\big)$, and hence of $T_{C_p}(C)$, attains the minimal possible value if the last map is surjective. The expected dimension of $H^0\big(S, \mathcal{J}_Z(C)\big)$ hence is

$$\operatorname{expdim}_{\mathbb{C}} H^0\big(S, \mathcal{J}_Z(C)\big) = \dim|L| + 1 - \deg(Z),$$

and it suffices to calculate $\deg(Z)$. If C is unitangential we may assume that C locally in p is given by $f = y^m + h.o.t.$, so that

$$\mathcal{O}_{Z,p} = \mathbb{C}\{x, y\}/\langle y^{m-1}\rangle + \langle x, y\rangle^m,$$

and hence $\deg(Z) = \frac{(m+1)\cdot m}{2} - 1$. If C is not unitangential, then we may assume that it locally in p is given by an equation f such that $f_m = \operatorname{jet}_m(f) = x^\mu \cdot y^\nu \cdot g$, where x and y do not divide g, but μ and ν are at least one. Suppose now that the partial derivatives of f_m are not linearly independent, then we may assume $\frac{\partial f_m}{\partial x} \equiv \alpha \cdot \frac{\partial f_m}{\partial y}$ and thus

$$\mu y g \equiv \alpha\nu x g + \alpha x y \cdot \frac{\partial g}{\partial y} - x y \cdot \frac{\partial g}{\partial x},$$

which would imply that y divides g in contradiction to our assumption. Thus the partial derivatives of f_m are linearly independent, which shows that

$$\deg(Z) = \dim_{\mathbb{C}}\Big(\mathbb{C}\{x, y\}/\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\rangle + \langle x, y\rangle^m\Big) = \frac{(m+1)\cdot m}{2} - 2.$$

Example 1.12
Let us consider the Example 1.5 in the case where $S = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(d)$. We will show that $\mathcal{L}_m$ is then *smooth of the expected dimension*. Note that

$\pi(\mathcal{L}_m)$ will only be smooth at C if C has an ordinary m-fold point, that is, if all tangents are different.

Given $C_p = (C,p) \in \mathcal{L}_m$ we may pass to a suitable affine chart containing p as origin and assume that $H^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\big)$ is parametrised by polynomials

$$F_{\underline{a}} = f + \sum_{i+j=0}^{d} a_{i,j} \cdot x^i y^j,$$

where f is the equation of C in this chart. The closure of $\pi(\mathcal{L}_m)$ in $|L|$ locally at C is then given by several equations, say $F_1, \ldots, F_k \in \mathbb{C}[a_{i,j} | i+j = 0, \ldots, d]$, in the coefficients $a_{i,j}$. We get these equations by eliminating the variables x and y from the ideal defined by

$$\left\langle \frac{\partial^{i+j} F_{\underline{a}}}{\partial x^i y^j} \;\middle|\; i+j = 0, \ldots, m-1 \right\rangle.$$

And $\mathcal{L}_m$ is locally in C_p described by the equations

$$F_1 = 0, \ldots, F_k = 0, \quad \frac{\partial^{i+j} F_{\underline{a}}}{\partial x^i y^j} = 0, \quad i+j = 0, \ldots, m-1.$$

However, the Jacoby matrix of these equations with respect to the variables $x, y, a_{i,j}$ contains a diagonal submatrix of size $\frac{m\cdot(m+1)}{2}$ with ones on the diagonal, so that its rank is at least $\frac{m\cdot(m+1)}{2}$, which – taking into account that $|L| = \mathbb{P}\big(H^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\big)\big)$ – implies that the tangent space to $\mathcal{L}_m$ at C_p has codimension at least $\frac{m\cdot(m+1)}{2} - 1$ in the tangent space of $\mathcal{L}$. By Proposition 1.11 we thus have

$$\begin{aligned}
\dim_{C_p}(\mathcal{L}_m) \leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}_m) &\leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}) - \frac{m\cdot(m+1)}{2} + 1 \\
&= \dim(\mathcal{L}) - \frac{m\cdot(m+1)}{2} + 1 \\
&= \dim|L| - \frac{m\cdot(m+1)}{2} + 2 \\
&= \operatorname{expdim}_{C_p}(\mathcal{L}_m) \leq \dim_{C_p}(\mathcal{L}_m),
\end{aligned}$$

which shows that $\mathcal{L}_m$ is smooth at C_p of the expected dimension.

PAPER VI

Triple-Point Defective Regular Surfaces

Abstract: In this paper we study the linear series $|L - 3p|$ of hyperplane sections with a triple point p of a surface Σ embedded via a very ample line bundle L for a *general* point p. If this linear series does not have the expected dimension we call (Σ, L) *triple-point defective*. We show that on a triple-point defective *regular* surface through a general point every hyperplane section has either a triple component or the surface is rationally ruled and the hyperplane section contains twice a fibre of the ruling.

This paper is a joint work with Luca Chiantini, Siena, [**ChM07a**].

1. Introduction

Throughout this note, Σ will be a smooth projective surface, $K = K_\Sigma$ will denote the canonical class and L will be a divisor class on Σ such that L and $L - K$ are both *very ample*.

The classical *interpolation problem* for the pair (Σ, L) is devoted to the study of the varieties:

$$V^{gen}_{m_1,\ldots,m_n} = \big\{C \in |L| \;\big|\; p_1, \ldots, p_n \in \Sigma \text{ general},\ \operatorname{mult}_{p_i}(C) \geq m_i\big\}.$$

In a more precise formulation, we start from the incidence variety:

$$\mathcal{L}_{m_1,\ldots,m_n} = \{(C, (p_1, \ldots, p_n)) \in |L| \times \Sigma^n \mid \operatorname{mult}_{p_i}(C) \geq m_i\}$$

together with the canonical projections:

$$\begin{array}{ccc} \mathcal{L}_{m_1,\ldots,m_n} & \xrightarrow{\alpha} & \Sigma^n \\ \downarrow{\scriptstyle\beta} & & \\ |L| = \mathbb{P}_{\mathbb{C}}(H^0(L)^*) & & \end{array} \tag{1.1}$$

As for the map α, the fibre over a fixed point $(p_1, \ldots, p_n) \in \Sigma^n$ is just the linear series $|L - m_1p_1 - \cdots - m_np_n|$ of effective divisors in $|L|$ having a point of multiplicity at least m_i at p_i. These fibres being irreducible, we deduce that if α is *dominant* then $\mathcal{L}_{m_1,\ldots,m_n}$ has a unique irreducible component, say $\mathcal{L}^{gen}_{m_1,\ldots,m_n}$, which dominates Σ. The closure of its image

$$V_{m_1,\ldots,m_n} := V_{m_1,\ldots,m_n}(\Sigma, L) := \overline{\beta(\mathcal{L}^{gen}_{m_1,\ldots,m_n})}$$

under β is an irreducible closed subvariety of $|L|$, a *Severi variety* of (Σ, L).

Imposing a point of multiplicity m_i corresponds to killing $\binom{m_i+1}{2}$ partial derivatives, so that

$$\dim |L - m_1 p_1 - \cdots - m_n p_n| \geq \max \left\{ -1, \dim |L| - \sum_{i=1}^{n} \binom{m_i + 1}{2} \right\},$$

and we expect that the previous inequality is in fact an equality, for the choice of general points $p_1, \ldots, p_n \in \Sigma$.

When this is not the case, then the surface is called *defective* and is endowed with some special structure.

The case when $m_i = 2$ for all i has been classically considered (and solved) by Terracini, who classified in **[Ter22]** double–point defective surfaces. In any event, the first example of such a defective surface which is smooth is the Veronese surface, for which $n = 2$.

It is indeed classical that imposing multiplicity two at a general point to a very ample line bundle $|L|$ always yields three independent conditions, so that $\dim |L - 2p| = \dim |L| - 3$ and the corresponding Severi variety has codimension 1 in $|L|$.

Furthermore, when Σ is double-point defective, then any general curve $C \in |L - 2p_1 - \cdots - 2p_n|$ has a double component passing through each point p_i.

When the multiplicities grow, the situation becomes completely different. Even in the case $\Sigma = \mathbb{P}_{\mathbb{C}}^2$, the situation is not understood and there are several, still unproved conjecture on the structure of defective embeddings (see **[Cil01]** for an introductory survey).

When Σ is a more complicated surface, it turns out that even imposing just one point of multiplicity 3, one may expect to obtain a defective behaviour.

Example 1.1
Let $\Sigma = \mathbb{F}_e \xrightarrow{\pi} \mathbb{P}_{\mathbb{C}}^1$ be a Hirzebruch surface, $e \geq 0$. We denote by F a fibre of π and by C_0 the section of π of minimal self intersection $C_0^2 = -e$ – both of which are smooth rational curves. The general element C_1 in the linear system $|C_0 + eF|$ will be a section of π which does not meet C_0 (see e.g. **[FuP00]**, Theorem 2.5).

Consider now the divisor $L = 2 \cdot F + C_1 = (2 + e) \cdot F + C_0$. Then for a general $p \in \Sigma$ there are curves $C_p \in |C_1 - p|$ and there is a unique curve $F_p \in |F - p|$, in particular $p \in F_p \cap C_p$. For each choice of C_p we have

$$2F_p + C_p \in |L - 3p|.$$

Since $F.L = 1 = F.(L - F)$ we see that every curve in $|L - 3p|$ must contain F_p as a double component, i.e.

$$|L - 3p| = 2F_p + |C_1 - p|.$$

Moreover, since $p \in \Sigma$ is general we have (see [**FuP00**], Lemma 2.10)

$$\dim |C_1 - p| = \dim |C_1| - 1 = h^0\big(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}\big) + h^0\big(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(e)\big) - 2 = e$$

and, using the notation from above,

$$\dim(V_3) \geq \dim |C_1 - p| + 2 = e + 2.$$

However,

$$\dim |L| = h^0\big(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(2)\big) + h^0\big(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(2+e)\big) - 1 = e + 5,$$

and thus

$$\operatorname{expdim}(V_3) = \dim |L| - 4 = e + 1 < e + 2 = \dim(V).$$

We say, $(\mathbb{F}_e, L)$ is *triple-point defective*, see Definition 1.2.

Note, moreover, that

$$(L - K)^2 = (4F + 3C_0)^2 = 24 > 16.$$

$\square$

It is interesting to observe that, even though, in the previous example, the general element of $|L - 3p|$ is non reduced, still the map β of Diagram (1.1) has finite general fibers, since the general element of $|L - 3p|$ has no triple components.

The aim of this note is to investigate the structure of pairs (Σ, L) for which the linear system $|L - 3p|$ for $p \in \Sigma$ general has dimension bigger that the expected value $\dim |L| - 6$, or equivalently, the variety $\mathcal{L}_3^{gen}$, defined as in Diagram (1), has dimension bigger than $\dim |L| - 4$.

Definition 1.2
We say that the pair (Σ, L) is *triple-point defective* or, in classical notation, that (Σ, L) *satisfies one Laplace equation* if

$$\dim |L - 3p| > \max\{-1, \dim |L| - 6\} = \operatorname{expdim} |L - 3p|$$

for $p \in \Sigma$ general.

Remark 1.3
Going back to Diagram (1), one sees that (Σ, L) is triple-point defective if and only if either:

- $\dim |L| \leq 5$ and the projection $\alpha : \mathcal{L}_3 \to \Sigma$ dominates, or
- $\dim |L| > 5$ and the general fibre of the map α has dimension at least $\dim |L| - 5$.

In particular, (Σ, L) is triple-point defective if and only if the map α is *dominant* and

$$\dim(\mathcal{L}_3^{gen}) > \dim|L| - 4.$$

The particular case in which the general fiber of the map β in Diagram (1) is positive-dimensional, (i.e. the general member of V_3 contains a triple component through p) has been investigated in [**Cas22**], [**FrI01**], and [**BoC05**]. We will recall the classification of such surfaces in Theorem 2.2 below.

Even when β is generically finite, one of the major subjects in algebraic interpolation theory, namely Segre's conjecture on defective linear systems *in the plane*, says in our situation that, when (Σ, L) is triple-point defective, then the general element of $|L-3p|$ must be non-reduced, with a double component through p (exactly as in the case of Hirzebruch surfaces).

We are able to show, under some assumptions, that this part of Segre's conjecture holds, even in the more general setting of *regular* surfaces.

Indeed our main result is:

Theorem 1.4
Let Σ be a regular *surface, and suppose that with the notation in* (1.1) *α is dominant. Let L be a very ample line bundle on Σ, such that $L-K$ is also very ample. Assume $(L-K)^2 > 16$ and (Σ, L) is triple-point defective.*

Then Σ is rationally ruled in the embedding defined by L. Moreover a general curve $C \in |L-3p|$ contains the fibre of the ruling through p as fixed component with multiplicity at least two.

Remark 1.5
In a forthcoming paper [**ChM07b**] we classify all triple-point defective linear systems L on ruled surfaces satisfying the assumptions of Theorem 1.4, and it follows from this classification that the linear system $|L-3p|$ will contain the fibre of the ruling through p precisely with multiplicity two as a fixed component. In particular, the map β will automatically be generically finite.

Our method is based on the observation that, when (Σ, L) is triple-point defective, then at a general point $p \in \Sigma$ there exists a non-reduced scheme Z_p supported at the point, such that

$$h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

By Serre's construction, this yields the existence of a rank 2 bundle $\mathcal{E}_p$ with first Chern class $L-K$, with a global section whose zero-locus is a subscheme of length at most 4, supported at p. Moreover the assumption $(L-K)^2 > 16$ implies that $\mathcal{E}_p$ is *Bogomolov unstable*, thus it has a destabilizing divisor A. By exploiting the properties of A and $B = L-K-A$, we obtain the result.

In a sort of sense, one of the main points missing for the proof of Segre's conjecture is a natural geometric construction for the non–reduced divisor which must be part of any defective linear system.

For double-point defective surfaces, the non–reduced component comes from contact loci of hyperplanes (see [**ChC02**]).

In our setting, the non–reduced component is essentially given by the effective divisor B above, which comes from a destabilizing divisor of the rank 2 bundle.

The result, applied to the blowing up of $\mathbb{P}_{\mathbb{C}}^{2}$, leads to the following partial proof of Segre's conjecture on defective linear systems in the plane.

Corollary 1.6
Fix multiplicities $m_1 \leq m_2 \leq \cdots \leq m_n$. Let H denote the class of a line in $\mathbb{P}_{\mathbb{C}}^{2}$ and assume that, for $p_1, \ldots, p_n$ general in $\mathbb{P}_{\mathbb{C}}^{2}$, the linear system $M = rH - m_1p_1 - \cdots - m_np_n$ is defective, i.e.

$$\dim |M| > \max \left\{ -1, \binom{r+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} \right\}.$$

Let $\pi : \Sigma \longrightarrow \mathbb{P}_{\mathbb{C}}^{2}$ be the blowing up of $\mathbb{P}_{\mathbb{C}}^{2}$ at the points $p_2, \ldots, p_n$ and set $L := r\pi^*H - m_2E_2 - \cdots - m_nE_n$, where $E_i = \pi^*(p_i)$ is the i-th exceptional divisor. Assume that L is very ample on Σ, of the expected dimension $\binom{r+2}{2} - \sum_{i=2}^{n} \binom{m_i+1}{2}$, and that also $L - K$ is very ample on Σ, with $(L-K)^2 > 16$. Assume, finally, $m_1 \leq 3$.

Then $m_1 = 3$ and the general element of M is non-reduced. Moreover L embeds Σ as a ruled surface.

Proof: Just apply the Main Theorem 1.4 to the pair (Σ, L). □

The reader can easily check that the previous result is exactly the translation of Segre's and Harbourne–Hirschowitz's conjectures on defective linear systems in the plane, for the case of a *minimally* defective system with lower multiplicity 3. The (-1)–curve predicted by Harbourne–Hirschowitz conjecture, in this situation, is just the pull-back of a line of the ruling.

Although the conditions "L and $L - K$ very ample" is not mild, we believe that the previous result could strengthen our believe in the general conjecture. Combining results in [**Xu95**] and [**Laz97**] Corollary. 2.6 one can give numerical conditions on r and the m_i such that L respectively $L - K$ are very ample.

The paper is organized as follows.

The case where β is not generically finite is pointed out in Theorem 2.2 in Section 2. In Section 3 we reformulate the problem as an h^1-vanishing problem. The Sections 4 to 7 are devoted to the proof of the main result: in Section 4 we use Serre's construction and Bogomolov instability in order to show that triple-point defectiveness leads to the existence of very special divisors A and B on our surface; in Section 5 we show that $|B|$ has no fixed component; in Section 6 we then list properties of B and we use these in Section 7 to classify the regular triple-point defective surfaces.

2. Triple components

In this section, we consider what happens when, in Diagram (1), the general fiber of β is positive-dimensional, in other words, when the general member of V_3 contains a triple component through p.

This case has been investigated (and essentially solved) in [**Cas22**], and then rephrased in modern language in [**FrI01**] and [**BoC05**].

Although not strictly necessary for the sequel, as our arguments do not make any use of the generic finiteness of β, (and so we will not assume this), for the sake of completeness we recall in this section some example and the classification of pairs (Σ, L) which are triple-point defective, and such that a general curve $L_p \in |L - 3p|$ has a triple component through p.

The family $\mathcal{L}_3$ of pairs $(L, p) \in |L| \times \Sigma$ where $L \in |L - 3p|$ has dimension bounded below by $\dim |L| - 4$, and in Remark 1.3 it has been pointed out that (Σ, L) is triple-point defective exactly when α is dominant and the bound is not attained.

Notice however that $\dim |L| - 4$ is *not* necessarily a bound for the dimension of the subvariety $V_3 \subset |L|$, the image of $\mathcal{L}_3$ under β. The following example (exploited in [**LaM02**]) shows that one may have $\dim(V_3) < \dim |L| - 4$ even when (Σ, L) is *not* triple-point defective.

Example 2.1 ((see [**LaM02**]))
Let Σ be the blowing up of $\mathbb{P}_{\mathbb{C}}^2$ at 8 general points $q_1, \ldots, q_8$ and L corresponds to the system of curves of degree nine in $\mathbb{P}_{\mathbb{C}}^2$, with a triple point at each q_i.

$\dim |L| = 6$, but for $p \in \Sigma$ general, the unique divisor in $|L - 3p|$ coincides with the cubic plane curve through $q_1, \ldots, q_8, p$, counted three times. As there exists only a (non-linear) 1-dimensional family of such divisors in $|L|$, then $\dim(V_3) = 1 < \dim |L| - 4$. On the other hand, these divisors have a triple component, so that the general fibre of β has dimension 1, hence $\dim(\mathcal{L}_3) = 2 = \dim |L| - 4$.

The classification of triple-point defective pairs (Σ, L) for which the map β is not generically finite is the following.

Theorem 2.2
Suppose that (Σ, L) is triple-point defective. Then for $p \in \Sigma$ general, the general member of $|L - 3p|$ contains a triple component through p if and only if Σ lies

in a threefold W which is a scroll in planes and moreover W is developable, i.e. the tangent space to W is constant along the planes.

Proof: (HINT) First, since we assume that Σ is triple-point defective and embedded in $\mathbb{P}_{\mathbb{C}}^r$ via L, then the hyperplanes π that meet Σ in a divisor $H = \Sigma \cap \pi$ with a triple point at a general $p \in \Sigma$, intersect in a $\mathbb{P}_{\mathbb{C}}^4$. Thus we may project down Σ to $\mathbb{P}_{\mathbb{C}}^5$ and work with the corresponding surface.

In this setting, through a general $p \in \Sigma$ one has only one hyperplane π with a triple contact, and π has a triple contact with Σ along the fibre C of β. Thus V_3 is a curve.

If H', H'' are two consecutive infinitesimally near points to H on V_3, then C also belongs to $H \cap H' \cap H''$. Thus C is a plane curve and Σ is fibred by a 1-dimensional family of plane curves. This determines the threefold scroll W.

The tangent line to V_3 determines in $(\mathbb{P}_{\mathbb{C}}^5)^*$ a pencil of hyperplanes which are tangent to Σ at any point of C, since this is the infinitesimal deformation of a family of hyperplanes with a triple contact along any point of C. Thus there is a $\mathbb{P}_{\mathbb{C}}^4 = H_C$ which is tangent to Σ along C.

Assume that C is not a line. Then C spans a $\mathbb{P}_{\mathbb{C}}^2 = \pi_C$ fibre of W, moreover the tangent space to W at a general point of C is spanned by π_C and $T_{\Sigma,P}$, hence it is constantly equal to H_C. Since C spans π_C, then it turns out that the tangent space to W is constant at any point of π_C, i.e. W is developable.

When C is a line, then arguing as above one finds that all the tangent planes to Σ along C belong to the same $\mathbb{P}_{\mathbb{C}}^3$. This is enough to conclude that Σ sits in some developable 3-dimensional scroll.

Conversely, if Σ is contained in the developable scroll W, then at a general point p, with local coordinates x, y, the tangent space t to W at p contains the derivatives $p, p_x, p_y, p_{xx}, p_{xy}$ (here x is the direction of the tangent line to C). Thus the $\mathbb{P}_{\mathbb{C}}^4$ spanned by t, p_{yy} intersects Σ in a triple curve along C. $\square$

3. The Equimultiplicity Ideal

If L_p is a curve in $|L - 3p|$ we denote by $f_p \in \mathbb{C}\{x_p, y_p\}$ an equation of L_p in local coordinates x_p and y_p at p. If $\mathrm{mult}_p(L_p) = 3$, the ideal sheaf $\mathcal{J}_{Z_p}$ whose stalk at p is the equimultiplicity ideal

$$\mathcal{J}_{Z_p,p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p \rangle^3$$

of f_p defines a zero-dimensional scheme $Z_p = Z_p(L_p)$ concentrated at p, and the tangent space $T_{(L_p,p)}(\mathcal{L}_3)$ of $\mathcal{L}_3$ at (L_p, p) satisfies (see **[Mar06]** Example 10)

$$T_{(L_p,p)}(\mathcal{L}_3) \cong \big(H^0\big(\Sigma, \mathcal{J}_{Z_p}(L_p)\big)/H^0(\Sigma, \mathcal{O}_\Sigma)\big) \oplus \mathcal{K},$$

where $\mathcal{K}$ is zero unless L_p is unitangential at p, in which case $\mathcal{K}$ is a one-dimensional vector space.

In particular, $\mathcal{L}_3$ is smooth at (L_p, p) of the expected dimension (see **[Mar06]** Proposition 11)

$$\operatorname{expdim}(\mathcal{L}_3) = \dim |L| - 4$$

as soon as

$$h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) = 0.$$

We thus have the following proposition.

Proposition 3.1
Let Σ be regular and suppose that α is surjective, then (Σ, L) is not triple-point defective if

$$h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) = 0$$

for general $p \in \Sigma$ and $L_p \in |L|$ with $\operatorname{mult}_p(L_p) = 3$.

Moreover, if L is non-special the above h^1-vanishing is also necessary for the non-triple-point-defectiveness of (Σ, L).

4. The Basic Construction

From now on we assume that for $p \in \Sigma$ general $\exists\, L_p \in |L|$ s.t.

$$h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

Then by Serre's construction for a subscheme $Z_p' \subseteq Z_p$ with ideal sheaf $\mathcal{J}_p = \mathcal{J}_{Z_p'}$ of minimal length such that $h^1\big(\Sigma, \mathcal{J}_p(L)\big) \neq 0$ there is a rank two bundle $\mathcal{E}_p$ on Σ and a section $s \in H^0(\Sigma, \mathcal{E}_p)$ whose 0-locus is Z_p', giving the exact sequence

$$0 \to \mathcal{O}_\Sigma \to \mathcal{E}_p \to \mathcal{J}_p(L - K) \to 0. \tag{4.1}$$

The Chern classes of $\mathcal{E}_p$ are

$$c_1(\mathcal{E}_p) = L - K \quad \text{and} \quad c_2(\mathcal{E}_p) = \operatorname{length}(Z_p').$$

Moreover, Z_p' is automatically a complete intersection.

We would now like to understand what $\mathcal{J}_p$ is depending on $\operatorname{jet}_3(f_p)$, which in suitable local coordinates will be one of those in Table (4.2). For this we first of all note that the very ample divisor L separates all subschemes of Z_p of length at most two. Thus Z_p' has length at least 3, and due to Lemma 4.1 below we are in one of the following situations:

$\operatorname{jet}_3(f_p)$	$\mathcal{J}_{Z_p,p}$	$\operatorname{length}(Z_p)$	$\mathcal{J}_p$	$c_2(\mathcal{E}_p)$
$x_p^3 - y_p^3$	$\langle x_p^2, y_p^2\rangle$	4	$\langle x_p^2, y_p^2\rangle$	4
$x_p^2 y_p$	$\langle x_p^2, x_p y_p, y_p^3\rangle$	4	$\langle x_p, y_p^3\rangle$	3
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3\rangle$	5	$\langle x_p^2, y_p^2\rangle$	4
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3\rangle$	5	$\langle x_p, y_p^3\rangle$	3

(4.2)

Lemma 4.1
If $f \in R = \mathbb{C}\{x,y\}$ *with* $\operatorname{jet}_3(f) \in \{x^3 - y^3, x^2y, x^3\}$*, and if* $I = \langle g, h\rangle \lhd R$ *such that* $\dim_{\mathbb{C}}(R/I) \geq 3$ *and* $\big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\big\rangle + \langle x, y\rangle^3 \subseteq I$*, then we may assume that we are in one of the following cases:*

(a) $I = \langle x^2, y^2\rangle$ *and* $\operatorname{jet}_3(f) \in \{x^3 - y^3, x^3\}$*, or*

(b) $I = \langle x, y^3\rangle$ *and* $\operatorname{jet}_3(f) \in \{x^2y, x^3\}$.

Proof: If $>$ is any *local degree* ordering on R, then the Hilbert-Samuel functions of R/I and of $R/L_>(I)$ coincide, where $L_>(I)$ denotes the leading ideal of I (see e.g. **[GrP02]** Proposition 5.5.7). In particular, $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}(R/L_>(I))$ and thus

$$L_>(I) \in \big\{\langle x^2, xy^2, y^3\rangle, \langle x^2, xy, y^2\rangle, \langle x^2, xy, y^3\rangle, \langle x^2, y^2\rangle, \langle x, y^3\rangle\big\},$$

since $\langle x^2, xy^2, y^3\rangle \subset I$.

Taking $>$, for a moment, to be the local degree ordering on R with $y > x$ we deduce at once that I does not contain any power series with a linear term in y. For the remaining part of the proof $>$ will be the local degree ordering on R with $x > y$.

1st Case: $L_>(I) = \langle x^2, xy^2, y^3\rangle$ or $L_>(I) = \langle x^2, xy, y^2\rangle$. Thus the graph of the slope $H^0_{R/I}$ of the Hilbert-Samuel function of R/I would be as shown in Figure 1, which contradicts the fact that I is a complete intersection due to **[Iar77]** Theorem 4.3.

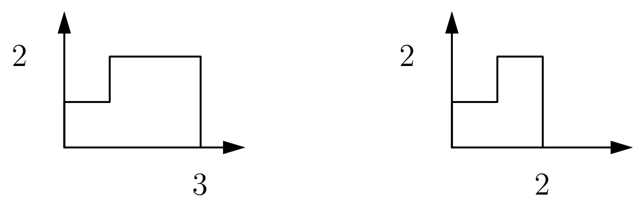


FIGURE 1. The graphs of $H^0_{R/\langle x^2, xy^2, y^3\rangle}$ respectively of $H^0_{R/\langle x^2, xy, y^2\rangle}$.

<u>2nd Case:</u> $L_>(I) = \langle x^2, xy, y^3 \rangle$. Then we may assume

$$g = x^2 + \alpha \cdot y^2 + h.o.t. \quad \text{and} \quad h = xy + \beta \cdot y^2 + h.o.t..$$

Since $x^2 \in I$ there are power series $a, b \in R$ such that

$$x^2 = a \cdot g + b \cdot h.$$

Thus the leading monomial of a is one, a is a unit and $g \in \langle x^2, h \rangle$. We may therefore assume that $g = x^2$. Moreover, since the intersection multiplicity of g and h is $\dim_{\mathbb{C}}(R/I) = 4$, g and h cannot have a common tangent line in the origin, i. e. $\beta \neq 0$. Thus, since $g = x^2$, we may assume that $h = xy + y^2 \cdot u$ with $u = \beta + h.o.t$ a unit.

In new coordinates $\widetilde{x} = x \cdot \sqrt{u}$ and $\widetilde{y} = y \cdot \frac{1}{\sqrt{u}}$ we have

$$I = \langle \widetilde{x}^2, \widetilde{x}\widetilde{y} + \widetilde{y}^2 \rangle.$$

Note that by the coordinate change $\mathrm{jet}_3(f)$ only changes by a constant, that $\frac{\partial f}{\partial \widetilde{x}}, \frac{\partial f}{\partial \widetilde{y}} \in I$ and that $\langle \widetilde{x}, \widetilde{y} \rangle^3 \subset I$, but $\widetilde{x}\widetilde{y}, \widetilde{y}^2 \notin I$. Thus $\mathrm{jet}_3(f) = x^3$.

Setting now $\bar{x} = \widetilde{x}$ and $\bar{y} = \widetilde{x} + 2\widetilde{y}$, then $\bar{y}^2 = \widetilde{x}^2 + 4 \cdot (\widetilde{x}\widetilde{y} + \widetilde{y}^2) \in I$ and thus, considering colengths,

$$I = \langle \bar{x}^2, \bar{y}^2 \rangle.$$

Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning.

<u>3rd Case:</u> $L_>(I) = \langle x^2, y^2 \rangle$. Then we may assume

$$g = x^2 + \alpha \cdot xy + h.o.t. \quad \text{and} \quad h = y^2 + h.o.t.$$

As in the second case we deduce that w.l.o.g. $g = x^2$ and thus $h = y^2 \cdot u$, where u is a unit. But then $I = \langle x^2, y^2 \rangle$.

<u>4th Case:</u> $L_>(I) = \langle x, y^3 \rangle$. Then we may assume

$$g = x + h.o.t. \quad \text{and} \quad h = y^3 + h.o.t.$$

since there is no power series in I involving a linear term in y. In new coordinates $\widetilde{x} = g$ and $\widetilde{y} = y$ we have

$$I = \langle \widetilde{x}, \widetilde{h} \rangle,$$

and we may assume that $\widetilde{h} = \widetilde{y}^3 \cdot u$, where u is a unit only depending on $\widetilde{y}$. Hence, $I = \langle \widetilde{x}, \widetilde{y}^3 \rangle$. Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning. $\square$

$\boxed{\text{From now on we assume that } (L-K)^2 > 16.}$

Thus

$$c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0,$$

and hence $\mathcal{E}_p$ is Bogomolov unstable. The Bogomolov instability implies the existence of a unique divisor A_p which destabilizes $\mathcal{E}_p$. (See e. g. [**Fri98**] Section 9, Corollary 2.) In other words, setting $B_p = L - K - A_p$, i. e.

$$A_p + B_p = L - K, \tag{4.3}$$

there is an immersion

$$0 \to \mathcal{O}_\Sigma(A_p) \to \mathcal{E}_p \tag{4.4}$$

where $(A_p - B_p)^2 \geq c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0$ and $(A_p - B_p).H > 0$ for every ample H. From this we deduce the following properties:

(a) $\mathcal{E}_p(-A_p)$ has a section that vanishes along a subscheme $\widetilde{Z}_p$ of codimension 2, and

$$A_p.B_p \leq \text{length}(Z_p'). \tag{4.5}$$

The previous immersion gives rise to a short exact sequence:

$$0 \to \mathcal{O}_\Sigma(A_p) \to \mathcal{E}_p \to \mathcal{J}_{\widetilde{Z}_p}(B_p) \to 0. \tag{4.6}$$

(b) The divisor B_p is effective and we may assume that $Z_p' \subset B_p$.

(c) $A_p - B_p$ is big, and hence $\dim\big(|k \cdot (A_p - B_p)|\big) = const \cdot k^2 + o(k) > 0$ for $k >> 0$. In particular

$$(A_p - B_p).M > 0 \tag{4.7}$$

if M is big and nef or if M is an irreducible curve with $M^2 \geq 0$ or if M is effective without fixed part.

(d) A_p is big.

Proof: (a) Sequence (4.6) is a consequence of Serre's construction. The first assertion now follows from Sequence (4.6), and Equation (4.5) is a consequence of

$$(A_p - B_p)^2 \geq c_1(\mathcal{E}_p)^2 - c_2(\mathcal{E}_p) = (A_p + B_p)^2 - 4 \cdot \text{length}(Z_p').$$

(b) Observe that $\big(2A_p - (L - K)\big).H > 0$ for any ample line bundle H, and thus

$$-A_p.H < -\frac{(L - K_\Sigma).H}{2} < 0.$$

Thus $H^0\big(\mathcal{O}_\Sigma(-A_p)\big) = 0$ and twisting the sequence (4.1) with $-A_p$ we are done.

(c) Since $(A_p - B_p)^2 > 0$ and $(A_p - B_p).H > 0$ for some ample H Riemann-Roch shows that $A_p - B_p$ is big, i. e. $\dim\big(|k \cdot (A_p - B_p)|\big)$ grows with k^2. The remaining part follows from Lemma 4.2.

(d) This follows since $A_p - B_p$ is big and B_p is effective.

□

Lemma 4.2
Let R be a big divisor. If M is big and nef or if M is an irreducible curve with $M^2 \geq 0$ or if M is an effective divisor without fixed component, then $R.M > 0$.

Proof: If R is big, then $\dim |k \cdot R|$ grows with k^2. Thus for $k >> 0$ we can write $k \cdot R = N' + N''$ where N' is ample and N'' effective (possibly zero). To see this, note that for $k >> 0$ we can write $|k \cdot R| = |N'| + N''$, where N'' is the fixed part of $|k \cdot R|$ and $N' \cap C \neq \emptyset$ for every irreducible curve C. Then apply the Nakai-Moishezon Criterion to N' (see also [**Tan04**]).

Analogously, if M is big and nef, for $j >> 0$ we can write $j \cdot M = M' + M''$ where M' is ample and M'' is effective. Therefore,

$$R.M = \frac{1}{kj} \cdot \left(N'.M' + N'.M'' + N''.M\right) > 0,$$

since $N'.M' > 0$, $N'.M'' \geq 0$ and $N''.M \geq 0$.

Similarly, if M is irreducible and has non-negative self-intersection, then

$$R.M = \frac{1}{k} \cdot (N'.M + N''.M) > 0.$$

And if M is effective without fixed component, we can apply the previous argument to every component of M. □

Now let p move freely in Σ. Accordingly the scheme Z_p' moves, hence the effective divisor B_p containing Z_p' moves in an algebraic family $\mathcal{B} \subseteq |B|_a$ which is the closure of $\{B_p \mid p \in \Sigma, L_p \in |L - 3p|,$ both general$\}$ and which covers Σ. A priori this family $\mathcal{B}$ might have a *fixed part* C, so that for general $p \in \Sigma$ there is an effective divisor D_p moving in a fixed-part free algebraic family $\mathcal{D} \subseteq |D|_a$ such that

$$B_p = C + D_p.$$

Whenever we only refer to the algebraic class of A_p respectively B_p respectively D_p we will write A respectively B respectively D for short.

For these considerations we assume, of course, that $\text{length}(Z_p')$ is constant for $p \in \Sigma$ general, so either $\text{length}(Z_p') = 3$ or $\text{length}(Z_p') = 4$.

5. $C = 0$.

Our first aim is to show that actually $C = 0$ (see Lemma 5.5). But in order to do so we first have to consider the boundary case that $A_p.B_p = \text{length}(Z'_p)$.

Proposition 5.1
If $A_p.B_p = \text{length}(Z'_p)$, then there exists a non-trivial global section $0 \neq s \in H^0\big(B_p, \mathcal{J}_{Z'_p/B_p}(A_p)\big)$ whose zero-locus is Z'_p.

In particular, $A_p.D_p = A_p.B_p = \text{length}(Z'_p)$ and $A_p.C = 0$.

Proof: By Equation (4.6) we have

$$A_p.B_p = \text{length}(Z'_p) = c_2(\mathcal{E}_p) = A_p.B_p + \text{length}\left(\widetilde{Z}_p\right).$$

Thus $\widetilde{Z}_p = \emptyset$.

If we merge the sequences (4.1), (4.6), and the structure sequence of B twisted by B we obtain the following exact commutative diagram in Figure 2, where

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{O}_\Sigma \arrow[r] \arrow[d] & \mathcal{O}_\Sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_\Sigma(A_p) \arrow[r] \arrow[d] & \mathcal{E}_p \arrow[r] \arrow[d] & \mathcal{O}_\Sigma(B_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_\Sigma(A_p) \arrow[r] \arrow[d] & \mathcal{J}_{Z'_p/\Sigma}(A_p+B_p) \arrow[r] \arrow[d] & \mathcal{O}_{B_p}(B_p) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

FIGURE 2. The diagram showing $\mathcal{O}_{B_p} = \mathcal{J}_{Z'_p/B_p}(A_p)$.

$\mathcal{O}_{B_p}(B_p) = \mathcal{J}_{Z'_p/B_p}(A_p + B_p)$, or equivalently $\mathcal{O}_{B_p} = \mathcal{J}_{Z'_p/B_p}(A_p)$. Thus from the rightmost column we get a non-trivial global section, say s, of this bundle which vanishes precisely at Z'_p, since Z'_p is the zero-locus of the monomorphism of vector bundles $\mathcal{O}_\Sigma \hookrightarrow \mathcal{E}_p$. However, since p is general we have that $p \notin C$ and thus the restriction $0 \neq s_{|D_p} \in H^0\big(D_p, \mathcal{J}_{Z'_p/D_p}(A_p)\big)$ and it still vanishes precisely at Z'_p. Thus $A_p.D_p = \text{length}(Z'_p) = A_p.B_p$, and $A_p.C = A_p.B_p - A_p.D_p = 0$. □

Lemma 5.2
$A_p.B_p \geq 1 + B_p^2$.

Proof: Let $B = P + N$ be a Zariski decomposition of B, i. e. P and N are effective $\mathbb{Q}$-divisors such that in particular P is nef, $P.N = 0$ and $N^2 < 0$ unless $N = 0$.

If $N \neq 0$, then

$$0 < (A + B).N = A.N + N^2,$$

since $A + B$ is very ample and N is effective. Moreover, since P is nef and $A - B$ big we have $(A - B).P \geq 0$ and hence

$$A.P \geq B.P = P^2.$$

Combining these two inequalities we get

$$A.B = A.P + A.N > P^2 - N^2 > P^2 + N^2 = B^2.$$

If $N = 0$, then B is nef and, therefore, $B^2 \geq 0$. If, actually $B^2 > 0$, then B is big and nef, so that by (4.7) $(A - B).B > 0$. While if $B^2 = 0$ then

$$B^2 = 0 < B.(A + B) = A.B$$

since $A + B$ is very ample and B is effective. □

Lemma 5.3
Let $p \in \Sigma$ *be general and suppose* $\operatorname{length}(Z'_p) = 4$.

(a) *If* D_p *is irreducible, then* $\dim(\mathcal{D}) \geq 2$ *and* $D_p^2 \geq 3$.

(b) *If* D_p *is reducible but the part containing* p *is reduced, then either* D_p *has a component singular in* p *and* $D_p^2 \geq 5$ *or at least two components of* D_p *pass through* p *and* $D_p^2 \geq 2$.

(c) *If* $D_p^2 \leq 1$*, then* $D_p = k \cdot E_p$ *where* $k \geq 2$*,* E_p *is irreducible and* $E_p^2 = 0$*. In particular,* $D_p^2 = 0$.

Proof: (a) If D_p is irreducible, then $\dim(\mathcal{D}) \geq 2$, since D_p, containing Z'_p, is singular in p by Table (4.2) and since $p \in \Sigma$ is general. If through $p \in \Sigma$ general and a general $q \in D_p$ there is another $D' \in \mathcal{D}$, then due to the irreducibility of D_p

$$D_p^2 = D_p.D' \geq \operatorname{mult}_p(D_p) + \operatorname{mult}_q(D_p) \geq 3.$$

Otherwise, $\mathcal{D}$ is a two-dimensional involution whose general element is irreducible, so that by **[ChC02]** Theorem 5.10 $\mathcal{D}$ must be a linear system. This, however, contradicts the Theorem of Bertini, since the general element of $\mathcal{D}$ would be singular.

(b) Suppose $D_p = \sum_{i=1}^k E_{i,p}$ is reducible but the part containing p is reduced. Since D_p has no fixed component and p is general, each $E_{i,p}$ moves in an at least one-dimensional family. In particular $E_{i,p}^2 \geq 0$.

If some $E_{i,p}$, say $i = 1$, would be singular in p for $p \in \Sigma$ general we could argue as above that $E_{1,p}^2 \geq 3$. Moreover, either $E_{2,p}$ is algebraically equivalent to $E_{1,p}$ and $E_{2,p}^2 \geq 3$, or $E_{1,p}$ and $E_{2,p}$ intersect properly, since both vary in different, at least one-dimensional families. In any case we have

$$D_p^2 \geq (E_{1,p} + E_{2,p})^2 \geq 5.$$

Otherwise, at least two components, say $E_{1,p}$ and $E_{2,p}$ pass through p, since D_p is singular in p and no component passes through p with higher multiplicity. Hence, $E_{1,p}.E_{2,p} \geq 1$ and therefore

$$D_p^2 \geq 2 \cdot E_{1,p}.E_{2,p} \geq 2.$$

(c) From the above we see that D_p is not reduced in p. Let therefore $D_p \equiv_a kE_p + E'$ where $k \geq 2$, E_p passes through p and E' does not contain any component algebraically equivalent to E_p.
Suppose $E' \neq 0$. Since D_p has no fixed component both, E_p and E' vary in an at least one dimensional family covering Σ and must therefore intersect properly. In particular, $E_p.E' \geq 1$ and $1 \geq D_p^2 \geq 2k \cdot E_p.E' \geq 4$. Thus, $E' = 0$.
We therefore may assume that $D_p = kE_p$ with $k \geq 2$. Then $0 \leq E_p^2 = \frac{1}{k^2} \cdot D_p^2 \leq \frac{1}{4}$, which leaves only the possibility $E_p^2 = 0$, implying also $D_p^2 = 0$.

□

Lemma 5.4
Suppose that $R \subset \Sigma$ is an irreducible curve.

(a) *If $(L-K).R \in \{1,2\}$, then R is smooth, rational and $R^2 \leq (L-K).R-3 \leq -1$.*

(b) *If $(L-K).R = 3$, then $R^2 \leq 0$, and either R is a plane cubic or it is a smooth rational space curve.*

Proof: Note that Σ is embedded in some $\mathbb{P}_{\mathbb{C}}^n$ via $|L-K|$ and that $\deg(R) = (L-K).R$ is just the degree of R as a curve in $\mathbb{P}_{\mathbb{C}}^n$. Moreover, by the adjunction formula we know that

$$p_a(R) = \frac{R^2 + R.K}{2} + 1,$$

and since L is very ample we thus get

$$1 \leq L.R = (L-K).R + R.K = (L-K).R + 2 \cdot \big(p_a(R) - 1\big) - R^2. \qquad (5.1)$$

(a) If $\deg(R) \in \{1,2\}$, then R must be a smooth, rational curve. Thus we deduce from (5.1)

$$R^2 \leq (L-K).R - 3.$$

(b) If $\deg(R) = 3$, then R is either a plane cubic or a smooth space curve of genus 0. If $p_a(R) = 1$ then actually $L.R \geq 3$ since otherwise $|L|$ would embed R as a rational curve of degree 1 resp. 2 in some projective space. In any case we are therefore done with (5.1).

□

Lemma 5.5
$C = 0$.

Proof: Suppose $C \neq 0$ and r is the number of irreducible components of C. Since $\mathcal{D}$ has no fixed component we know by (4.5) that $(A - B).D > 0$, so that

$$A.D \geq B.D + 1 = D.C + D^2 + 1 \tag{5.2}$$

or equivalently

$$D.C \leq A.D - D^2 - 1. \tag{5.3}$$

Moreover, since $A + B$ is very ample we have $r \leq (A + B).C = A.C + D.C + C^2$ and thus

$$A.C + D.C = (A + B).C - C^2 \geq r - C^2. \tag{5.4}$$

1st Case: $C^2 \leq 0$. Then (5.4) together with (5.2) gives

$$A.B = A.C + A.D \geq A.C + D.C + D^2 + 1 \geq r + (-C^2) + D^2 + 1 \geq 2, \tag{5.5}$$

or the slightly stronger inequality

$$\operatorname{length}(Z'_p) \geq A.B \geq (A + B).C + (-C^2) + D^2 + 1. \tag{5.6}$$

2nd Case: $C^2 > 0$. Then by Lemma 5.2 simply

$$\operatorname{length}(Z'_p) \geq A.B \geq B^2 + 1 = D^2 + 2 \cdot C.D + C^2 + 1 \geq 2. \tag{5.7}$$

Since all the summands involved in the right hand side of (5.5) and (5.7) are non-negative and since by Lemma 5.3 the case $D^2 = 1$ cannot occur when $\operatorname{length}(Z'_p) = 4$, we are left considering the cases shown in Figure 3, where for the additional information (the last four columns) we take Proposition 5.1 and Lemma 5.3 into account.

Let us first and for a while consider the situation $\operatorname{length}(Z'_p) = 4$ and $D^2 = 0$, so that by Lemma 5.3 $D = kE$ for some irreducible curve E with $k \geq 2$ and $E^2 = 0$. Applying Lemma 5.4 to E we see that $(A + B).E \geq 3$, and thus

$$6 \leq 3k \leq (A + B).D = A.D + C.D. \tag{5.8}$$

If in addition $A.D \leq 4$, then (5.3) leads to

$$6 \leq A.D + C.D \leq 4 + C.D \leq 7, \tag{5.9}$$

	length(Z'_p)	D^2	C^2	$C.D$	r	$A.B$	$A.D$	$A.C$	D
1)	4	0	-2		1	4	4	0	$kE, k \geq 2$
2)	4	0	-1		2	4	4	0	$kE, k \geq 2$
3)	4	0	0		3	4	4	0	$kE, k \geq 2$
4)	4	0	-1		1	$3, 4$			$kE, k \geq 2$
5)	4	2	0		1	4	4	0	
6)	4	0	0		2	$3, 4$			$kE, k \geq 2$
7)	4	0	0		1	$2, 3, 4$			$kE, k \geq 2$
8)	3	0	-1		1	3	3	0	
9)	3	0	0		2	3	3	0	
10)	3	1	0		1	3	3	0	
11)	3	0	0		1	$2, 3$			
12)	4	0	1	1		4	4	0	$kE, k \geq 2$
13)	4	2	1	0		4	4	0	
14)	4	0	1	0		$2, 3, 4$			$kE, k \geq 2$
15)	4	0	2	0		$3, 4$			$kE, k \geq 2$
16)	3	1	1	0		3	3	0	
17)	3	0	1	0		$2, 3$			

FIGURE 3. The cases to be considered.

which is only possible for $k = 2$, $C.E = 1$ and

$$C.D = k \cdot C.E = 2. \tag{5.10}$$

This outrules Case 12.

In Cases 1, 2 and 3 we have $A.D = 4$, and we can apply (5.10), which by (5.4) then gives the contradiction

$$2 = A.C + C.D \geq r - C^2 = 3.$$

If, still under the assumption $\mathrm{length}(Z'_p) = 4$ and $D^2 = 0$, we moreover assume $2 \geq C^2 \geq 0$ then by Lemma 5.2

$$3 \geq B^2 = 2 \cdot C.D + C^2 \geq 2 \cdot C.D \geq 0,$$

and thus $C.D \leq 1$ and $C.D + C^2 \leq 2$, which due to (5.8) implies $A.D \geq 5$. But then by Proposition 5.1 we have $A.B \leq 3$ and hence $A.C = A.B - A.D \leq -2$, which leads to the contradiction

$$(A+B).C = A.C + D.C + C^2 \leq 0, \tag{5.11}$$

since $A + B$ is very ample. This outrules the Cases 6, 7, 14 and 15.

In Case 4 Lemma 5.4 applied to C shows

$$2 \leq (A+B).C = A.C + D.C + C^2. \tag{5.12}$$

If in this situation $A.B = 4$, then Proposition 5.1 shows $A.C = 0$ and $A.D = A.B = 4$, and therefore (5.10) leads a contradiction, since the right hand side of Equation (5.12) is $A.C + D.C + C^2 = 0 + 2 - 1 = 1$. We, therefore, conclude that $A.B = 3$, and as above we get from Lemma 5.2

$$2 \geq B^2 = 2 \cdot C.D + C^2 = 2k \cdot C.E - 1 \geq 4 \cdot C.E - 1 \geq -1,$$

which is only possible for $C.E = C.D = 0$. But then (5.12) implies $A.C \geq 3$, and since A is big and D has no fixed component Lemma 4.2 gives the contradiction

$$1 \leq A.D = A.B - A.C \leq 0.$$

This finishes the cases where $\operatorname{length}(Z_p') = 4$ and $D^2 = 0$.

In Cases 5, 10 and 11 we apply Lemma 5.4 to the irreducible curve C with $C^2 = 0$ and find

$$(A+B).C \geq 3.$$

In Cases 5 and 10 Equation (5.6) then gives the contradiction

$$4 \geq A.B \geq 3 - C^2 + D^2 + 1 \geq 5,$$

and similarly in Case 11 we get

$$3 \geq A.B \geq 3 - C^2 + D^2 + 1 = 4.$$

In very much the same way we get in Case 8

$$(A+B).C \geq 2$$

and the contradiction

$$3 \geq A.B \geq 2 - C^2 + D^2 + 1 = 4.$$

Let us next have a look at the Cases 16 and 17. Consider the decomposition of the general $D = \sum_{i=1}^{s} E_i$ into irreducible components, none of which is fixed. In Case 16 we have $D^2 = 0$, and thus $E_i.E_j = 0$ for all i, j, while in Case 17 we have $D^2 = 1$ and we thus may assume $E_1^2 = 1$ and $E_i.E_j = 0$ for all $(i,j) \neq (1,1)$. Moreover, in both cases $C.D = 0$ and thus $C.E_i = 0$ for all i. Applying Lemma 5.4 to E_i we find

$$A.E_i = (A+B).E_i - E_1.E_i \geq 3,$$

and by (5.4) we get

$$A.C = A.C + D.C \geq r - C^2 \geq 0. \tag{5.13}$$

But then

$$3 \geq A.B = A.C + \sum_{i=1}^{s} A.E_i \geq 3s,$$

which implies $s = 1$ and $A.C = 0$. From (5.13) we deduce that $r = C^2 = 1$, and thus C is irreducible with $C^2 = 1$. Similarly in Case 13 we have by (5.4)

$$0 = A.C + D.C \geq r - C^2 = r - 1 \geq 0,$$

and again C is irreducible with $C^2 = 1$. Applying now Lemma 5.4 to C we get in each of these three cases the contradiction

$$4 \leq (A + B).C = A.C + D.C + C^2 = 1.$$

This outrules the Cases 13, 16, and 17.

It remains to consider Case 9. Here we deduce from (5.6) that

$$2 \geq (A + B).C \geq r = 2,$$

and hence

$$2 = (A + B).C = A.C + D.C + C^2 = D.C.$$

But then Lemma 5.2 leads to the final contradiction

$$2 = A.B - 1 \geq B^2 = D^2 + 2 \cdot D.C + C^2 = 4.$$

□

It follows that $B_p = D_p$, $\mathcal{B} = \mathcal{D}$, and that B_p is nef.

6. The General Case

Let us review the situation and recall some notation. We are considering a divisor L such L and $L - K$ are very ample with $(L - K)^2 > 16$, and such that for a general point $p \in \Sigma$ the general element $L_p \in |L - 3p|$ has no triple component through p and that the equimultiplicity ideal of L_p in p in suitable local coordinates is one of the ideals in Table (4.2) – and for all p the ideals have the same length. Moreover, we know that there is an algebraic family $\mathcal{B} = \overline{\{B_p \mid p \in \Sigma\}} \subset |B|_a$ without fixed component such that for a general point $p \in \Sigma$

$$B_p \in |\mathcal{J}_{Z'_p/\Sigma}(L - K - A_p)|,$$

where Z'_p is the equimultiplicity scheme of L_p and A_p is the unique divisor linearly equivalent to $L - K - B_p$ such that B_p and A_p destabilize the vector bundle $\mathcal{E}_p$ in (4.1). Keeping these notations in mind we can now consider the two cases that either $\operatorname{length}(Z'_p) = 4$ or $\operatorname{length}(Z'_p) = 3$.

Proposition 6.1
Let $p \in \Sigma$ *be general and suppose that* $\mathrm{length}(Z'_p) = 4$. *Then* $B_p = E_p + F_p$, E_p *and* F_p *are irreducible, smooth, elliptic curves,* $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$, $A.E_p = A.F_p = 2$, $L.E_p = L.F_p = 3$, $A.B = 4$, $K.E_p = K.F_p = 0$, *and* $\exists\ s \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ *such that* $Z'_p = \{s = 0\}$.

Moreover, neither $|E|_a$ *and* $|F|_a$ *is a linear system, but they both induce an elliptic fibration with section on* Σ *over an elliptic curve.*

Proof: Since $A^2 > 0$ we can apply the Hodge Index Theorem (see e.g. [**BHPV04**]), and since $(A+B)^2 \geq 17$ by assumption and $A.B \leq 4$ by Equation (4.5) we deduce

$$16 \geq (A.B)^2 \geq A^2 \cdot B^2 = \big((A+B)^2 - 2A.B - B^2\big).B^2$$
$$\geq (9 - B^2) \cdot B^2. \quad (6.1)$$

In Section 5 we have shown that $B = D$ is nef, and thus Lemma 5.2 together with Equation (6.1) shows

$$0 \leq B^2 \leq 2. \quad (6.2)$$

Then, however, Lemma 5.3 implies that B_p must be reducible.

Let us first consider the case that the part of B_p through p is reduced. Then by Lemma 5.3, Lemma 5.2, and Equations (4.5) and (6.2) we know that $B_p = E_p + F_p + R$, where E_p and F_p are irreducible and smooth in p. In particular, $E_p.F_p \geq 1$, and thus

$$2 \geq B^2 = E_p^2 + 2 \cdot E_p.F_p + F_p^2 + 2 \cdot (E_p + F_p).R + R^2$$
$$\geq 2 + 2 \cdot (E_p + F_p).R.$$

Since $E_p.F_p = 1$ and since the components E_p and F_p vary in at least one-dimensional families and R has no fixed component, $(E_p + F_p).R \geq 1$, unless $R = 0$. This would however give a contradiction, so $R = 0$. Therefore necessarily, $B_p = E_p + F_p$, $E_p.F_p = 1$, and $E_p^2 = F_p^2 = 0$. Then by Lemma 5.4 $(A+B).E_p \geq 3$ and $(A+B).F_p \geq 3$, so that

$$4 \geq A.B \geq (A+B).E_p + (A+B).F_p - B^2 \geq 4$$

implies $E_p.A_p = 2 = F_p.A_p$ and $(A+B).E_p = 3 = (A+B).F_p$. Applying Lemma 5.4 once more, we see that

$$p_a(E_p) \leq 1 \quad \text{and} \quad p_a(F_p) \leq 1. \quad (6.3)$$

We claim that in p the curve L_p can share at most with one of E_p or F_p a common tangent, and it can do so at most with multiplicity one. For this consider local coordinates (x_p, y_p) as in the Table (4.2). Since $\mathrm{length}(Z'_p) = 4$ we know that $\mathcal{J}_{Z'_p,p} = \langle x_p^2, y_p^2 \rangle$ does not contain $x_p y_p$, and since $B_p = E_p + F_p \in$

$|\mathcal{J}_{Z'_p}(L-K-A)|$, where E_p and F_p are smooth in p, we deduce that in local coordinates their equations are

$$x_p + a \cdot y_p + h.o.t. \quad \text{respectively} \quad x_p - a \cdot y_p + h.o.t.,$$

where $a \neq 0$. By Table (4.2) the local equation f_p of L_p has either $\mathrm{jet}_3(f_p) = x_p^3$ and has thus no common tangent with either E_p or F_p, or $\mathrm{jet}_3(f_p) = x_p^3 - y_p^3$ and it is divisible at most once by one of $x_p - ay_p$ or $x_p + ay_p$.

In particular, E_p can at most once be a component of L_p, and we deduce

$$E_p.K_\Sigma = E_p.L_p - E_p.A_p - E_p.B_p = E_p.L_p - 3 \geq \begin{cases} 0, & \text{if } E_p \not\subset L_p, \\ -1, & \text{if } E_p \subset L_p. \end{cases}$$

But then, since the genus is an integer,

$$p_a(E_p) = \frac{E_p^2 + E_p.K_\Sigma}{2} + 1 = \frac{E_p.K_\Sigma}{2} + 1 \geq 1,$$

in which case (6.3) gives $p_a(E_p) = 1$. This shows, in particular, that

$$K.E_p = 0 \quad \text{and} \quad L_p.E_p = 3.$$

By symmetry the same holds for F_p.

Since $E_p^2 = 0$ the family $|E|_a$ is a pencil and induces an elliptic fibration on Σ (see [**Kei01**] App. B.1). In particular, the generic element E_p in $|E|_a$ must be smooth (see e.g. [**BHPV04**] p. 110). And with the same argument the generic element F_p in $|F|_a$ is smooth.

Suppose now that $|E|_a$ is a linear system. Since $E.F = 1$ and for $q \in F_p$ general $E_q \cap F_p = \{q\}$ the linear system $|\mathcal{O}_{F_p}(E)|$ is a $\mathfrak{g}_1^1$ on the smooth curve F_p implying that F_p is rational contradicting $p_a(F_p) = 1$. Thus $|E|_a$ is not linear, and analogously $|F|_a$ is not.

It remains to consider the case that B_p is not reduced in p. Using the notation of the proof of Lemma 5.3 we write $B_p \equiv k \cdot E_p + E'$ with $k \geq 2$, E_p irreducible passing through p and E' not containing any component algebraically equivalent to E_p. We have seen there (see p. 105) that $E' \neq 0$ implies $B_p^2 \geq 4$ in contradiction to Lemma 5.2. We may therefore assume $B_p = k \cdot E_p$ with $E_p^2 \geq 0$. If $E_p^2 \geq 1$, then again $B_p^2 \geq 4$. Thus $E_p^2 = 0$. Applying Lemma 5.4 to E_p we get

$$3 \leq (A+B).E_p = A.E_p,$$

and hence the contradiction

$$4 \geq A.B = k \cdot A.E_p \geq 6.$$

Therefore, B_p must be reduced in p. $\square$

Proposition 6.2
Let $p \in \Sigma$ be general and suppose that $\operatorname{length}(Z_p') = 3$. Then B_p is an irreducible, smooth, rational curve in the pencil $|B|_a$ with $B^2 = 0$, $A.B = 3$ and $\exists\ s \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s.

In particular, $\Sigma \to |B|_a$ is a ruled surface and $2B_p$ is a fixed component of $|L - 3p|$.

Proof: Since $A^2 > 0$ we can apply the Hodge Index Theorem (see e.g. [**BHPV04**]), and since $(A+B)^2 \geq 17$ by assumption and $A.B \leq 3$ by Equation (4.5) we deduce

$$9 \geq (A.B)^2 \geq A^2 \cdot B^2 = \big((A+B)^2 - 2A.B - B^2\big) \geq (11 - B^2) \cdot B^2.$$

Since in Section 5 we have shown that B is nef, this inequality together with Lemma 5.2 implies

$$B^2 = 0. \tag{6.4}$$

Once we have shown that B_p is irreducible and reduced, we then know that $|B|_a$ is a pencil and induces a fibration on Σ whose fibres are the elements of $|B|_a$ (see [**Kei01**] App. B.1). In particular, the general element of $|B|_a$, which is B_p, is smooth (see [**BHPV04**] p. 110).

Let us therefore first show that B_p is irreducible and reduced. Since $\mathcal{B}$ has no fixed component we know for each irreducible component B_i of $B_p = \sum_{i=1}^r B_i$ that $B_i^2 \geq 0$, and hence by Lemma 5.4 that $(A+B).B_i \geq 3$. Thus by (4.5) and (6.4)

$$3 \cdot r \leq (A+B).B = A.B + B^2 = A.B \leq 3,$$

which shows that B_p is irreducible and reduced and that $A.B = 3$. Moreover, $(A+B).B = 3$, and Lemma 5.4 implies that

$$p_a(B_p) \leq 1. \tag{6.5}$$

Since $A.B = 3 = \operatorname{length}(Z_p')$ Proposition 5.1 implies that there is a section $s_p \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s_p, which is just $3p$. Note that for $p \in \Sigma$ general and $q \in B_p$ general we have $B_p = B_q$ since $|B|_a$ is a pencil, and thus by the construction of B_p and B_q we also have

$$A_p \sim_l L - K - B_p = L - K - B_q \sim_l A_q.$$

But if A_p and A_q are linearly equivalent, then so are the divisors s_p and s_q induced on the curve $B_p = B_q$. The curve B_p therefore contains a linear series $|\mathcal{O}_{B_p}(A_p)|$ of degree three which contains $3q$ for a general point $q \in B_p$. If B_p was an elliptic curve, then $|\mathcal{O}_{B_p}(A_p)|$ would necessarily have to be a $\mathfrak{g}_3^2$ embedding B_p as a plane curve of degree three and the general point q would be an inflexion point. But that is clearly not possible. Thus

$$p_a(B_p) = 0,$$

and by the adjunction formula we get

$$K.B = 2p_a(B) - 2 - B^2 = -2. \tag{6.6}$$

Note also, that $Z'_p \subset B_p$ in view of Table (4.2) implies that B_p and L_p have a common tangent in p. Suppose that B_p and L_p have no common component, i. e. $B_p \not\subset L_p$, then

$$3 \leq \text{mult}_p(B_p) \cdot \text{mult}_p(L_p) < L.B = A.B + B^2 + K.B = 3 + K.B = 1,$$

which contradicts (6.6). Thus, B_p is at least once contained in L_p. Moreover, if $2B_p \not\subset L_p$ then by Table (4.2) $L'_p := L_p - B_p$ has multiplicity two in p, and it still has a common tangent with B_p in p, so that

$$3 \leq L'_p.B_p = L.B - B^2 = A.B + K.B = 3 + K.B = 1 \tag{6.7}$$

again is impossible. We conclude finally, that B_p is at least twice contained in L_p

Note finally, since $\dim |B|_a = 1$ there is a unique curve B_p in $|B|_a$ which passes through p, i. e. it does not depend on the choice of L_p, so that in these cases B_p respectively $2B_p$ is actually a fixed component of $|L - 3p|$. □

7. Regular Surfaces

Theorem 7.1 (“If Σ is regular, then Σ is a rationally ruled surface.”)
More precisely, let Σ be a regular surface and L a line bundle on Σ such that L and $L-K$ are very ample. Suppose that $(L-K)^2 > 16$ and that for a general $p \in \Sigma$ the linear system $|L - 3p|$ contains a curve L_p which has no triple component through p, but such that $h^1\big(\mathcal{J}_{Z_p}(L)\big) \neq 0$ where Z_p is the equimultiplicity scheme of L_p at p.

Then there is a rational ruling $\pi : \Sigma \to \mathbb{P}^1_{\mathbb{C}}$ of Σ such that L_p contains the fibre over $\pi(p)$ with multiplicity two.

Proof: Let us suppose that Σ is regular, so that each algebraic family is indeed a linear system, and let $p \in \Sigma$ be general.

The case $\text{length}(Z'_p) = 4$ is excluded since by Proposition 6.1 the algebraic families $|E|_a$ and $|F|_a$ would have to be linear systems. Thus necessarily $\text{length}(Z'_p) = 3$, and we are done by Proposition 6.2. □

PAPER VII

Triple-Point Defective Ruled Surfaces

Abstract: In [**ChM07a**] we studied triple-point defective very ample linear systems on regular surfaces, and we showed that they can only exist if the surface is ruled. In the present paper we show that we can drop the regularity assumption, and we classify the triple-point defective very ample linear systems on ruled surfaces.

This paper is a joint work with Luca Chiantini, Siena, [**ChM07a**].

1. Introduction

Let Σ be a smooth projective surface, $K = K_\Sigma$ the canonical class and L a divisor class on Σ

We study a classical interpolation problem for the pair (Σ, L), namely whether for a general point $p \in \Sigma$ the linear system $|L-3p|$ has the expected dimension

$$\operatorname{expdim}|L-3p| = \max\{-1, \dim|L| - 6\}.$$

If this is not the case we call the pair (Σ, L) *triple-point defective*.

This paper is indeed a continuation of [**ChM07a**], where some classification of triple point defective pairs is achieved, under the assumptions:

$$L, L-K \textit{ very ample, and } (L-K)^2 > 16,$$

conditions that we will take all over the paper.

With these assumptions, the main result of [**ChM07a**] says that all triple-point defective *regular* surfaces are rationally ruled.

We tackled the problem by considering $|L-3p|$ as fibres of the the map α in the following diagram,

$$|L| = \mathbb{P}_{\mathbb{C}}(H^0(L)^*) \xleftarrow{\ \beta\ } \mathcal{L}_3 \xrightarrow{\ \alpha\ } \Sigma \tag{1.1}$$

where $\mathcal{L}_3$ denotes the incidence variety

$$\mathcal{L}_3 = \{(C,p) \in |L| \times \Sigma \mid \operatorname{mult}_p(C) \geq 3\}$$

and α and β are the obvious projections.

Assuming that for a general point $p \in \Sigma$ there is a curve in L_p with a triple-point in p – and hence α surjective, we considered then the *equimultiplicity scheme* Z_p of a curve $L_p \in |L - 3p|$ defined by

$$\mathcal{J}_{Z_p,p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p \rangle^3.$$

One easily sees that (Σ, L) triple-point defective necessarily implies that

$$h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

Non–zero elements in $H^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big)$ determine by Serre duality a non–trivial extension $\mathcal{E}_p$ of $\mathcal{J}_{Z_p}(L-K)$ by $\mathcal{O}_\Sigma$, which turns out to be a rank 2 bundle on the surface. Due to the assumption $(L-K)^2 > 16$, $\mathcal{E}_p$ is Bogomolov unstable. We then exploited the destabilizing divisor A_p of $\mathcal{E}_p$ in order to obtain the above mentioned result.

For non–regular surfaces, the argument of **[ChM07a]** shows the following lemma (see **[ChM07a]**, Prop. 17 and Prop. 18):

Proposition 1.1
Suppose that, with the notation in (1.1)*, α is surjective, and suppose as usual that L and $L-K$ are very ample with $(L-K)^2 > 16$.*

For p general in Σ and for $L_p \in |L-3p|$ general, call Z_p' the minimal subscheme of the equimultiplicity scheme Z_p of L_p such that

$$h^1\big(\Sigma, \mathcal{J}_{Z_p'}(L)\big) \neq 0.$$

Then either:

1) $\mathrm{length}(Z_p') = 3$ *and Σ is ruled; or*
2) $\mathrm{length}(Z_p') = 4$ *and, for $p \in \Sigma$ general, there are smooth, elliptic curves E_p and F_p in Σ through p such that $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ and $L.E_p = L.F_p = 3$. In particular, both $|E|_a$ and $|F|_a$ induce an elliptic fibration with section on Σ over an elliptic curve.*

This is our starting point. We will in this paper show that the latter case actually cannot occur, and we will classify the triple-point defective linear systems L as above on ruled surfaces. It will in particular follow that the fibre of the ruling is contained exactly twice, and thus that the map β above is generically finite.

Our main results are:

Theorem 1.2
Suppose that the pair (Σ, L) is triple-point defective where L and $L-K$ are very ample with $(L-K)^2 > 16$. Then Σ admids a ruling $\pi : \Sigma \to C$.

For the classification, call C_0 a section of the ruled surface Σ, $\mathfrak{e}$ the line bundle on the base curve given by the determinant of the defining bundle, and call E_i the exceptional divisors (see pp. 122 and 125 for a more precise setting of the notation):

Theorem 1.3
Assume that $\pi : \Sigma \to C$ is a ruled surface and that the pair (Σ, L) is triple-point defective, where L and $L - K$ are very ample with $(L-K)^2 > 16$.

Then π is minimal, i.e. Σ is geometrically ruled, and for a general point $p \in \Sigma$ the linear system $|L - 3p|$ contains a fibre of the ruling as fixed component with multiplicity two.

Moreover, in the previous notation, the line bundle L is of type $C_0 + \pi^*\mathfrak{b}$ for some divisor $\mathfrak{b}$ on C such that $\mathfrak{b} + \mathfrak{e}$ is very ample.

In Section 2 we will first show that a surface Σ admitting two elliptic fibrations as required by Proposition 1.1 would necessarily be a product of two elliptic curves and the triple-point defective linear system would be of type $(3, 3)$. We then show that such a system is never triple-point defective, setting the first part of the main theorem.

In Section 3 we classify the triple-point defective linear systems on ruled surfaces, thus completing our main results.

2. Products of Elliptic Curves

In the above setting, consider a triple-point defective tuple (Σ, L) where the equimultiplicity scheme Z_p (see **[ChM07a]**) of a general element $L_p \in |L - 3p|$ admitted a complete intersection subscheme Z_p' of length *four* with

$$h^1\big(\Sigma, \mathcal{J}_{Z_p'}(L)\big) \neq 0.$$

As explained in the introduction, Prop. 1.1, after **[ChM07a]** we know that, for $p \in \Sigma$ general, there are smooth, elliptic curves E_p and F_p in Σ through p such that $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ and $L.E_p = L.F_p = 3$.

In particular, both $|E|_a$ and $|F|_a$ induce an elliptic fibration with section on Σ over an elliptic curve.

We will now show that this situation indeed cannot occur. Namely, for general p and L_p there cannot exist such a scheme Z_p'.

Lemma 2.1
Suppose that the surface Σ has two elliptic fibrations $\pi : \Sigma \longrightarrow E_0$ and $\pi' : \Sigma \longrightarrow F_0$ with general fibre E respectively F satisfying $E.F = 1$.

Then E_0 and F_0 are elliptic curves, and Σ is the blow-up of a product of two elliptic curves $\Sigma' = E \times E_0 \cong E \times F$.

Proof: Since $E.F = 1$ we have that F is a section of π, and thus $F \cong E_0$ via π. In particular, E_0 and, similarly, F_0 are elliptic curves.

It is well known that there are no non–constant maps from a rational curve to a curve of positive genus ([**Har77**], IV.2.5.4). Thus any exceptional curve of Σ sits in some fiber. Thus we can reach relatively minimal models of π and π' by successively blowing down exceptional -1-curves which belong to fibres of both π and π', i.e. we have the following commutative diagram

$$\begin{tikzcd} \Sigma \arrow[dr, "\phi"] \arrow[drr, bend left, "\pi"] \arrow[ddr, bend right, "\pi'"'] & & \\ & \Sigma' \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\pi}'"] & E_0 \\ & F_0 & \end{tikzcd}$$

where Σ' is actually a minimal surface. Since a general fibre of π or π' is not touched by the blowing-down ϕ we may denote the general fibres of $\widetilde{\pi}$ and $\widetilde{\pi}'$ again by E respectively F, and we still have $E.F = 1$.

We will now try to identify the minimal surface Σ' in the classification of minimal surfaces.

By [**Fri98**] Ex. 7.9 the canonical divisor $K_{\Sigma'}$ is numerically trivial, since Σ' is a minimal surface admitting two elliptic fibrations over elliptic curves.

But then we can apply [**Fri98**] Ex. 7.7, and since the base curve E_0 of the fibration $\widetilde{\pi}$ is elliptic we see that the invariant $d = \deg(L) = \deg\left((R^1\pi_*\mathcal{O}_{\Sigma'})^{-1}\right)$ of the relatively minimal fibration $\widetilde{\pi}$ mentioned in [**Fri98**] Cor. 7.17 is zero, so that the same corollary implies that the fibration has at most multiple fibres with smooth reduction as singular fibres. However, since $\widetilde{\pi}$ has a section F there are no multiple fibres, and thus all fibres of $\widetilde{\pi}$ are smooth.

Moreover, since the canonical divisor of Σ' is numerically trivial it is in particular nef, and by [**Fri98**] Thm. 10.5 we get that the Kodaira dimension $\kappa(\Sigma')$ of Σ' is zero.

Moreover, by [**Fri98**] Cor. 7.16 the surface Σ' has second Chern class $c_2(\Sigma') = 0$, since the invariant $d = \deg\left((R^1\pi_*\mathcal{O}_{\Sigma'})^{-1}\right) = 0$ as already mentioned above. Thus by the Enriques-Kodaira Classification (see e.g. [**BHPV04**] Thm. 10.1.1) Σ' must either be a torus or hyperelliptic (where the latter is sometimes also called bielliptic). A bielliptic surface has precisely two elliptic fibrations, but one of them is a fibration over a $\mathbb{P}_{\mathbb{C}}^1$ and only one is over an elliptic curve (see e.g. [**Rei97**] Thm. E.7.2). Thus Σ' is not bielliptic. Moreover, if Σ' is a torus then $K_{\Sigma'}$ is trivial and thus so is $(R^1\pi_*\mathcal{O}_{\Sigma'})^{-1}$, which by [**Fri98**] Cor. 7.21 implies that Σ' is a product of the base curve with a fibre. $\square$

Lemma 2.1 implies that in order to show that the situation of Proposition 1.1 cannot occur, we have to understand products of elliptic curves.

Let us, therefore, consider a surface $\Sigma = C_1 \times C_2$ which is the product of two smooth elliptic curves.

Let us set some notation. We will use some results by **[Kei01]** Appendices G.b and G.c in the sequel.

The surface Σ is naturally equipped with two projections $\pi_i : \Sigma \longrightarrow C_i$. If $\mathfrak{a}$ is a divisor on C_2 of degree a and $\mathfrak{b}$ is a divisor on C_1 of degree b then the divisor $\pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b} \sim_a aC_1 + bC_2$, where by abuse of notation we denote by C_1 a fixed fibre of π_2 and by C_2 a fixed fibre of π_1. Moreover, K_Σ is trivial, and given two divisors $D \sim_a aC_1 + bC_2$ and $D' \sim_a a'C_1 + b'C_2$ then the intersection product is

$$D.D' = (aC_1 + bC_2).(a'C_1 + b'C_2) = a \cdot b' + a' \cdot b.$$

We will consider first the case

$$L = \pi_2^*(\mathfrak{a}) + \pi_1^*(\mathfrak{b})$$

where both $\mathfrak{b}$ on C_1 and $\mathfrak{a}$ on C_2 are divisors of degree 3. The dimension of the linear system $|L|$ is $\dim |L| = 8$, and thus for a point $p \in \Sigma$ the expected dimension is $\operatorname{expdim} |L - 3p| = \dim |L| - 6 = 2$.

Notice that a divisor of degree three on an elliptic curve is always very ample and embeds the curve as a smooth cubic in $\mathbb{P}_{\mathbb{C}}^2$. Since the smooth plane cubics are classified by their normal forms $xz^2 - y \cdot (y - x) \cdot (y - \lambda \cdot x)$ with $\lambda \neq 0$ the following example reflects the behaviour of any product of elliptic curves embedded via a linear system of bidegree $(3, 3)$.

Example 2.2
Consider two smooth plane cubics

$$C_1 = V\big(xz^2 - y \cdot (y - z) \cdot (y - az)\big)$$

and

$$C_2 = V\big(xz^2 - y \cdot (y - z) \cdot (y - bz)\big).$$

The surface $\Sigma = C_1 \times C_2$ is embedded into $\mathbb{P}_{\mathbb{C}}^8$ via the Segre embedding

$$\phi : \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2 \longrightarrow \mathbb{P}_{\mathbb{C}}^8 : ((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \mapsto (x_0y_0 : \ldots : x_2y_2).$$

We may assume that both curves contain the point $p = (1 : 0 : 0)$ as a general non-inflexion point, and the point (p, p) is mapped by the Segre embedding to $\phi(p, p) = (1 : 0 : \ldots : 0)$. If we denote by $z_{i,j}$, $i, j \in \{0, 1, 2\}$, the coordinates on $\mathbb{P}_{\mathbb{C}}^8$ as usual, then the maximal ideal locally at $\phi(p, p)$ is generated by $z_{0,2}$ and $z_{2,0}$, i.e. these are local coordinates of Σ at $\phi(p, p)$. A standard basis computation shows that locally at $\phi(p,p)$ the coordinates $z_{i,j}$ satisfy modulo the ideal of Σ and up to multiplication by a unit the following congruences (note, $z_{0,0} = 1$)

$$
\begin{aligned}
z_{0,1} &\equiv \frac{1}{b} \cdot z_{0,2}^2, & z_{1,0} &\equiv \frac{1}{a} \cdot z_{2,0}^2, & z_{1,1} &\equiv \frac{1}{ab} \cdot z_{0,2}^2 \cdot z_{2,0}^2,\\
z_{1,2} &\equiv \frac{1}{a} \cdot z_{0,2} \cdot z_{2,0}^2, & z_{2,1} &\equiv \frac{1}{b} \cdot z_{0,2}^2 \cdot z_{2,0}, & z_{2,2} &\equiv z_{0,2} \cdot z_{2,0}.
\end{aligned}
$$

Thus a hyperplane section $H = a_{0,0}z_{0,0} + \ldots + a_{2,2}z_{2,2}$ of Σ is locally in $\phi(p,p)$ modulo $\mathfrak{m}^3 = \langle z_{0,2}, z_{2,0}\rangle^3$ given by

$$H \equiv a_{0,0} + a_{0,2}z_{0,2} + a_{2,0}z_{2,0} + \frac{a_{0,1}}{b} \cdot z_{0,2}^2 + \frac{a_{1,0}}{a} \cdot z_{2,0}^2 + a_{2,2}z_{0,2}z_{2,0},$$

and hence the family of hyperplane sections having multiplicity at least three in $\phi(p,p)$ is given by

$$a_{0,0} = a_{0,1} = a_{1,0} = a_{0,2} = a_{2,0} = a_{2,2} = 0.$$

But then the family has parameters $a_{1,1}, a_{1,2}, a_{2,1}$, and its dimension coincides with the expected dimension 2. Moreover, the 3-jet of a hyperplane section H through $\phi(p,p)$ with multiplicity at least three is

$$\operatorname{jet}_3(H) \equiv z_{0,2} \cdot z_{2,0} \cdot \left(\frac{a_{1,2}}{a} \cdot z_{2,0} + \frac{a_{2,1}}{b} \cdot z_{0,2}\right),$$

which shows that for a general choice of $a_{2,1}$ and $a_{1,2}$ the point $\phi(p,p)$ is an ordinary triple point.

Remark 2.3

We actually can say very precisely what it means that p is general in the product, namely that neither $\pi_1(p)$ is a inflexion point of C_1, nor $\pi_2(p)$ is a inflexion point of C_2.

Indeed, since $\mathfrak{a}$ is very ample of degree three, for each point $p \in \Sigma$ there is a unique point $q_a \in C_2$ such that $q_a + 2 \cdot \pi_2(p) \sim_l \mathfrak{a}$. When $\pi_2(p)$ is a inflexion point of C_2, then $q_a = \pi_2(p)$ and thus the two-dimensional family

$$3C_{1,\pi_2(p)} + |\pi^*(\mathfrak{b})| \subset |L - 3p|$$

gives a superabundance of the dimension of $|L - 3p|$ by one.

Similarly one can argue when $\pi_1(p)$ is a inflexion point of C_1.

Now we are ready for the proof of Theorem 1.2.

Proof of Theorem 1.2: By Proposition 1.1, it is enough to prove that when Σ has two elliptic fibrations as in the proposition, then Σ is not triple–point defective.

By Lemma 2.1, Σ is the blow-up $\pi : \Sigma \longrightarrow \Sigma'$ of a product $\Sigma' = C_1 \times C_2$ of two elliptic curves, and we may assume that the curves E_p and F_p in Proposition 1.1 are the fibres of π_1 respectively π_2.

Our first aim will be to show that actually $\Sigma = \Sigma'$. For this note that

$$\operatorname{Pic}(\Sigma) = \bigoplus_{i=1}^{k} E_i \oplus \pi^* \operatorname{Pic}(\Sigma'),$$

where the E_i are the total transforms of the exceptional curves arising throughout the blow-up, i.e. the E_i are (not necessarily irreducible) rational curves with self-intersection $E_i^2 = -1$ and such that $E_i.E_j = 0$ for $i \neq j$ and $E_i.\pi^*(C) = 0$ for any curve C on Σ'. In particular, since $K_{\Sigma'}$ is trivial we have that $K_\Sigma = \sum_{i=1}^k E_i$, and if $L = \pi^*L' - \sum_{i=1}^k e_i E_i$ then $L - K = \pi^* L' - \sum_{i=1}^k (e_i+1)E_i$. We therefore have

$$16 < (L-K)^2 = (L')^2 - \sum_{i=1}^{k} (e_i+1)^2,$$

or equivalently

$$(L')^2 \geq 17 + \sum_{i=1}^{k} (e_i+1)^2 \geq 17 + 4k, \tag{2.1}$$

where the latter inequality is due to the fact that $e_i = L.E_i > 0$ since L is very ample. By the assumption of Proposition 1.1 we know that $L'.C_1 = L.E_p = 3$ and $L'.C_2 = L.F_p = 3$, and therefore by **[Har77]** Ex. V.1.9

$$(L')^2 \leq 2 \cdot (L'.C_1) \cdot (L'.C_2) = 18. \tag{2.2}$$

But (2.1) and (2.2) together imply that no exceptional curve exists, i.e. $\Sigma = \Sigma'$.

Since now Σ is a product of two elliptic curves, by **[LaB92]** we know that the Picard number $\rho = \rho(\Sigma)$ satisfies $2 \leq \rho \leq 4$, and the Néron-Severi group can be generated by the two general fibres C_1 and C_2 together with certain graphs C_j, $3 \leq j \leq \rho$, of morphisms $\varphi_j : C_1 \longrightarrow C_2$. In particular, $C_j.C_2 = 1$ and $C_j.C_1 = \deg(\varphi_j) \geq 1$ for $3 \leq j \leq \rho$. Moreover, these graphs have self intersecting zero. If we now assume that $L \sim_a \sum_{j=1}^{\rho} a_i C_i$ then

$$L^2 = 2 \cdot \sum_{i<j} a_i \cdot a_j \cdot (C_i.C_j)$$

is divisible by 2, and since $L = L - K$ with $(L-K)^2 > 16$ we deduce with **[Har77]** Ex. V.1.9 that

$$L^2 = (L-K)^2 = 18 = 2 \cdot (L.C_1) \cdot (L.C_2),$$

and thus that

$$L \sim_a 3C_1 + 3C_2,$$

or in equivalently, that

$$L = \pi_2^* \mathfrak{a} + \pi_1^* \mathfrak{b}$$

for some divisors $\mathfrak{a}$ on C_2 and $\mathfrak{b}$ on C_1, both of degree 3. That is, we are in the situation of Example 2.2, and we showed there that (Σ, L) then is not triple-point defective. $\square$

Remark 2.4
Notice that, in practice, since

$$h^1(\Sigma, L) = h^0(C_1, \mathfrak{b}) \cdot h^1(C_2, \mathfrak{a}) + h^0(C_2, \mathfrak{a}) \cdot h^1(C_1, \mathfrak{b}) = 0,$$

the non-triple-point defectiveness shows that for general $p \in \Sigma$ and $L_p \in |L - 3p|$ no Z'_p as in the assumptions of Proposition 1.1 can have length 4.

3. Geometrically Ruled Surfaces

Let $\Sigma = \mathbb{P}_{\mathbb{C}}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalized bundle $\mathcal{E}$ (in the sense of **[Har77]** V.2.8.1). The Néron-Severi group of Σ is

$$\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus f\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $f \cong \mathbb{P}_{\mathbb{C}}^1$ is a fixed fibre of π, C_0 a fixed section of π with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}(\mathcal{E})}(1)$, and $e = -\deg(\mathfrak{e}) \geq -g$ where $\mathfrak{e} = \Lambda^2\mathcal{E}$. If $\mathfrak{b}$ is a divisor on C we will write $\mathfrak{b}f$ for the divisor $\pi^*(\mathfrak{b})$ on Σ, and so for the canonical divisor we have

$$K_\Sigma \sim_l -2C_0 + (K_C + \mathfrak{e}) \cdot f \sim_a -2C_0 + (2g - 2 - e)f,$$

where $g = g(C)$ is the genus of the base curve C.

Example 3.1
Let $\mathfrak{b}$ be a divisor on C such that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are both very ample and such that $\mathfrak{b}$ is non-special. If C is rational we should in addition assume that $\deg(\mathfrak{b}) + \deg(\mathfrak{b} + \mathfrak{e}) \geq 6$. Then the divisor $L = C_0 + \mathfrak{b}f$ is very ample (see e.g. **[FuP00]** Prop. 2.15) of dimension

$$\dim |L| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 1$$

Moreover, for any point $p \in \Sigma$ we then have (see **[FuP00]** Cor. 2.13)

$$\dim |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f| = \dim |\, C_0 + \mathfrak{b}f| - 4 = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 5,$$

and we have for p general

$$\dim |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f - p\,| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 6.$$

For this note that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ very ample implies that this number is non-negative – in the rational case we need the above degree bound.

If we denote by $f_p = \pi^*\big(\pi(p)\big)$ the fibre of π over $\pi(p)$, then by Bézout and since $L.f_p = (L - f_p).f_p = 1$ we see that $2f_p$ is a fixed component of $|L - 3p|$ and we have

$$|L - 3p| = 2f_p + |\, C_0 + (\mathfrak{b} - 2\pi(p)) \cdot f - p\,|,$$

so that

$$\begin{aligned}\dim|L-3p| &= h^0(C,\mathfrak{b})+h^0(C,\mathfrak{b}+\mathfrak{e})-6=\dim|L|-5\\ &> \dim|L|-6=\operatorname{expdim}|L-3p|.\end{aligned}$$

This shows that (Σ,L) is triple-point defective and $|L-3p|$ contains a fibre of the ruling as double component. Moreover, for a general p the linear series $|L-3p|$ cannot contain a fibre of the ruling more than twice due to the above dimension count for $|\,C_0+(\mathfrak{b}-2\pi(p))\cdot f-p\,|$.

Next we are showing that a geometrically ruled surface is indeed triple-point defective with respect to a line bundle L which fulfills our assumptions, and in Corollary 3.6 we will see that this is not the case for non-geometrically ruled surfaces.

Proposition 3.2
On every geometrically ruled surface $\Sigma=\mathbb{P}_C(\mathcal{E})\xrightarrow{\pi}C$ there exists some very ample line bundle L such that the pair (Σ,L) is triple–point defective, and moreover also $L-K$ is very ample with $(L-K)^2>16$.

Proof: It is enough to take $L=C_0+\mathfrak{b}f$, with $b=\deg(\mathfrak{b})=3a$ such that $a,a-e,a+e,a-2g+2+e,a-2g+2-e$ are all bigger or equal than $2g+1$.

Indeed in this case $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are both very ample. For $p\in C$ general, we also have that both $\mathfrak{b}-p$ and $\mathfrak{b}+\mathfrak{e}-p$ are non-special. It follows that L is very ample (by **[Har77]** Ex. V.2.11.b) and (Σ,L) is triple point defective, by the previous example. Moreover, in this situation we have:

$$L-K\sim_l 3C_0+\big(\mathfrak{b}-K_C-\mathfrak{e}\big)\cdot f.$$

Hence

$$(L-K)^2=\big(3C_0+(\deg(\mathfrak{b})-2g+2+e)\cdot f\big)^2\geq 18>16.$$

Finally, if we fix a divisor $\mathfrak{a}$ of degree a on C, then $L-K$ is the sum of the divisors $C_0+\big(\mathfrak{a}-K_C\big)\cdot f$, $C_0+\big(\mathfrak{a}-\mathfrak{e}\big)\cdot f$, $C_0+\mathfrak{a}f$, which are very ample (**[Har77]** Ex. V.2.11). Thus $L-K$ is very ample. $\square$

Next, let us describe which linear systems L on a ruled surface Σ determine a triple-point defective pair (Σ,L).

We will show that example 3.1 describes, in most cases, the only possibilities. In order to do so we first have to consider the possible algebraic classes of irreducible curves with self-intersection zero on a ruled surface.

Lemma 3.3
Let $B\in|bC_0+b'f|_a$ be an irreducible curve with $B^2=0$ and $\dim|B|_a\geq 0$, then we are in one of the following cases:

(a.1) $B\sim_a f$,

(a.2) $e = 0$, $b \geq 1$, $B \sim_a bC_0$, *and* $|B|_a = |B|_l$, *or*

(a.3) $e < 0$, $b \geq 2$, $b' = \frac{b}{2}e < 0$, $B \sim_a bC_0 + \frac{b}{2}ef$ *and* $|B|_a = |B|_l$.

Moreover, if $b = 1$, *then* $\Sigma \cong C_0 \times \mathbb{P}_{\mathbb{C}}^1$.

Proof: See **[Kei01]** App. Lemma G.2. □

We can now classify the triple-point defective linear systems on a geometrically ruled surface. In order to do so we should recall the result of **[ChM07a]** Prop. 18.

Proposition 3.4
Suppose that, with the notation in (1.1)*,* α *is surjective, and suppose that* L *and* $L - K$ *are very ample with* $(L-K)^2 > 16$. *Moreover, suppose that for* $p \in \Sigma$ *general and for* $L_p \in |L - 3p|$ *general the equimultiplicity scheme* Z_p *of* L_p *has a subscheme* Z_p' *of length* 3 *such that* $h^1\big(\Sigma, \mathcal{J}_{Z_p'}(L)\big) \neq 0$.

Then for $p \in \Sigma$ *general there is an irreducible, smooth, rational curve* B_p *in a pencil* $|B|_a$ *with* $B^2 = 0$, $(L-K).B = 3$ *and* $L - K - B$ *big.*

In particular, $\Sigma \to |B|_a$ *is a ruled surface and* $2B_p$ *is a fixed component of* $|L - 3p|$.

Theorem 3.5
With the above notation let $\pi : \Sigma \to C$ *be a geometrically ruled surface, and let* L *be a line bundle on* Σ *such that* L *and* $L - K$ *are very ample. Suppose that* $(L-K)^2 > 16$ *and that for a general* $p \in \Sigma$ *the linear system* $|L - 3p|$ *contains a curve* L_p *such that* $h^1\big(\Sigma, \mathcal{J}_{Z_p}(L)\big) \neq 0$ *where* Z_p *is the equimultiplicity scheme of* L_p *at* p.

Then $L = C_0 + \mathfrak{b} \cdot f$ *for some divisor* $\mathfrak{b}$ *on* C *such that* $\mathfrak{b} + \mathfrak{e}$ *is very ample and* $|L - 3p|$ *contains a fibre of* π *as fixed component with multiplicity two. Moreover, if* $e \geq -1$ *then* $\deg(\mathfrak{b}) \geq 2g + 1$ *and we are in the situation of Example 3.1.*

Proof: As in the proof of **[ChM07a]** Thm. 19, since the case in which the length of Z_p is 4 has been ruled out in Remark 2.4, we only have to consider the situations in Proposition 3.4 above.

Using the notation there we have a divisor $A := L - K - B \sim_a aC_0 + a'f$ and a curve $B \sim_a bC_0 + b'f$ satisfying certain numerical properties, in particular $p_a(B) = 0$, $B^2 = 0$, and $a > 0$ since A is big. Moreover,

$$3 = A.B = -eab + ab' + a'b \tag{3.1}$$

and

$$a \cdot (2a' - ae) = A^2 = (L-K)^2 - 2 \cdot A.B - B^2 \geq 17 - 2 \cdot A.B - B^2 = 11. \tag{3.2}$$

By Lemma 3.3 there are three possibilities for B to consider. If $e < 0$ and $B \sim_a bC_0 + \frac{eb}{2} \cdot f$ with $b \geq 2$, then Riemann-Roch leads to the impossible equation

$$-2 = 2p_a(B) - 2 = B.K = (2g-2) \cdot b.$$

If $e = 0$ and $B \sim_a bC_0$, then similarly Riemann-Roch shows

$$-2 = B.K = (2g-2) \cdot b,$$

which now implies that $b = 1$ and $g = 0$. In particular, $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $L \sim_a A + B + K \sim_a (a-1) \cdot C_0 + f$, since $3 = A.B = a'$. But this is then one of the cases of Example 3.1.

Finally, if $B \sim_a f$ then (3.1) gives $a = 3$, and thus

$$L \sim_a A + B + K \sim_a C_0 + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) \cdot f,$$

where $A = 3C_0 + \mathfrak{a}' \cdot f$. Moreover, by the assumptions of Case (b) the linear system $|L - 3p|$ contains the fibre of the ruling over p as double fixed component, and since L is very ample it induces on C the very ample divisor $\mathfrak{e} + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e})$. Note also, that (3.2) implies that

$$a' - 2 - e \geq \frac{e}{2},$$

and thus for $e \geq -1$ we have

$$\deg(\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) = 2g + 1 + (a' - 2 - e) \geq 2g + 1,$$

so that then the assumptions of Example 3.1 are fulfilled. This finishes the proof. □

If $\pi : \Sigma \longrightarrow C$ is a ruled surface, then there is a (not necessarily unique (if $g(C) = 0$)) minimal model

$$\begin{tikzcd} \Sigma \arrow[dr, "\phi"'] \arrow[drr, bend left, "\pi"] & & \\ & \Sigma' \arrow[r, "\widetilde{\pi}"] & C, \end{tikzcd}$$

and the Néron-Severi group of Σ is

$$\mathrm{NS}(\Sigma) = C_0 \cdot \mathbb{Z} \oplus f \cdot \mathbb{Z} \oplus \bigoplus_{i=1}^{k} E_i \cdot \mathbb{Z},$$

where f is a general fibre of π, C_0 is the total transform of section of $\widetilde{\pi}$, and the E_i are the total transforms of the exceptional divisors of the blow-up ϕ. Moreover, for the Picard group of Σ we just have to replace $f \cdot \mathbb{Z}$ by $\pi^* \operatorname{Pic}(C)$. We may, therefore, represent a divisor class A on Σ as

$$L = a \cdot C_0 + \pi^* \mathfrak{b} - \sum_{i=1}^{k} c_i E_i. \tag{3.3}$$

Corollary 3.6
Suppose that (Σ, L) *is a tuple as in Proposition 1.1 with ruling* $\pi : \Sigma \to C$*, and suppose that the Néron-Severi gruop of* Σ *is as described before with general fibre* $f = B_p$*.*

Then Σ *is minimal,* $L = C_0 + \pi^*\mathfrak{b}$ *for some divisor* $\mathfrak{b}$ *on* C *such that* $\mathfrak{b} + \mathfrak{e}$ *is very ample and* $|L - 3p|$ *contains a fibre of* π *as fixed component with multiplicity two.*

Proof: Let $L = C_0 + \pi^*\mathfrak{b} - \sum_{i=1}^k c_i E_i$, as described in (3.3). Then

$$L - K = (a+2) \cdot C_0 + \pi^*(\mathfrak{b} - K_C - \mathfrak{e}) - \sum_{i=1}^k (c_i + 1) \cdot E_i,$$

and thus considering Proposition 3.4

$$3 = (L-K).B = a + 2.$$

The very ampleness of L implies thus that $c_i > 0$ for all i. But then, if Σ is not minimal and f' is the strict transform of a fiber of the minimal model, meeting some E_i, then $L \cdot f' \leq 0$, a contradiction. $\square$

By [**ChM07a**] we get Theorem 1.3 as an immediate corollary.

Part C

PAPER VIII

Standard Bases in $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

Abstract: In this paper we study standard bases for submodules of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$ respectively of their localisation with respect to a $\underline{t}$-local monomial ordering. The main step is to prove the existence of a division with remainder generalising and combining the division theorems of Grauert-Hironaka and Mora. Everything else then translates naturally. Setting either $m = 0$ or $n = 0$ we get standard bases for polynomial rings respectively for power series rings as a special case. We then apply this technique to show that the t-initial ideal of an ideal over the Puiseux series field can be read of from a standard basis of its generators. This is an important step in the constructive proof that each point in the tropical variety of such an ideal admits a lifting.

[**Mar07**]

The paper follows the lines of [**GrP02**] and [**DeS07**] generalising the results where necessary. Basically, the only original parts for the standard bases are the proofs of Theorem 2.1 and Theorem 3.3, but even here they are easy generalisations of Grauert-Hironaka's respectively Mora's Division Theorem (the latter in the form stated and proved first by Greuel and Pfister, see [**GGM$^+$94**], [**GrP96**]; see also [**Mor82**], [**Grä94**]). The paper should therefore rather be seen as a unified approach for the existence of standard bases in polynomial and power series rings, and it was written mostly due to the lack of a suitable reference for the existence of standard bases in $K[[t]][x_1,\ldots,x_n]$ which are needed when dealing with tropical varieties. Namely, when we want to show that every point in the tropical variety of an ideal J defined over the field of Puiseux series exhibits a lifting to the variety of J, then, assuming that J is generated by elements in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$, we need to know that we can compute the so-called t-initial ideal of J by computing a standard basis of the ideal defined by the generators in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$ (see Theorem 6.10 and [**JMM07**]).

An important point is that if the input data is polynomial in both $\underline{t}$ and $\underline{x}$ then we can actually compute the standard basis since a standard basis computed in $K[t_1,\ldots,t_m]_{\langle t_1,\ldots,t_m\rangle}[x_1,\ldots,x_n]$ will do (see Corollary 4.7). This was previously known for the case where there are no x_i (see [**GrP96**]).

In this paper we treat only formal power series, while Grauert (see [**Gra72**]) and Hironaka (see [**Hir64**]) considered convergent power series with respect to certain valuations which includes the formal case. It should be rather straight forward how to adjust Theorem 2.1 accordingly. Many authors contributed to the further development (see e.g. [**Bec90**] for a standard basis criterion in the power series ring) and to generalisations of the theory, e.g. to algebraic power series (see e.g. [**Hir77**], [**AMR77**], [**ACH05**]) or to differential operators (see e.g. [**GaH05**]). This list is by no means complete.

In Section 1 we introduce the basic notions. Section 2 is devoted to the proof of the existence of a determinate division with remainder for polynomials in $K[[t_1,\ldots,t_m]][x_1,\ldots,x_m]^s$ which are homogeneous with respect to the x_i. This result is then used in Section 3 to show the existence of weak divisions with remainder for all elements of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_m]^s$. In Section 4 we introduce standard bases and prove the basics for these, and we prove Schreyer's Theorem and, thus Buchberger's Criterion in Section 5. Finally, in Section 6 we apply standard bases to study t-initial ideals of ideals over the Puiseux series field.

1. Basic Notation

Throughout the paper K will be any field, $R = K[[t_1,\ldots,t_m]]$ will denote the ring of formal power series over K and

$$R[x_1,\ldots,x_n] = K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]$$

denotes the ring of polynomials in the indeterminates $x_1,\ldots,x_n$ with coefficients in the power series ring R. We will in general use the short hand notation $\underline{x} = (x_1,\ldots,x_n)$ and $\underline{t} = (t_1,\ldots,t_m)$, and the usual multi index notation

$$\underline{t}^\alpha = t_1^{\alpha_1}\cdots t_m^{\alpha_m} \quad \text{and} \quad \underline{x}^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n},$$

for $\alpha = (\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m$ and $\beta = (\beta_1,\ldots,\beta_n)\in\mathbb{N}^n$.

Definition 1.1
A *monomial ordering* on

$$\mathrm{Mon}(\underline{t},\underline{x}) = \left\{\underline{t}^\alpha\cdot\underline{x}^\beta \;\middle|\; \alpha\in\mathbb{N}^m,\beta\in\mathbb{N}^n\right\}$$

is a total ordering $>$ on $\mathrm{Mon}(\underline{t},\underline{x})$ which is compatible with the semi group structure of $\mathrm{Mon}(\underline{t},\underline{x})$, i.e. such that for all $\alpha,\alpha',\alpha''\in\mathbb{N}^m$ and $\beta,\beta',\beta''\in\mathbb{N}^n$

$$\underline{t}^\alpha\cdot\underline{x}^\beta \;>\; \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'} \quad\Longrightarrow\quad \underline{t}^{\alpha+\alpha''}\cdot\underline{x}^{\beta+\beta''} \;>\; \underline{t}^{\alpha'+\alpha''}\cdot\underline{x}^{\beta'+\beta''}.$$

We call a monomial ordering $>$ on $\mathrm{Mon}(\underline{t},\underline{x})$ $\underline{t}$*-local* if its restriction to $\mathrm{Mon}(\underline{t})$ is *local*, i.e. $t_i < 1$ for all $i = 1,\ldots,m$. We call a $\underline{t}$-local monomial ordering on

$\mathrm{Mon}(\underline{t}, \underline{x})$ a *$\underline{t}$-local weighted degree ordering* if there is a $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$ such that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $\beta, \beta' \in \mathbb{N}^n$

$$w \cdot (\alpha, \beta) > w \cdot (\alpha', \beta') \implies \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'},$$

where $w \cdot (\alpha, \beta) = w_1 \cdot \alpha_1 + \ldots + w_m \cdot \alpha_m + w_{m+1} \cdot \beta_1 + \ldots + w_n \cdot \beta_n$ denotes the standard scalar product. We call w a *weight vector* of $>$.

Example 1.2
The *$\underline{t}$-local lexicographical ordering* $>_{lex}$ on $\mathrm{Mon}(\underline{t}, \underline{x})$ is defined by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \ > \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$$

if and only if

$$\exists\, j \in \{1, \ldots, n\} \ : \ \beta_1 = \beta'_1, \ldots, \beta_{j-1} = \beta'_{j-1}, \ \text{ and } \ \beta_j > \beta'_j,$$

or

$$\big(\beta = \beta' \text{ and } \exists\, j \in \{1, \ldots, m\} \ : \ \alpha_1 = \alpha'_1, \ldots, \alpha_{j-1} = \alpha'_{j-1}, \alpha_j < \alpha'_j\big).$$

Example 1.3
Let $>$ be any $\underline{t}$-local ordering and $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$, then $\underline{t}^\alpha \cdot \underline{x}^\beta >_w \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$ if and only if $w \cdot (\alpha, \beta) > w \cdot (\alpha', \beta')$ or

$$\big(w \cdot (\alpha, \beta) = w \cdot (\alpha', \beta') \ \text{ and } \ \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}\big)$$

defines a $\underline{t}$-local weighted degree ordering $>_w$ on $\mathrm{Mon}(\underline{t}, \underline{x})$ with weight vector w.

Even if we are only interested in standard bases of ideals we have to pass to submodules of free modules in order to have syzygies at hand for the proof of Buchberger's Criterion via Schreyer orderings.

Definition 1.4
We define

$$\mathrm{Mon}^s(\underline{t}, \underline{x}) := \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m, i = 1, \ldots, s\big\},$$

where $e_i = (\delta_{ij})_{j=1,\ldots,s}$ is the vector with all entries zero except the i-th one which is one. We call the elements of $\mathrm{Mon}^s(\underline{t}, \underline{x})$ *module monomials* or simply *monomials*.

For $p, p' \in \mathrm{Mon}^s(\underline{t}, \underline{x}) \cup \{0\}$ the notion of divisibility and of the lowest common multiple $\mathrm{lcm}(p, p')$ are defined in the obvious way.

Given a monomial ordering on $\mathrm{Mon}(\underline{t}, \underline{x})$, a *$\underline{t}$-local monomial ordering* on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ with respect to $>$ is a total ordering $>_m$ on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ which is strongly compatible with the operation of the multiplicative semi group $\mathrm{Mon}(\underline{t}, \underline{x})$ on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ in the sense that

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \ >_m \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j \quad \Longrightarrow \quad \underline{t}^{\alpha+\alpha''} \cdot \underline{x}^{\beta+\beta''} \cdot e_i \ >_m \ \underline{t}^{\alpha'+\alpha''} \cdot \underline{x}^{\beta'+\beta''} \cdot e_j$$

and

$$\underline{t}^\alpha\cdot\underline{x}^\beta > \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'} \iff \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i >_m \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_i$$

for all $\beta,\beta',\beta''\in\mathbb{N}^n$, $\alpha,\alpha',\alpha''\in\mathbb{N}^m, i,j\in\{1,\dots,s\}$.

Note that due to the second condition the ordering $>_m$ on $\operatorname{Mon}^s(\underline{t},\underline{x})$ determines the ordering $>$ on $\operatorname{Mon}(\underline{t},\underline{x})$ uniquely, and we will therefore usually not distinguish between them, i.e. we will use the same notation $>$ also for $>_m$, and we will not specify the monomial ordering on $\operatorname{Mon}(\underline{t},\underline{x})$ in advance, but instead refer to it as the *induced monomial ordering on* $\operatorname{Mon}(\underline{t},\underline{x})$.

We call a monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ *$\underline{t}$-local* if the induced monomial ordering on $\operatorname{Mon}(\underline{t},\underline{x})$ is so.

We call a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ a *$\underline{t}$-local weight ordering* if there is a $w=(w_1,\dots,w_{m+n+s})\in\mathbb{R}^m_{\leq 0}\times\mathbb{R}^n\times\mathbb{R}^s$ such that for all $\alpha,\alpha'\in\mathbb{N}^m$, $\beta,\beta'\in\mathbb{N}^n$ and $i,j\in\{1,\dots,s\}$

$$w\cdot(\alpha,\beta,e_i)>w\cdot(\alpha',\beta',e_j) \implies \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j,$$

and we call w a *weight vector* of $>$.

Example 1.5

Let $w\in\mathbb{R}^m_{\leq 0}\times\mathbb{R}^{n+s}$ and let $>$ be any $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ such that the induced $\underline{t}$-local monomial ordering on $\operatorname{Mon}(\underline{t},\underline{x})$ is a $\underline{t}$-local weighted degree ordering with respect to the weight vector $(w_1,\dots,w_{m+n})$. Then

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i >_w \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j$$

if and only if

$$w\cdot(\alpha,\beta,e_i)>w\cdot(\alpha',\beta',e_j)$$

or

$$\big(w\cdot(\alpha,\beta,e_i)=w\cdot(\alpha',\beta',e_j) \text{ and } \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j\big)$$

defines a $\underline{t}$-local weight monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ with weight vector w. In particular, there exists such a monomial ordering.

Remark 1.6

In the following we will mainly be concerned with monomial orderings on $\operatorname{Mon}^s(\underline{t},\underline{x})$ and with submodules of free modules over $R[\underline{x}]$, but all these results specialise to $\operatorname{Mon}(\underline{t},\underline{x})$ and ideals by just setting $s=1$. □

For a $\underline{t}$-local monomial ordering we can introduce the notions of leading monomial and leading term of elements in $R[\underline{x}]^s$.

Definition 1.7

Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$. We call

$$0\neq f=\sum_{i=1}^s\sum_{|\beta|=0}^d\sum_{|\alpha|=0}^\infty a_{\alpha,\beta,i}\cdot\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i\in R[\underline{x}]^s,$$

with $a_{\alpha,\beta,i} \in K$, $|\beta| = \beta_1 + \ldots + \beta_n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_m$, the *distributive representation* of f, $\mathcal{M}_f := \{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0\}$ the set of *monomials of* f and $\mathcal{T}_f := \{a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0\}$ the set of *terms of* f.

Moreover, $\mathrm{lm}_>(f) := \max\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in \mathcal{M}_f\}$ is called the *leading monomial* of f. Note again, that this maximum exists since the number of β's occurring in f and the number of i's is finite and the ordering is local with respect to $\underline{t}$.

If $\mathrm{lm}_>(f) = \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ then we call $\mathrm{lc}_>(f) := a_{\alpha,\beta,i}$ the *leading coefficient* of f, $\mathrm{lt}_>(f) := a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ its *leading term*, and $\mathrm{tail}_>(f) := f - \mathrm{lt}_>(f)$ its *tail*.

For the sake of completeness we define $\mathrm{lm}_>(0) := 0,\ \mathrm{lt}_>(0) := 0,\ \mathrm{lc}_>(0) := 0,\ \mathrm{tail}_>(f) = 0$, and $0 < \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \quad \forall\, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n, i \in \mathbb{N}$.

Finally, for a subset $G \subseteq R[\underline{x}]^s$ we call the submodule

$$L_>(G) = \langle \mathrm{lm}_>(f) \mid f \in G \rangle \leq K[\underline{t}, \underline{x}]^s$$

of the free module $K[\underline{t}, \underline{x}]^s$ over the polynomial ring $K[\underline{t}, \underline{x}]$ generated by all the leading monomials of elements in G the *leading submodule* of G.

We know that in general a standard basis of an ideal respectively submodule I will not be a generating set of I itself, but only of the ideal respectively submodule which I generates in the localisation with respect to the monomial ordering. We therefore introduce this notion here as well.

Definition 1.8
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}(\underline{t}, \underline{x})$, then $S_> = \{u \in R[\underline{x}] \mid \mathrm{lt}_>(u) = 1\}$ is the *multiplicative set associated to* $>$, and $R[\underline{x}]_> = S_>^{-1} R[\underline{x}] = \left\{ \frac{f}{u} \,\middle|\, f \in R[\underline{x}], u \in S_> \right\}$ is the *localisation of* $R[\underline{x}]$ *with respect to* $>$.

If $>$ is a $\underline{t}$-local monomial ordering with $x_i > 1$ for all $i = 1, \ldots, n$ (e.g. $>_{lex}$ from Example 1.2), then $S_> \subset R^*$, and therefore $R[\underline{x}]_> = R[\underline{x}]$.

It is straight forward to extend the notions of leading monomial, leading term and leading coefficient to $R[\underline{x}]_>$ and free modules over this ring.

Definition 1.9
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $g = \frac{f}{u} \in R[\underline{x}]_>^s$ with $u \in S_>$, and $G \subseteq R[\underline{x}]_>^s$. We then define the *leading monomial*, the *leading coefficient* respectively the *leading term* of g as

$$\mathrm{lm}_>(g) := \mathrm{lm}_>(f), \quad \mathrm{lc}_>(g) := \mathrm{lc}_>(f), \quad \text{resp.} \quad \mathrm{lt}_>(g) := \mathrm{lt}_>(f),$$

and the *leading ideal* (if $s = 1$) respectively *leading submodule* of G

$$L_>(G) = \langle \mathrm{lm}_>(h) \mid h \in G \rangle \leq K[\underline{t}, \underline{x}]^s.$$

These definitions are independent of the chosen representative, since if $g = \frac{f}{u} = \frac{f'}{u'}$ then $u' \cdot f = u \cdot f'$, and hence

$$\mathrm{lt}_>(f) = \mathrm{lt}_>(u') \cdot \mathrm{lt}_>(f) = \mathrm{lt}_>(u' \cdot f) = \mathrm{lt}_>(u \cdot f') = \mathrm{lt}_>(u) \cdot \mathrm{lt}_>(f') = \mathrm{lt}_>(f').$$

Remark 1.10
Note that the leading submodule of a submodule in $R[\underline{x}]^s_>$ is a submodule in a free module over the polynomial ring $K[\underline{t},\underline{x}]$ over the base field, and note that for $J \leq R[\underline{x}]^s_>$ we obviously have $L_>(J) = L_>(J \cap R[\underline{x}]^s)$, and similarly for $I \leq R[\underline{x}]^s$ we have $L_>(I) = L_>\big(\langle I\rangle_{R[\underline{x}]_>}\big)$, since every element of $\langle I\rangle_{R[\underline{x}]_>}$ is of the form $\frac{f}{u}$ with $f \in I$ and $u \in S_>$.

In order to be able to work either theoretically or even computationally with standard bases it is vital to have a division with remainder and possibly an algorithm to compute it. We will therefore generalise Grauert-Hironaka's and Mora's Division with remainder. For this we first would like to consider the different qualities a division with remainder may satisfy.

Definition 1.11
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, and let $A = R[\underline{x}]$ or $A = R[\underline{x}]_>$, where we consider the latter as a subring of $K[[\underline{t},\underline{x}]]$ in order to have the notion of terms of elements at hand.

Suppose we have $f, g_1, \ldots, g_k, r \in A^s$ and $q_1, \ldots, q_k \in A$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{1.1}$$

With the notation $r = \sum_{j=1}^{s} r_j \cdot e_j$, $r_1, \ldots, r_s \in A$, we say that (1.1) satisfies with respect to $>$ the condition

(ID1): iff $\mathrm{lm}_>(f) \geq \mathrm{lm}_>(q_i \cdot g_i)$ for all $i = 1, \ldots, k$,
(ID2): iff $\mathrm{lm}_>(g_i) \not| \ \mathrm{lm}_>(r)$ for $i = 1, \ldots, k$, unless $r = 0$,
(DD1): iff for $j < i$ no term of $q_i \cdot \mathrm{lm}_>(g_i)$ is divisible by $\mathrm{lm}_>(g_j)$,
(DD2): iff no term of r is divisible by $\mathrm{lm}_>(g_i)$ for $i = 1, \ldots, k$.
(SID2): iff $\mathrm{lm}_>(g_i) \not| \ \mathrm{lm}_>(r_j \cdot e_j)$ unless $r_j = 0$ for all i and j.

Here, "ID" stands for *indeterminate division with remainder* while "DD" means *determinate division with remainder* and the "S" in (SID2) represents *strong*. Accordingly, we call a representation of f as in (1.1) a *determinate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if it satisfies (DD1) and (DD2), while we call it an *indeterminate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if it satisfies (ID1) and (ID2). In any of these cases we call r a *remainder* or a *normal form* of f with respect to $(g_1, \ldots, g_k)$.

If the remainder in a division with remainder of f with respect to $(g_1, \ldots, g_k)$ is zero we call the representation of f a *standard representation*.

Finally, if $A = R[\underline{x}]$ then for $u \in S_>$ we call a division with remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ also a *weak division with remainder* of f with respect to $(g_1, \ldots, g_k)$, a remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak normal form* of f with respect to $(g_1, \ldots, g_k)$, and a standard representation of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak standard representation* of f with respect to $(g_1, \ldots, g_k)$.

It is rather obvious to see that (DD2) $\Longleftarrow$ (SID2)$\Longleftarrow$ (ID2), that (DD1)+(ID2) $\Longleftarrow$ (ID1), and that the coefficients and the remainder of a division satisfying (DD1) and (DD2) is uniquely determined.

We first want to generalise Grauert-Hironaka's Division with Remainder to the case of elements in $R[\underline{x}]$ which are homogeneous with respect to $\underline{x}$. We therefore introduce this notion in the following definition.

Definition 1.12
Let $f = \sum_{i=1}^{s} \sum_{|\beta|=0}^{d} \sum_{\alpha \in \mathbb{N}^m} a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in R[\underline{x}]^s$.

(a) We call $\deg_{\underline{x}}(f) := \max\left\{|\beta| \;\middle|\; a_{\alpha,\beta,i} \neq 0\right\}$ the $\underline{x}$*-degree* of f.

(b) $f \in R[\underline{x}]^s$ is called $\underline{x}$*-homogeneous of* $\underline{x}$*-degree* d if all terms of f have the same $\underline{x}$-degree d. We denote by $R[\underline{x}]_d^s$ the R-submodule of $R[\underline{x}]^s$ of $\underline{x}$-homogeneous elements. Note that by this definition 0 is $\underline{x}$-homogeneous of degree d for all $d \in \mathbb{N}$.

(c) If $>$ is a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ then we call

$$\operatorname{ecart}_>(f) := \deg_{\underline{x}}(f) - \deg_{\underline{x}}\left(\operatorname{lm}_>(f)\right) \geq 0$$

the *ecart* of f. It in some sense measures the failure of the homogeneity of f.

2. Determinate Division with Remainder in $K[[\underline{t}]][\underline{x}]_d^s$

We are now ready to show that for $\underline{x}$-homogeneous elements in $R[\underline{x}]$ there exists a determinate division with remainder. We follow mainly the proof of Grauert-Hironaka's Division Theorem as given in [**DeS07**].

Theorem 2.1 (HDDwR)
Let $f, g_1, \ldots, g_k \in R[\underline{x}]^s$ be $\underline{x}$-homogeneous, then there exist uniquely determined $q_1, \ldots, q_k \in R[\underline{x}]$ and $r \in R[\underline{x}]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfying (DD1), (DD2) and

(DDH): *$q_1, \ldots, q_k, r$ are $\underline{x}$-homogeneous of $\underline{x}$-degrees* $\deg_{\underline{x}}(q_i) = \deg_{\underline{x}}(f) - \deg_{\underline{x}}\left(\operatorname{lm}_>(g_i)\right)$ *respectively* $\deg_{\underline{x}}(r) = \deg_{\underline{x}}(f)$.

Proof: The result is obvious if the g_i are terms, and we will reduce the general case to this one. We set $f_0 = f$ and for $\nu > 0$ we define recursively

$$f_\nu = f_{\nu-1} - \sum_{i=1}^k q_{i,\nu}\cdot g_i - r_\nu = \sum_{i=1}^k q_{i,\nu}\cdot \big(\big(-\operatorname{tail}(g_i)\big),$$

where the $q_{i,\nu} \in R[\underline{x}]$ and $r_\nu \in R[\underline{x}]^s$ are such that

$$f_{\nu-1} = q_{1,\nu}\cdot \operatorname{lt}_>(g_1) + \ldots + q_{k,\nu}\cdot \operatorname{lt}_>(g_k) + r_\nu \tag{2.1}$$

satisfies (DD1), (DD2) and (DDH). Note that such a representation of $f_{\nu-1}$ exists since the $\operatorname{lt}_>(g_i)$ are terms.

We want to show that f_ν, $q_{i,\nu}$ and r_ν all converge to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, that is that for each $N \geq 0$ there exists a $\mu_N \geq 0$ such that for all $\nu \geq \mu_N$

$$f_\nu, r_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{resp.} \quad q_{i,\nu} \in \langle t_1,\ldots,t_m\rangle^N.$$

By Lemma 2.3 there is $\underline{t}$-local weight ordering $>_w$ such that

$$\operatorname{lm}_>(g_i) = \operatorname{lm}_{>_w}(g_i) \quad \text{for all } i = 1,\ldots,k.$$

If we replace in the above construction $>$ by $>_w$, we still get the same sequences $(f_\nu)_{\nu=0}^\infty$, $(q_{i,\nu})_{\nu=1}^\infty$ and $(r_\nu)_{\nu=1}^\infty$, since for the construction of $q_{i,\nu}$ and r_ν only the leading monomials of the g_j are used. In particular, (2.1) will satisfy (DD1), (DD2) and (DDH) with respect to $>_w$. Due to (DDH) f_ν is again $\underline{x}$-homogeneous of $\underline{x}$-degree equal to that of $f_{\nu-1}$, and since (DD1) and (DD2) imply (ID1) we have

$$\begin{aligned}\operatorname{lm}_{>_w}(f_{\nu-1}) &\geq \max\{\operatorname{lm}_{>_w}(q_{i,\nu})\cdot \operatorname{lm}_{>_w}(g_i) \mid i = 1,\ldots,k\}\\ &> \max\big\{\operatorname{lm}_{>_w}(q_{i,\nu})\cdot \operatorname{lm}_{>_w}\big(-\operatorname{tail}(g_i)\big) \bigm| i = 1,\ldots,k\big\} \geq \operatorname{lm}_{>_w}(f_\nu).\end{aligned}$$

It follows from Lemma 2.4 that f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, i.e. for given N there is a μ_N such that

$$f_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{for all } \nu \geq \mu.$$

But then, by construction for $\nu > \mu_N$

$$r_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s$$

and

$$q_{i,\nu} \in \langle t_1,\ldots,t_m\rangle^{N-d_i},$$

where $d_i = \deg\big(\operatorname{lm}_>(g_i)\big) - \deg_{\underline{x}}\big(\operatorname{lm}_>(g_i)\big)$ is independent of ν. Thus both, r_ν and $q_{i,\nu}$, converge as well to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology.

But then

$$q_i := \sum_{\nu=1}^\infty q_{i,\nu} \in R[\underline{x}] \quad \text{and} \quad r := \sum_{\nu=1}^\infty r_\nu \in R[\underline{x}]^s$$

are $\underline{x}$-homogeneous of $\underline{x}$-degrees $\deg_{\underline{x}}(q_i) = \deg_{\underline{x}}(f) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ respectively $\deg_{\underline{x}}(r) = \deg_{\underline{x}}(f)$ unless they are zero, and

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfies (DD1), (DD2) and (DDH).

The uniqueness of the representation is obvious. $\square$

The following lemmata contain technical results used throughout the proof of the previous theorem.

Lemma 2.2
If $>$ is a monomial ordering on $\mathrm{Mon}^s(\underline{z})$ with $\underline{z} = (\underline{t}, \underline{x})$, and $M \subset \mathrm{Mon}^s(\underline{z})$ is finite, then there exists $w \in \mathbb{Z}^{m+n+s}$ with

$$w_i < 0, \quad \textit{if } z_i < 1, \quad \textit{and} \quad w_i > 0, \quad \textit{if } z_i > 1,$$

such that for $\underline{z}^\gamma \cdot e_i, \underline{z}^{\gamma'} \cdot e_j \in M$ we have

$$\underline{z}^\gamma \cdot e_i \;>\; \underline{z}^{\gamma'} \cdot e_j \quad \Longleftrightarrow \quad w \cdot (\gamma, e_i) \;>\; w \cdot (\gamma', e_j).$$

In particular, if $>$ is $\underline{t}$-local then every $\underline{t}$-local weight ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ with weight vector w coincides on M with $>$.

Proof: The proof goes analogous to [**GrP02**, Lemma 1.2.11], using [**Bay82**, (1.7)] (for this note that in the latter the requirement that $>$ is a well-ordering is superfluous). $\square$

Lemma 2.3
Let $>$ be a $\underline{t}$-local ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and let $g_1, \ldots, g_k \in R[\underline{x}]^s$ be $\underline{x}$-homogeneous (not necessarily of the same degree), then there is a $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ such that any $\underline{t}$-local weight ordering with weight vector w, say $>_w$, induces the same leading monomials as $>$ on $g_1, \ldots, g_k$, i.e.

$$\mathrm{lm}_>(g_i) = \mathrm{lm}_{>_w}(g_i) \quad \textit{for all } i = 1, \ldots, k.$$

Proof: Consider the monomial ideals $I_i = \langle \mathcal{M}_{\mathrm{tail}(g_i)} \rangle$ in $K[\underline{t}, \underline{x}]$ generated by all monomials of $\mathrm{tail}(g_i)$, $i = 1, \ldots, k$. By Dickson's Lemma (see e.g. [**GrP02**, Lemma 1.2.6]) I_i is generated by a finite subset, say $B_i \subset \mathcal{M}_{\mathrm{tail}(g_i)}$, of the monomials of $\mathrm{tail}(g_i)$. If we now set

$$M = B_1 \cup \ldots \cup B_k \cup \{\mathrm{lm}_>(g_1), \ldots, \mathrm{lm}_>(g_k)\},$$

then by Lemma 2.2 there is $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ such that any $\underline{t}$-local weight ordering, say $>_w$, with weight vector w coincides on M with $>$. Let now $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ be any monomial occurring in $\mathrm{tail}(g_i)$. Then there is a monomial $\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \in B_i$ such that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \;\big|\; \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

which in particular implies that $e_\nu = e_\mu$. Since g_i is $\underline{x}$-homogeneous it follows first that $|\beta| = |\beta'|$ and thus that $\beta = \beta'$. Moreover, since $>_w$ is $\underline{t}$-local it follows that $\underline{t}^{\alpha'} \geq_w \underline{t}^\alpha$ and thus that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu.$$

But since $>$ and $>_w$ coincide on $\{\mathrm{lm}_>(g_i)\} \cup B_i \subset M$ we necessarily have that

$$\mathrm{lm}_>(g_i) \;>_w\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

and hence $\mathrm{lm}_{>_w}(g_i) = \mathrm{lm}_>(g_i)$. □

Lemma 2.4
Let $>$ be a $\underline{t}$-local weight ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ with weight vector $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$, and let $(f_\nu)_{\nu\in\mathbb{N}}$ be a sequence of $\underline{x}$-homogeneous elements of fixed $\underline{x}$-degree d in $R[\underline{x}]^s$ such that

$$\mathrm{lm}_>(f_\nu) > \mathrm{lm}_>(f_{\nu+1}) \quad \textit{for all} \;\; \nu \in \mathbb{N}.$$

Then f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, i.e.

$$\forall\, N \geq 0 \;\exists\, \mu_N \geq 0 \;:\; \forall\, \nu \geq \mu_N \;\; \textit{we have} \;\; f_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s.$$

In particular, the element $\sum_{\nu=0}^\infty f_\nu \in R[\underline{x}]^s_d$ exists.

Proof: Since $w_1,\ldots,w_m < 0$ the set of monomials

$$M_k = \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \;\big|\; w\cdot(\alpha,\beta,e_i) > -k, |\beta| = d\big\}.$$

is finite for a any fixed $k \in \mathbb{N}$.

Let $N \geq 0$ be fixed, set $\tau = \max\{|w_1|,\ldots,|w_{m+n+s}|\}$ and $k := (N + nd + 1)\cdot\tau$, then for any monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ of $\underline{x}$-degree d

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \not\in M_k \quad\Longrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s, \tag{2.2}$$

since

$$\sum_{i=1}^m \alpha_i \cdot w_i \leq -k - \sum_{i=1}^n \beta_i \cdot w_{m+i} - w_{m+n+j} \leq -k + (nd+1)\cdot\tau$$

and thus

$$|\alpha| = \sum_{i=1}^m \alpha_i \geq \sum_{i=1}^m \alpha_i \cdot \frac{-w_i}{\tau} \geq \frac{k}{\tau} - nd - 1 = N.$$

Moreover, since M_k is finite and the $\mathrm{lm}_>(f_\nu)$ are pairwise different there are only finitely many ν such that $\mathrm{lm}_>(f_\nu) \in M_k$. Let μ be maximal among those ν, then by (2.2)

$$\mathrm{lm}_>(f_\nu) \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{for all} \quad \nu > \mu.$$

But since $>$ is a $\underline{t}$-local weight ordering we have that $\mathrm{lm}_>(f_\nu) \not\in M_k$ implies that no monomial of f_ν is in M_k, and thus $f_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s$ for all $\nu > \mu$ by (2.2). This shows that f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology.

Since f_ν converges to zero in the $\langle t_1, \ldots, t_m\rangle$-adic topology, for every monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ there is only a finite number of ν's such that $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ is a monomial occurring in f_ν. Thus the sum $\sum_{\nu=0}^\infty f_\nu$ exists and is obviously $\underline{x}$-homogeneous of degree d. $\square$

From the proof of Theorem 2.1 we can deduce an algorithm for computing the determinate division with remainder up to arbitrary order, or if we don't require termination then it will "compute" the determinate division with remainder completely. Since for our purposes termination is not important, we will simply formulate the non-terminating algorithm.

Algorithm 2.5 (HDDwR)

INPUT: (f, G) with $G = \{g_1, \ldots, g_k\}$ and $f, g_1, \ldots, g_k \in R[\underline{x}]^s$ $\underline{x}$-homogeneous, $>$ a $\underline{t}$-local monomial ordering

OUTPUT: $(q_1, \ldots, q_k, r) \in R[\underline{x}]^k \times R[\underline{x}]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is a homogeneous determinate division with remainder of f satisfying (DD1), (DD2) and (DDH).

INSTRUCTIONS:

- $f_0 := f$
- $r := 0$
- FOR $i = 1, \ldots, k$ DO $q_i := 0$
- $\nu := 0$
- WHILE $f_\nu \neq 0$ DO
  - $q_{0,\nu} := 0$
  - FOR $i = 1, \ldots, k$ DO
    - $h_{i,\nu} := \sum_{p \in \mathcal{T}_{f_\nu} \,:\, \mathrm{lm}_>(g_i) \mid p} p$
    - $q_{i,\nu} := \frac{h_{i,\nu}}{\mathrm{lt}_>(g_i)}$
    - $q_i := q_i + q_{i,\nu}$
  - $r_\nu := f_\nu - q_{1,\nu} \cdot \mathrm{lt}_>(g_1) - \ldots - q_{k,\nu} \cdot \mathrm{lt}_>(g_k)$
  - $r := r + r_\nu$
  - $f_{\nu+1} := f_\nu - q_{1,\nu} \cdot g_1 - \ldots - q_{k,\nu} \cdot g_k - r_\nu$
  - $\nu := \nu + 1$

Remark 2.6

If $m = 0$, i.e. if the input data $f, g_1, \ldots, g_k \in K[\underline{x}]^s$, then Algorithm 2.5 terminates since for a given degree there are only finitely many monomials of this degree and therefore there cannot exist an infinite sequence of homogeneous polynomials $(f_\nu)_{\nu \in \mathbb{N}}$ of the same degree with

$$\mathrm{lm}_>(f_1) > \mathrm{lm}_>(f_2) > \mathrm{lm}_>(f_3) > \ldots.$$

3. Division with Remainder in $K[[\underline{t}]][\underline{x}]^s$

We will use the existence of homogeneous determinate divisions with remainder to show that in $R[\underline{x}]^s$ weak normal forms exist. In order to be able to apply this existence result we have to homogenise, and we need to extend our monomial ordering to the homogenised monomials.

Definition 3.1
Let $\underline{x}_h = (x_0, \underline{x}) = (x_0, \ldots, x_n)$.

(a) For $0 \neq f \in R[\underline{x}]^s$. We define the *homogenisation* f^h of f to be

$$f^h := x_0^{\deg_{\underline{x}}(f)} \cdot f\left(\underline{t}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in R[\underline{x}_h]^s_{\deg_{\underline{x}}(f)}$$

and $0^h := 0$. If $T \subset R[\underline{x}]^s$ then we set $T^h := \{f^h \mid f \in T\}$.

(b) We call the $R[\underline{x}]$-linear map $d : R[\underline{x}_h]^s \longrightarrow R[\underline{x}]^s : g \mapsto g^d := g_{|x_0=1}$ the *dehomogenisation* with respect to x_0.

(c) Given a $\underline{t}$-local monomial ordering $>$ on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ we define a $\underline{t}$-local monomial ordering $>_h$ on $\mathrm{Mon}^s(\underline{t}, \underline{x}_h)$ by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot x_0^a \cdot e_i \;>_h\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot x_0^{a'} \cdot e_j$$

if and only if

$$|\beta| + a > |\beta'| + a'$$

or

$$\big(|\beta| + a = |\beta'| + a' \text{ and } \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j\big),$$

and we call it the *homogenisation of* $>$.

In the following remark we want to gather some straight forward properties of homogenisation and dehomogenisation.

Remark 3.2
Let $f, g \in R[\underline{x}]^s$ and $F \in R[\underline{x}_h]^s_k$. Then:

(a) $f = (f^h)^d$.

(b) $F = (F^d)^h \cdot x_0^{\deg_{\underline{x}_h}(F) - \deg_{\underline{x}}(F^d)}$.

(c) $\mathrm{lm}_{>_h}(f^h) = x_0^{\mathrm{ecart}(f)} \cdot \mathrm{lm}_>(f)$.

(d) $\mathrm{lm}_{>_h}(g^h) \,|\, \mathrm{lm}_{>_h}(f^h) \Longleftrightarrow \mathrm{lm}_>(g) \,|\, \mathrm{lm}_>(f) \;\wedge\; \mathrm{ecart}(g) \leq \mathrm{ecart}(f)$.

(e) $\mathrm{lm}_{>_h}(F) = x_0^{\mathrm{ecart}(F^d) + \deg_{\underline{x}_h}(F) - \deg_{\underline{x}}(f)} \cdot \mathrm{lm}_>(F^d)$.

Theorem 3.3 (Division with Remainder)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s$. Then any $f \in R[\underline{x}]^s$ has a weak division with remainder with respect to $g_1, \ldots, g_k$.

Proof: The proof follows from the correctness and termination of Algorithm 3.4, which assumes the existence of the homogeneous determinate division with remainder from Theorem 2.1 respectively Algorithm 2.5. $\square$

The following algorithm relies on the HDDwR-Algorithm, and it only terminates under the assumption that we are able to produce homogeneous determinate divisions with remainder, which implies that it is not an algorithm that can be applied in practise.

Algorithm 3.4 (DwR - Mora's Division with Remainder)
INPUT: (f, G) with $G = \{g_1, \ldots, g_k\}$ and $f, g_1, \ldots, g_k \in R[\underline{x}]^s$, $>$ a $\underline{t}$-local monomial ordering
OUTPUT: $(u, q_1, \ldots, q_k, r) \in S_> \times R[\underline{x}]^k \times R[\underline{x}]^s$ such that

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is a weak division with remainder of f.
INSTRUCTIONS:

- $T := (g_1, \ldots, g_k)$
- $D := \{g_i \in T \mid \mathrm{lm}_>(g_i) \text{ divides } \mathrm{lm}_>(f)\}$
- IF $f \neq 0$ AND $D \neq \emptyset$ DO
  - IF $e := \min\{\mathrm{ecart}_>(g_i) \mid g_i \in D\} - \mathrm{ecart}_>(f) > 0$ THEN
    * $(Q_1', \ldots, Q_k', R') := \mathrm{HDDwR}\big(x_0^e \cdot f^h, (\mathrm{lt}_{>_h}(g_1^h), \ldots, \mathrm{lt}_{>_h}(g_k^h))\big)$
    * $f' := \big(x_0^e \cdot f^h - \sum_{i=1}^k Q_i' \cdot g_i^h\big)^d$
    * $(u'', q_1'', \ldots, q_{k+1}'', r) := \mathrm{DwR}\big(f', (g_1, \ldots, g_k, f)\big)$
    * $q_i := q_i'' + u'' \cdot {Q_i'}^d, \quad i = 1, \ldots, k$
    * $u := u'' - q_{k+1}''$
  - ELSE
    * $(Q_1', \ldots, Q_k', R') := \mathrm{HDDwR}\big(f^h, (g_1^h, \ldots, g_k^h)\big)$
    * $(u, q_1'', \ldots, q_{k+1}'', r) := \mathrm{DwR}\big((R')^d, T\big)$
    * $q_i := q_i'' + u \cdot {Q_i'}^d, \quad i = 1, \ldots, k$
- ELSE $(u, q_1, \ldots, q_k, r) = (1, 0, \ldots, 0, f)$

Proof: Let us first prove the *termination*. For this we denote the numbers, ring elements and sets, which occur in the ν-th recursion step by a subscript ν, e.g. e_ν, f_ν or T_ν. Since

$$T_1^h \subseteq T_2^h \subseteq T_3^h \subseteq \ldots$$

also their leading submodules in $K[\underline{t}, \underline{x}_h]^s$ form an ascending chain

$$L_{>_h}(T_1^h) \subseteq L_{>_h}(T_2^h) \subseteq L_{>_h}(T_3^h) \subseteq \ldots,$$

and since the polynomial ring is noetherian there must be an N such that

$$L_{>_h}(T_\nu^h) = L_{>_h}(T_N^h) \quad \forall\, \nu \geq N.$$

If $g_{i,N} \in T_N$ such that $\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N)$ with $\mathrm{ecart}_>(g_{i,N}) \leq \mathrm{ecart}_>(f_N)$, then

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h).$$

We thus have either $\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$ for some $g_i \in D^N \subseteq T^{N+1}$ or $f_N \in T_{N+1}$, and hence

$$\mathrm{lm}_{>_h}(f_N^h) \in L_{>_h}(T_{N+1}^h) = L_{>_h}(T_N^h).$$

This ensures the existence of a $g_{i,N} \in T_N$ such that

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$$

which in turn implies that

$$\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N),$$

$e_N \leq \mathrm{ecart}_>(g_{i,N}) - \mathrm{ecart}_>(f_N) \leq 0$ and $T_N = T_{N+1}$. By induction we conclude

$$T_\nu = T_N \quad \forall\, \nu \geq N,$$

and

$$e_\nu \leq 0 \quad \forall\, \nu \geq N. \tag{3.1}$$

Since in the N-th recursion step we are in the first "ELSE" case we have $(R_N')^d = f_{N+1}$, and by the properties of HDDwR we know that for all $g \in T_N$

$$x_0^{\mathrm{ecart}_>(g)} \cdot \mathrm{lm}_>(g) = \mathrm{lm}_{>_h}(g^h) \;\not|\; \mathrm{lm}_{>_h}(R_N')$$

and that

$$\mathrm{lm}_{>_h}(R_N') = x_0^a \cdot \mathrm{lm}_{>_h}(f_{N+1}^h) = x_0^{a+\mathrm{ecart}_>(f_{N+1})} \cdot \mathrm{lm}_>(f_{N+1})$$

for some $a \geq 0$. It follows that, whenever $\mathrm{lm}_>(g) \mid \mathrm{lm}_>(f_{N+1})$, then necessarily

$$\mathrm{ecart}_>(g) > a + \mathrm{ecart}_>(f_{N+1}) \geq \mathrm{ecart}_>(f_{N+1}). \tag{3.2}$$

Suppose now that $f_{N+1} \neq 0$ and $D_{N+1} \neq \emptyset$. Then we may choose $g_{i,N+1} \in D_{N+1} \subseteq T_{N+1} = T_N$ such that

$$\mathrm{lm}_>(g_{i,N+1}) \mid \mathrm{lm}_>(f_{N+1})$$

and

$$e_{N+1} = \mathrm{ecart}_>(g_{i,N+1}) - \mathrm{ecart}_>(f_{N+1}).$$

According to (3.1) e_{N+1} is non-positive, while according to (3.2) it must be strictly positive. Thus we have derived a contradiction which shows that either $f_{N+1} = 0$ or $D_{N+1} = \emptyset$, and in any case the algorithm stops.

Next we have to prove the *correctness*. We do this by induction on the number of recursions, say N, of the algorithm.

If $N = 1$ then either $f = 0$ or $D = \emptyset$, and in both cases

$$1 \cdot f = 0 \cdot g_1 + \ldots + 0 \cdot g_k + f$$

is a weak division with remainder of f satisfying (ID1) and (ID2). We may thus assume that $N > 1$ and $e = \min\{\mathrm{ecart}_>(g) \mid g \in D\} - \mathrm{ecart}_>(f)$.

If $e \leq 0$ then by Theorem 2.1

$$f^h = Q_1' \cdot g_1^h + \ldots + Q_k' \cdot g_k^h + R'$$

satisfies (DD1), (DD2) and (DDH). (DD1) implies that for each $i = 1, \ldots, k$ we have

$$x_0^{\mathrm{ecart}_>(f)} \cdot \mathrm{lm}_>(f) = \mathrm{lm}_{>_h}(f^h) \geq \\ \mathrm{lm}_{>_h}(Q_i') \cdot \mathrm{lm}_{>_h}(g_i^h) = x_0^{a_i + \mathrm{ecart}_>(g_i)} \cdot \mathrm{lm}_> \left({Q_i'}^d\right) \cdot \mathrm{lm}_>(g_i)$$

for some $a_i \geq 0$, and since f^h and $Q_i' \cdot g_i^h$ are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree by (DDH) the definition of the homogenised ordering implies that necessarily

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_> \left({Q_i'}^d\right) \cdot \mathrm{lm}_>(g_i) \quad \forall\, i = 1, \ldots, k.$$

Note that

$$(R')^d = \left(f^h - \sum_{i=1}^k Q_i' \cdot g_i^h\right)^d = f - \sum_{i=1}^k {Q_i'}^d \cdot g_i,$$

and thus

$$\mathrm{lm}_> \left((R')^d\right) = \mathrm{lm}_> \left(f - \sum_{i=1}^k {Q_i'}^d \cdot g_i\right) \leq \mathrm{lm}_>(f).$$

Moreover, by induction

$$u \cdot (R')^d = q_1'' \cdot g_1 + \ldots q_k'' \cdot g_k + r$$

satisfies (ID1) and (ID2). But (ID1) implies that

$$\mathrm{lm}_>(f) \;\geq\; \mathrm{lm}_> \left((R')^d\right) \;\geq\; \mathrm{lm}_>(q_i'' \cdot g_i),$$

so that

$$u \cdot f = \sum_{i=1}^k \left(q_i'' + u \cdot {Q_i'}^d\right) \cdot g_i + r$$

satisfies (ID1) and (ID2).

It remains to consider the case $e > 0$. Then by Theorem 2.1

$$x_0^e \cdot f^h = Q_1' \cdot \mathrm{lt}_{>_h}(g_1^h) + \ldots + Q_k' \cdot \mathrm{lt}_{>_h}(g_k^h) + R' \tag{3.3}$$

satisfies (DD1), (DD2) and (DDH). (DD1) and (DD2) imply (ID1) for this representation, which means that for some $a_i \geq 0$

$$x_0^{e+\mathrm{ecart}_>(f)} \cdot \mathrm{lm}_>(f) = \mathrm{lm}_{>_h}(x_0^e \cdot f^h) \geq \\ \mathrm{lm}_{>_h}(Q_i') \cdot \mathrm{lm}_{>_h} \left(\mathrm{lt}_{>_h}(g_i^h)\right) = x_0^{a_i + \mathrm{ecart}_>(g_i)} \cdot \mathrm{lm}_>({Q_i'}^d) \cdot \mathrm{lm}_>(g_i),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree with by (DDH) we again necessarily have

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_>\left(Q_i'^d\right) \cdot \mathrm{lm}_>(g_i).$$

Moreover, by induction

$$u'' \cdot \left(f - \sum_{i=1}^{k} Q_i'^d \cdot g_i\right) = \sum_{i=1}^{k} q_i'' \cdot g_i + q_{k+1}'' \cdot f + r \tag{3.4}$$

satisfies (ID1) and (ID2).

Since $\mathrm{lt}_>(u'') = 1$ we have

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_>\left(q_i'' + u'' \cdot Q_i'^d\right) \cdot \mathrm{lm}_>(g_i),$$

for $i = 1, \ldots, k$ and therefore

$$(u'' - q_{k+1}'') \cdot f = \sum_{i=1}^{k} \left(q_i'' + u'' \cdot Q_i'^d\right) \cdot g_i + r$$

satisfies (ID1) and (ID2) as well. It remains to show that $u = u'' - q_{k+1}'' \in S_>$, or equivalently that

$$\mathrm{lt}_>(u'' - q_{k+1}'') = 1.$$

By assumption there is a $g_i \in D$ such that $\mathrm{lm}_>(g_i) \mid \mathrm{lm}_>(f)$ and $\mathrm{ecart}_>(g_i) - \mathrm{ecart}_>(f) = e$. Therefore, $\mathrm{lm}_{>_h}(g_i^h) \mid x_0^e \cdot \mathrm{lm}_{>_h}(f^h)$ and thus in the representation (3.3) the leading term of $x_0^e \cdot f^h$ has been cancelled by some $Q_j' \cdot \mathrm{lt}_{>_h}(g_j^h)$, which implies that

$$\mathrm{lm}_{>_h}(f^h) > \mathrm{lm}_{>_h}\left(f^h - \sum_{i=1}^{k} Q_i' \cdot g_i^h\right),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree, unless the right hand side is zero, we must have

$$\mathrm{lm}_>(f) > \mathrm{lm}_>\left(f - \sum_{i=1}^{k} Q_i'^d \cdot g_i\right) \geq \mathrm{lm}_>(q_{k+1}'' \cdot f),$$

where the latter inequality follows from (ID1) for (3.4). Thus however $\mathrm{lm}_>(q_{k+1}'') < 1$, and since $\mathrm{lm}_>(u'') = 1$ we conclude that

$$\mathrm{lt}_>(u'' - q_{k+1}'') = \mathrm{lt}_>(u'') = 1.$$

This finishes the proof. $\square$

Remark 3.5

As we have pointed out our algorithms are not useful for computational purposes since Algorithm 2.5 does not in general terminate after a finite number of steps. If, however, the input data are in fact polynomials in $\underline{t}$ and $\underline{x}$, then we can replace the t_i by x_{n+i} and apply Algorithm 3.4 to $K[x_1, \ldots, x_{n+m}]^s$, so that it terminates due to Remark 2.6 the computed weak division with remainder

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is then *polynomial* in the sense that $u, q_1, \ldots, q_k \in K[\underline{t}, \underline{x}]$ and $r \in K[\underline{t}, \underline{x}]^s$. In fact, Algorithm 3.4 is then only a variant of the usual Mora algorithm.

In the proof of Schreyer's Theorem we will need the existence of weak divisions with remainder satisfying (SID2), the proof is the same as [**GrP02**, Remark 2.3.4].

Corollary 3.6
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s_>$. Then any $f \in R[\underline{x}]^s_>$ has a division with remainder with respect to $g_1, \ldots, g_k$ satisfying (SID2).

4. Standard Bases in $K[[\underline{t}]][\underline{x}]^s$

Definition 4.1
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $I \leq R[\underline{x}]^s$ and $J \leq R[\underline{x}]^s_>$ be submodules. A *standard basis of* I is a finite subset $G \subset I$ such that $L_>(I) = L_>(G)$. A *standard basis of* J is a finite subset $G \subset J$ such that $L_>(J) = L_>(G)$. A finite subset $G \subseteq R[\underline{x}]^s_>$ is called a *standard basis* with respect to $>$ if G is a standard basis of $\langle G \rangle \leq R[\underline{x}]^s_>$.

The existence of standard bases is immediate from Hilbert's Basis Theorem.

Proposition 4.2
If $>$ is a $\underline{t}$-local monomial ordering then every submodule of $R[\underline{x}]^s$ and of $R[\underline{x}]^s_>$ has a standard basis.

Standard bases are so useful since they are generating sets for submodules of $R[\underline{x}]^s_>$ and since submodule membership can be tested by division with remainder.

Proposition 4.3
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $I, J \leq R[\underline{x}]^s_>$ submodules, $G = (g_1, \ldots, g_k) \subset J$ a standard basis of J and $f \in R[\underline{x}]^s_>$ with division with remainder $f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$. Then:

(a) *$f \in J$ if and only if $r = 0$.*

(b) *$J = \langle G \rangle$.*

(c) *If $I \subseteq J$ and $L_>(I) = L_>(J)$, then $I = J$.*

Proof: Word by word as in [**GrP02**, Lemma 1.6.7]. □

In order to work, even theoretically, with standard bases it is vital to have a good criterion to decide whether a generating set is standard basis or not. In order to formulate Buchberger's Criterion it is helpful to have the notion of an *s-polynomial*.

Definition 4.4
Let $>$ be a $\underline{t}$-local monomial ordering on $R[\underline{x}]^s$ and $f,g\in R[\underline{x}]^s$. We define the *s-polynomial* of f and g as

$$\operatorname{spoly}(f,g):=\frac{\operatorname{lcm}\big(\operatorname{lm}_>(f),\operatorname{lm}_>(g)\big)}{\operatorname{lt}_>(f)}\cdot f-\frac{\operatorname{lcm}\big(\operatorname{lm}_>(f),\operatorname{lm}_>(g)\big)}{\operatorname{lt}_>(g)}\cdot g.$$

Theorem 4.5 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$, $J\leq R[\underline{x}]^s_>$ a submodule and $g_1,\ldots,g_k\in J$. The following statements are equivalent:

(a) $G=(g_1,\ldots,g_k)$ *is a standard basis of* J.

(b) *Every normal form with respect to* G *of any element in* J *is zero.*

(c) *Every element in* J *has a standard representation with respect to* G.

(d) $J=\langle G\rangle$ *and* $\operatorname{spoly}(g_i,g_j)$ *has a standard representation for all* $i<j$.

Proof: In Proposition 4.3 we have shown that (a) implies (b), and the implication (b) to (c) is trivially true. And, finally, if $f\in J$ has a standard representation with respect to G, then $\operatorname{lm}_>(f)\in L_>(G)$, so that (c) implies (a). Since $\operatorname{spoly}(g_i,g_j)\in J$ condition (d) follows from (c), and the hard part is to show that (d) implies actually (c). This is postponed to Theorem 5.3. □

Since for $G\subset R[\underline{x}]^s$ we have $L_>\big(\langle G\rangle_{R[\underline{x}]}\big)=L_>\big(\langle G\rangle_{R[\underline{x}]_>}\big)$ we get the following corollary.

Corollary 4.6 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k\in I\leq R[\underline{x}]^s$. Then the following statements are equivalent:

(a) $G=(g_1,\ldots,g_k)$ *is a standard basis of* I.

(b) *Every weak normal form with respect to* G *of any element in* I *is zero.*

(c) *Every element in* I *has a weak standard representation with respect to* G.

(d) $\langle I\rangle_{R[\underline{x}]_>}=\langle G\rangle_{R[\underline{x}]_>}$ *and* $\operatorname{spoly}(g_i,g_j)$ *has a weak standard representation for all* $i<j$.

When working with polynomials in $\underline{x}$ as well as in $\underline{t}$ we can actually compute divisions with remainder and standard bases (see Remark 3.5), and they are also standard bases of the corresponding submodules considered over $R[\underline{x}]$ by the following corollary.

Corollary 4.7
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ and let $G\subset K[\underline{t},\underline{x}]^s$ be finite. Then G is a standard basis of $\langle G\rangle_{K[\underline{t},\underline{x}]}$ if and only if G is a standard basis of $\langle G\rangle_{R[\underline{x}]}$.

Proof: Let $G=(g_1,\ldots,g_k)$. By Theorem 3.3 and Remark 3.5 each $\mathrm{spoly}(g_i,g_j)$ has a weak division with remainder with respect to G such that the coefficients and remainders involved are polynomials in $\underline{x}$ as well as in $\underline{t}$. But by Corollary 4.6 G is a standard basis of either of $\langle G\rangle_{K[\underline{t},\underline{x}]}$ and $\langle G\rangle_{R[\underline{x}]}$ if and only if all these remainders are actually zero. $\square$

And thus it makes sense to formulate the classical standard basis algorithm also for the case $R[\underline{x}]$.

Algorithm 4.8 (STD – Standard Basis Algorithm)
INPUT: $(f_1,\ldots,f_k)\in\big(R[\underline{x}]^s\big)^k$ and $>$ a $\underline{t}$-local monomial ordering.

OUTPUT: $(f_1,\ldots,f_l)\in\big(R[\underline{x}]^s\big)^l$ a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$.

INSTRUCTIONS:

- $G=(f_1,\ldots,f_k)$
- $P=\big((f_i,f_j)\;\big|\;1\leq i<j\leq k\big)$
- WHILE $P\neq\emptyset$ DO
  - Choose some pair $(f,g)\in P$
  - $P=P\setminus\{(f,g)\}$
  - $(u,\underline{q},r)=\mathrm{DwR}\big(\mathrm{spoly}(f,g),G\big)$
  - IF $r\neq 0$ THEN
    - $P=P\cup\{(f,r)\mid f\in G\}$
    - $G=G\cup\{r\}$

Remark 4.9
If the input of STD are polynomials in $K[\underline{t},\underline{x}]$ then the algorithm works in practise due to Remark 3.5, and it computes a standard basis G of $\langle f_1,\ldots,f_k\rangle_{K[\underline{t},\underline{x}]}$ which due to Corollary 4.7 is also a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$, since G still contains the generators $f_1,\ldots,f_k$.

Having division with remainder, standard bases and Buchberger's Criterion at hand one can, from a theoretical point of view, basically derive all the standard algorithms from computer algebra also for free modules over $R[\underline{x}]$ respectively $R[\underline{x}]_>$. Moreover, if the input is polynomial in $\underline{t}$ and $\underline{x}$, then the corresponding operations computed over $K[\underline{t},\underline{x}]_>$ will also lead to generating sets for the corresponding operations over $R[\underline{x}]_>$.

5. Schreyer's Theorem for $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

In this section we want to prove Schreyer's Theorem for $R[\underline{x}]^s$ which proves Buchberger's Criterion and shows at the same time that a standard basis of a submodule gives rise to a standard basis of the syzygy module defined by it with respect to a special ordering.

Definition 5.1 (Schreyer Ordering)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k \in R[\underline{x}]^s_>$. We define a *Schreyer ordering* with respect to $>$ and $(g_1,\ldots,g_k)$, say $>_S$, on $\mathrm{Mon}^k(\underline{t},\underline{x})$ by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \varepsilon_i \ >_S \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \varepsilon_j$$

if and only if

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \mathrm{lm}_>(g_i) \ > \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \mathrm{lm}_>(g_j)$$

or

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \mathrm{lm}_>(g_i) \ = \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \mathrm{lm}_>(g_j) \ \text{ and } \ i<j,$$

where $\varepsilon_i = (\delta_{ij})_{j=1,\ldots,k}$ is the canonical basis with i-th entry one and the rest zero.

Moreover, we define the *syzygy module of* $(g_1,\ldots,g_k)$ to be

$$\mathrm{syz}(g_1,\ldots,g_k) := \{(q_1,\ldots,q_k) \in R[\underline{x}]^k_> \mid q_1\cdot g_1 + \ldots + q_k \cdot g_k = 0\},$$

and we call the elements of $\mathrm{syz}(g_1,\ldots,g_k)$ *syzygies* of $g_1,\ldots,g_k$.

Remark 5.2
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k \in R[\underline{x}]^s_>$. Let us fix for each $i<j$ a division with remainder of $\mathrm{spoly}(g_i,g_j)$, say

$$\mathrm{spoly}(g_i,g_j) = \sum_{\nu=1}^k q_{i,j,\nu}\cdot g_\nu + r_{ij}, \tag{5.1}$$

and define

$$m_{ji} := \frac{\mathrm{lcm}\big(\mathrm{lm}_>(g_i),\mathrm{lm}_>(g_j)\big)}{\mathrm{lm}_>(g_i)},$$

so that

$$\mathrm{spoly}(g_i,g_j) = \frac{m_{ji}}{\mathrm{lc}_>(g_i)}\cdot g_i - \frac{m_{ij}}{\mathrm{lc}_>(g_j)}\cdot g_j.$$

Then

$$s_{ij} := \frac{m_{ji}}{\mathrm{lc}_>(g_i)}\cdot \varepsilon_i - \frac{m_{ij}}{\mathrm{lc}_>(g_j)}\cdot \varepsilon_j - \sum_{\nu=1}^k q_{i,j,\nu}\cdot\varepsilon_\nu \in R[\underline{x}]^k_>$$

has the property

$$s_{ij} \in \mathrm{syz}(g_1,\ldots,g_k) \quad\Longleftrightarrow\quad r_{ij}=0.$$

Theorem 5.3 (Schreyer)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $g_1,\ldots,g_k \in R[\underline{x}]^s_>$ and suppose that $\mathrm{spoly}(g_i,g_j)$ has a weak standard representation with respect to $G=(g_1,\ldots,g_k)$ for each $i<j$.

Then G is a standard basis, and with the notation in Remark 5.2 $\{s_{ij} \mid i<j\}$ is a standard basis of $\mathrm{syz}(g_1,\ldots,g_k)$ with respect to $>_S$.

Proof: The same as in [**GrP02**, Theorem 2.5.9]. □

6. Application to t-Initial Ideals

In this section we want to show that for an ideal J over the field of Puiseux series which is generated by elements in $K[[t^{\frac{1}{N}}]][\underline{x}]$ respectively in $K[t^{\frac{1}{N}},\underline{x}]$ the t-initial ideal (a notion we will introduce further down) with respect to $w \in \mathbb{Q}_{<0} \times \mathbb{Q}^n$ can be computed from a standard basis of the generators.

Definition 6.1

We consider for $0 \neq N \in \mathbb{N}$ the discrete valuation ring

$$R_N\left[\left[t^{\frac{1}{N}}\right]\right] = \left\{ \sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \;\middle|\; a_\alpha \in K \right\}$$

of power series in the unknown $t^{\frac{1}{N}}$ with *discrete valuation*

$$\operatorname{val}\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}}\right) = \operatorname{ord}_t\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}}\right) = \min\left\{\frac{\alpha}{N} \;\middle|\; a_\alpha \neq 0\right\} \in \frac{1}{N}\cdot\mathbb{Z},$$

and we denote by $L_N = \operatorname{Quot}(R_N)$ its quotient field. If $N \mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L = K\{\{t\}\} = \varinjlim L_N = \bigcup_{N\geq 0} L_N$$

the *field of (formal) Puiseux series* over K.

Remark 6.2

If $0 \neq N \in \mathbb{N}$ then $S_N = \{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, t^{\frac{2}{N}}, \ldots\}$ is a multiplicative subset of R_N, and obviously $L_N = S_N^{-1}R_N = \{t^{\frac{-\alpha}{N}} \cdot f \mid f \in R_N, \alpha \in \mathbb{N}\}$, since $R_N^* = \{\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \mid a_0 \neq 0\}$. The valuations of R_N extend to L_N, and thus L, by $\operatorname{val}\left(\frac{f}{g}\right) = \operatorname{val}(f) - \operatorname{val}(g)$ for $f, g \in R_N$ with $g \neq 0$.

Definition 6.3

For $0 \neq N \in \mathbb{N}$ if we consider $t^{\frac{1}{N}}$ as a variable, we get the set of monomials $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) = \left\{t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta \;\middle|\; \alpha \in \mathbb{N}, \beta \in \mathbb{N}^n\right\}$ in $t^{\frac{1}{N}}$ and $\underline{x}$. If $N \mid M$ then obviously $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) \subset \operatorname{Mon}\left(t^{\frac{1}{M}}, \underline{x}\right)$.

Remark and Definition 6.4

Let $0 \neq N \in \mathbb{N}$, $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.

We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta)\,\in\,\mathbb{N}^{n+1} \\ w\cdot(\frac{\alpha}{N},\beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+1}\\ w\cdot(\frac{\alpha}{N},\beta)=q}} K\cdot t^{\frac{\alpha}{N}}\cdot \underline{x}^{\beta}.$$

As a K-vector space the formal power series ring $K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$ is just

$$K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big] = \prod_{q\in\mathbb{R}} V_{q,w,N},$$

and we can thus write any power series $f\in K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$ in a unique way as

$$f=\sum_{q\in\mathbb{R}} f_{q,w} \quad \text{with} \quad f_{q,w}\in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f\in K\big[\big[t^{\frac{1}{N'}},\underline{x}\big]\big]$ for some other $0\neq N'\in\mathbb{N}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.

Moreover, if $0\neq f\in R_N[\underline{x}]\subset K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$, then there is a maximal $\hat{q}\in\mathbb{R}$ such that $f_{\hat{q},w}\neq 0$ and $f_{q,w}\in W_{q,w,N}$ for all $q\in\mathbb{R}$, since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w})=q\in\mathbb{R}$,

$$\operatorname{in}_w(f)=f_{\hat{q},w}\in K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial form* of f or the *initial form of f w.r.t. w*, and

$$\operatorname{ord}_w(f)=\hat{q}=\max\{\deg_w(f_{q,w})\mid f_{q,w}\neq 0\}$$

the *w-order* of f. For $I\subseteq R_N[\underline{x}]$ we call

$$\operatorname{in}_w(I)=\big\langle \operatorname{in}_w(f)\;\big|\; f\in I\big\rangle \trianglelefteq K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial ideal* of I. Note that its definition depends on N!

Moreover, we call

$$\text{t-in}_w(f)=\operatorname{in}_w(f)(1,\underline{x})=\operatorname{in}_w(f)_{|t=1}\in K[\underline{x}]$$

the *t-initial form of f w.r.t. w*, and if $f=t^{\frac{-\alpha}{N}}\cdot g\in L[\underline{x}]$ with $g\in R_N[\underline{x}]$ we set $\text{t-in}_w(f):=\text{t-in}_w(g)$. This definition does not depend on the particular representation of f. If $I\subseteq L[\underline{x}]$ is an ideal, then

$$\text{t-in}_w(I)=\langle \text{t-in}_w(f)\mid f\in I\rangle \lhd K[\underline{x}]$$

is the *t-initial ideal* of I, which does not depend on any N.

Note also that the product of two w-quasihomogeneous elements $f_{q,w}\cdot f_{q',w}\in V_{q+q',w,N}$, and in particular, $\operatorname{in}_w(f\cdot g)=\operatorname{in}_w(f)\cdot\operatorname{in}_w(g)$ for $f,g\in R_N[\underline{x}]$, and for $f,g\in L[\underline{x}]$ $\text{t-in}_w(f\cdot g)=\text{t-in}_w(f)\cdot\text{t-in}_w(g)$. An immediate consequence of this is the following lemma.

Lemma 6.5
If $0 \neq f = \sum_{i=1}^{k} g_i \cdot h_i$ *with* $f, g_i, h_i \in R_N[\underline{x}]$ *and* $\operatorname{ord}_w(f) \geq \operatorname{ord}_w(g_i \cdot h_i)$ *for all* $i = 1, \ldots, k$*, then*

$$\operatorname{in}_w(f) \in \big\langle \operatorname{in}_w(g_1), \ldots, \operatorname{in}_w(g_k) \big\rangle \lhd K\big[t^{\frac{1}{N}}, \underline{x}\big].$$

Proof: Due to the direct product decomposition we have that

$$\operatorname{in}_w(f) = f_{\hat{q},w} = \sum_{i=1}^{k} (g_i \cdot h_i)_{\hat{q},w}$$

where $\hat{q} = \operatorname{ord}_w(f)$. By assumption $\operatorname{ord}_w(g_i) + \operatorname{ord}_w(h_i) = \operatorname{ord}_w(g_i \cdot h_i) \leq \operatorname{ord}_w(f) = \hat{q}$ with equality if and only if $(g_i \cdot h_i)_{\hat{q},w} \neq 0$. In that case necessarily $(g_i \cdot h_i)_{\hat{q},w} = \operatorname{in}_w(g_i) \cdot \operatorname{in}_w(h_i)$, which finishes the proof. $\square$

In order to be able to apply standard bases techniques we need to fix a t-local monomial ordering which refines a given weight vector w.

Definition 6.6
Fix any *global* monomial ordering, say $>$, on $\operatorname{Mon}(\underline{x})$ and let $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$.

We define a t-local monomial ordering, say $>_w$, on $\operatorname{Mon}\big(t^{\frac{1}{N}}, \underline{x}\big)$ by

$$t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta} \;>_w\; t^{\frac{\alpha'}{N}} \cdot \underline{x}^{\beta'}$$

if and only if

$$w \cdot \left(\frac{\alpha}{N}, \beta\right) > w \cdot \left(\frac{\alpha'}{N}, \beta'\right)$$

or

$$w \cdot \left(\frac{\alpha}{N}, \beta\right) = w \cdot \left(\frac{\alpha'}{N}, \beta'\right) \text{ and } \underline{x}^{\beta} > \underline{x}^{\beta'}.$$

Note that this ordering is indeed t-local since $w_0 < 0$, and that it depends on w and on $>$, but assuming that $>$ is fixed we will refrain from writing $>_{w,>}$ instead of $>_w$.

Remark 6.7
If $N \mid M$ then $\operatorname{Mon}\big(t^{\frac{1}{N}}, \underline{x}\big) \subset \operatorname{Mon}\big(t^{\frac{1}{M}}, \underline{x}\big)$, as already mentioned. For $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we may thus consider the ordering $>_w$ on both $\operatorname{Mon}\big(t^{\frac{1}{N}}, \underline{x}\big)$ and on $\operatorname{Mon}\big(t^{\frac{1}{M}}, \underline{x}\big)$, and let us call them for a moment $>_{w,N}$ respectively $>_{w,M}$. It is important to note, that the restriction of $>_{w,M}$ to $\operatorname{Mon}\big(t^{\frac{1}{N}}, \underline{x}\big)$ coincides with $>_{w,N}$. We therefore omit the additional subscript in our notation.

We now fix some global monomial ordering $>$ on $\operatorname{Mon}(\underline{x})$, and given a vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we will throughout this section always denote by $>_w$ the monomial ordering from Definition 6.6.

Proposition 6.8
If $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ *and* $f \in R_N[\underline{x}]$ *with* $\mathrm{lt}_{>_w}(f) = 1$*, then* $\mathrm{in}_w(f) = 1$.

Proof: Suppose this is not the case then there exists a monomial of f, say $1 \neq t^\alpha \cdot \underline{x}^\beta \in \mathcal{M}_f$, such that $w \cdot (\alpha, \beta) \geq w \cdot (0, \ldots, 0) = 0$, and since $\mathrm{lm}_{>_w}(f) = 1$ we must necessarily have equality. But since $>$ is global $\underline{x}^\beta > 1$, which implies that also $t^\alpha \cdot \underline{x}^\beta >_w 1$, in contradiction to $\mathrm{lm}_{>_w}(f) = 1$. □

Proposition 6.9
Let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$*,* $I \trianglelefteq R_N[\underline{x}]$ *be an ideal, and let* $G = \{g_1, \ldots, g_k\}$ *be a standard basis of* I *with respect to* $>_w$ *then*

$$\mathrm{in}_w(I) = \big\langle\, \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k)\big\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big],$$

and in particular,

$$\text{t-in}_w(I) = \big\langle\, \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k)\big\rangle \trianglelefteq K[\underline{x}].$$

Proof: If G is standard basis of I then by Corollary 4.6 every element $f \in I$ has a weak standard representation of the form $u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k$, where $\mathrm{lt}_{>_w}(u) = 1$ and $\mathrm{lm}_{>_w}(u \cdot f) \geq \mathrm{lm}_{>_w}(q_i \cdot g_i)$. The latter in particular implies that

$$\mathrm{ord}_w(u \cdot f) = \deg_w\big(\mathrm{lm}_{>_w}(u \cdot f)\big) \geq \deg_w\big(\mathrm{lm}_{>_w}(q_i \cdot g_i)\big) = \mathrm{ord}_w(q_i \cdot g_i).$$

We conclude therefore by Lemma 6.5 and Proposition 6.8 that

$$\mathrm{in}_w(f) = \mathrm{in}_w(u \cdot f) \in \big\langle\, \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k)\big\rangle.$$

For the part on the t-initial ideals just note that if $f \in I$ then by the above $\mathrm{in}_w(f) = \sum_{i=1}^k h_i \cdot \mathrm{in}_w(g_i)$ for some $h_i \in K\big[t^{\frac{1}{N}}, \underline{x}\big]$, and thus

$$\text{t-in}_w(f) = \sum_{i=1}^k h_i(1, \underline{x}) \cdot \text{t-in}_w(g_i) \in \langle \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k)\rangle_{K[\underline{x}]}.$$

□

Theorem 6.10
Let $J \trianglelefteq L[\underline{x}]$ *and* $I \trianglelefteq R_N[\underline{x}]$ *be ideals with* $J = \langle I\rangle_{L[\underline{x}]}$*, let* $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$*, and let* G *be a standard basis of* I *with respect to* $>_w$*. Then*

$$\text{t-in}_w(J) = \text{t-in}_w(I) = \big\langle\, \text{t-in}_w(G)\big\rangle \lhd K[\underline{x}].$$

Proof: Since $R_N[\underline{x}]$ is noetherian, we may add a finite number of elements of I to G so as to assume that $G = (g_1, \ldots, g_k)$ generates I. Since by Proposition 6.9 we already know that the t-initial forms of any standard basis of I with respect to $>_w$ generate $\text{t-in}_w(I)$ this does not change the right hand side. But then by assumption $J = \langle G\rangle_{L[\underline{x}]}$, and given an element $f \in J$ we can write it as

$$f = \sum_{i=1}^k t^{\frac{-\alpha}{N \cdot M}} \cdot a_i \cdot g_i$$

for some $M >> 0$, $a_i \in R_{N\cdot M}$ and $\alpha \in \mathbb{N}$. It follows that

$$t^{\frac{\alpha}{N\cdot M}} \cdot f = \sum_{i=1}^{k} a_i \cdot g_i \in \langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Since G is a standard basis over $R_N[\underline{x}]$ with respect to $>_w$ on $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$ by Buchberger's Criterion 4.6 $\operatorname{spoly}(g_i, g_j)$, $i < j$, has a weak standard representation $u_{ij} \cdot \operatorname{spoly}(g_i, g_j) = \sum_{\nu=1}^{k} q_{ij\nu} \cdot g_\nu$ with $u_{ij}, q_{ij\nu} \in R_N[\underline{x}] \subseteq R_{N\cdot M}[\underline{x}]$ and $\operatorname{lt}_{>_w}(u_{ij}) = 1$. Taking Remark 6.7 into account these are also weak standard representations with respect to the corresponding monomial ordering $>_w$ on $\operatorname{Mon}(t^{\frac{1}{N\cdot M}}, \underline{x})$, and again by Buchberger's Criterion 4.6 there exists a weak standard representation $u \cdot t^{\frac{\alpha}{N\cdot M}} \cdot f = \sum_{i=1}^{k} q_i \cdot g_i$. By Propositions 6.5 and 6.8 this implies that

$$t^{\frac{\alpha}{N\cdot M}} \cdot \operatorname{in}_w(f) = \operatorname{in}_w\left(u \cdot t^{\frac{\alpha}{N\cdot M}} \cdot f\right) \in \big\langle \operatorname{in}_w(G)\big\rangle.$$

Setting $t = 1$ we get $\text{t-in}_w(f) = \left(t^{\frac{k}{N\cdot M}} \cdot \operatorname{in}_w(f)\right)_{|t=1} \in \big\langle \text{t-in}_w(G)\big\rangle$. □

Corollary 6.11
Let $J = \langle I'\rangle_{L[\underline{x}]}$ with $I' \trianglelefteq K\left[t^{\frac{1}{N}}, \underline{x}\right]$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and G is a standard basis of I' with respect to $>_w$ on $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$, then

$$\text{t-in}_w(J) = \text{t-in}_w(I') = \big\langle \text{t-in}_w(G)\big\rangle \trianglelefteq K[\underline{x}].$$

Proof: Enlarge G to a finite generating set G' of I', then G' is still a standard basis of I'. By Corollary 4.7 G' is then also a standard basis of

$$I := \langle G'\rangle_{R_N[\underline{x}]} = \langle f_1, \ldots, f_k\rangle_{R_N[\underline{x}]},$$

and Theorem 6.10 applied to I thus shows that

$$\text{t-in}(J) = \big\langle \text{t-in}_w(G')\big\rangle.$$

However, if $f \in G' \subset I'$ is one of the additional elements then it has a weak standard representation

$$u \cdot f = \sum_{g\in G} q_g \cdot g$$

with respect to G and $>_w$, since G is a standard basis of I'. Applying Propositions 6.5 and 6.8 then shows that $\operatorname{in}_w(f) \in \langle \operatorname{in}_w(G)\rangle$, which finishes the proof. □

Remark 6.12
Note that if $I \trianglelefteq R_N[\underline{x}]$ and $J = \langle I\rangle_{L[\underline{x}]}$, then

$$J \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}}\right\rangle^{\infty},$$

but the saturation is in general necessary.

Since $L_N \subset L$ is a field extension Corollary 6.13 implies $J \cap L_N[\underline{x}] = \langle I\rangle_{L_N[\underline{x}]}$, and it suffices to see that

$$\langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty.$$

If $I \cap S_N \neq \emptyset$ then both sides of the equation coincide with $R_N[\underline{x}]$, so that we may assume that $I \cap S_N$ is empty. Recall that $L_N = S_N^{-1} R_N$, so that if $f \in R_N[\underline{x}]$ with $t^{\frac{\alpha}{N}} \cdot f \in I$ for some α, then

$$f = \frac{t^{\frac{\alpha}{N}} \cdot f}{t^{\frac{\alpha}{N}}} \in \langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}].$$

Conversely, if $f = \frac{g}{t^{\frac{k}{N}}} \in \langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}]$ with $g \in I$, then $g = t^{\frac{\alpha}{N}} \cdot f \in I$ and thus f is in the right hand side.

Corollary 6.13
Let $F \subset F'$ be a field extension and $I \trianglelefteq F[\underline{x}]$. Then $I = \langle I\rangle_{F'[\underline{x}]} \cap F[\underline{x}]$.

Proof: The result is obvious if I is generated by monomials. For the general case fix any global monomial ordering $>$ on $\mathrm{Mon}(\underline{x})$ and set $I^e = \langle I\rangle_{F'[\underline{x}]}$. Since $I \subseteq I^e \cap F[\underline{x}] \subseteq I^e$ we also have

$$L_>(I) \subseteq L_>\big(I^e \cap F[\underline{x}]\big) \subseteq L_>(I^e) \cap F[\underline{x}]. \tag{6.1}$$

If we choose a standard basis $G = (g_1,\ldots,g_k)$ of I, then by Buchberger's Criterion G is also a Gröbner basis of I^e and thus

$$L_>(I) = \langle \mathrm{lm}_>(g_i) \mid i = 1,\ldots,k\rangle_{F[\underline{x}]}$$

and

$$L_>(I^e) = \langle \mathrm{lm}_>(g_i) \mid i = 1,\ldots,k\rangle_{F'[\underline{x}]} = \big\langle L_>(I)\big\rangle_{F'[\underline{x}]}.$$

Since the latter is a monomial ideal, we have

$$L_>(I^e) \cap F[\underline{x}] = L_>(I).$$

In view of (6.1) this shows that

$$L_>(I) = L_>\big(I^e \cap F[\underline{x}]\big),$$

and since $I \subseteq I^e \cap F[\underline{x}]$ this finishes the proof by Proposition 4.3. □

We can actually show more, namely, that for each $I \trianglelefteq R_N[\underline{x}]$ and each $M > 0$ (see Corollary 6.15)

$$\langle I\rangle_{R_{M\cdot N}[\underline{x}]} \cap R_N[\underline{x}] = I,$$

and if I is saturated with respect to $t^{\frac{1}{N}}$ then (see Corollary 6.18)

$$\mathrm{in}_w\left(\langle I\rangle_{R_{M\cdot N}[\underline{x}]}\right) = \big\langle\, \mathrm{in}_w(G)\big\rangle,$$

if G is a standard basis of I with respect to $>_w$.

For this we need the following simple observation.

Lemma 6.14
$R_{N\cdot M}[\underline{x}]$ is a free $R_N[\underline{x}]$-module with basis $\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\}$.

Corollary 6.15
If $I \trianglelefteq R_N[\underline{x}]$ then $\langle I\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_N[\underline{x}] = I$.

Proof: If $f = g \cdot h \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_N[\underline{x}]$ with $g \in I$ and $h \in R_{N\cdot M}[\underline{x}]$ then by Lemma 6.14 there are uniquely determined $h_i \in R_N$ such that $h = \sum_{i=0}^{M-1} h_i \cdot t^{\frac{i}{N\cdot M}}$, and hence $f = \sum_{i=0}^{M-1}(g \cdot h_i) \cdot t^{\frac{i}{N\cdot M}}$ with $g \cdot h_i \in R_N[\underline{x}]$. By assumption $f \in R_N[\underline{x}] = R_{N\cdot M}[\underline{x}] \cap \langle 1\rangle_{R_N[\underline{x}]}$ and by Lemma 6.14 we thus have $g \cdot h_i = 0$ for all $i = 1, \ldots, M-1$. But then $f = g \cdot h_0 \in I$. □

Lemma 6.16
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \langle t^{\frac{1}{N}}\rangle^\infty$, then for any $M \geq 1$

$$\langle I\rangle_{R_{N\cdot M}[\underline{x}]} = \langle I\rangle_{R_{N\cdot M}[\underline{x}]} : \langle t^{\frac{1}{N\cdot M}}\rangle^\infty.$$

Proof: Let $f, h \in R_{N\cdot M}[\underline{x}]$, $\alpha \in \mathbb{N}$, $g \in I$ such that

$$t^{\frac{\alpha}{N\cdot M}} \cdot f = g \cdot h. \tag{6.2}$$

We have to show that $f \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. For this purpose do division with remainder in order to get $\alpha = a \cdot M + b$ with $0 \leq b < M$. By Lemma 6.14 there are $h_i, f_i \in R_N[\underline{x}]$ such that $f = \sum_{i=0}^{M-1} f_i \cdot t^{\frac{i}{N\cdot M}}$ and $h = \sum_{i=0}^{M-1} h_i \cdot t^{\frac{i}{N\cdot M}}$. (6.2) then translates into

$$\sum_{i=0}^{M-1-b} t^{\frac{b+i}{N\cdot M}} \cdot t^{\frac{a}{N}} \cdot f_i + \sum_{i=M-b}^{M-1} t^{\frac{b+i-M}{N\cdot M}} \cdot t^{\frac{a+1}{N}} \cdot f_i = \sum_{i=0}^{M-1} g \cdot h_i \cdot t^{\frac{i}{N\cdot M}},$$

and since $\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\}$ is $R_N[\underline{x}]$-linearly independent we can compare coefficients to find $t^{\frac{a}{N}} \cdot f_i = g \cdot h_{b+i} \in I$ for $i = 0, \ldots, M-b-1$, and $t^{\frac{a+1}{N}} \cdot f_i = g \cdot h_{b+i-M} \in I$ for $i = M-b, \ldots, M-1$. In any case, since I is saturated with respect to $t^{\frac{1}{N}}$ by assumption we conclude that $f_i \in I$ for all $i = 0, \ldots, M-1$, and therefore $f \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. □

Corollary 6.17
Let $J \trianglelefteq L[\underline{x}]$ be an ideal such that $J = \langle J \cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of $J \cap R_N[\underline{x}]$ with respect to $>_w$.

Then for all $M \geq 1$

$$\text{in}_w\left(J \cap R_{N\cdot M}[\underline{x}]\right) = \left\langle \text{in}_w(G)\right\rangle \lhd K\left[t^{\frac{1}{N\cdot M}}, \underline{x}\right]$$

and

$$\text{t-in}_w\left(J \cap R_{N\cdot M}[\underline{x}]\right) = \left\langle \text{t-in}_w(G)\right\rangle = \text{t-in}_w\left(J \cap R_N[\underline{x}]\right) \lhd K[\underline{x}].$$

Proof: Enlarge G to a generating set G' of $I = J \cap R_N[\underline{x}]$ over $R_N[\underline{x}]$ by adding a finite number of elements of I. Then

$$\left\langle L_{>_w}(G')\right\rangle \subseteq \left\langle L_{>_w}(I)\right\rangle = \left\langle L_{>_w}(G)\right\rangle \subseteq \left\langle L_{>_w}(G')\right\rangle$$

shows that G' is still a standard basis of I with respect to $>_w$. So we can assume that $G = G'$.

By Proposition 6.9 it suffices to show that G is also a standard basis of $J \cap R_{N\cdot M}[\underline{x}]$. Since by assumption $J = \langle I\rangle_{L[\underline{x}]} = \langle G\rangle_{L[\underline{x}]}$, Corollary 6.13 implies that

$$J \cap L_{N\cdot M}[\underline{x}] = \langle G\rangle_{L_{N\cdot M}[\underline{x}]} = S^{-1}_{N\cdot M}\langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Moreover, by Remark 6.12 the ideal $I = \langle G\rangle_{R_N[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N}}$ and by Lemma 6.16 therefore also $\langle G\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, which implies that

$$J \cap R_{N\cdot M}[\underline{x}] = S^{-1}_{N\cdot M}\langle G\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_{N\cdot M}[\underline{x}] = \langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Since $G = (g_1,\ldots,g_k)$ is a standard basis of I every $\mathrm{spoly}(g_i,g_j)$, $i<j$, has a weak standard representation with respect to G and $>_w$ over $R_N[\underline{x}]$ by Buchberger's Criterion 4.6, and these are of course also weak standard representations over $R_{N\cdot M}[\underline{x}]$, so that again by Buchberger's Criterion G is a standard basis of $\langle G\rangle_{R_{N\cdot M}[\underline{x}]} = J \cap R_{N\cdot M}[\underline{x}]$. □

Corollary 6.18
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$, let $w \in \mathbb{R}_{<0}\times\mathbb{R}^n$, and let G be a standard basis of I with respect to $>_w$.

Then for all $M \geq 1$

$$\mathrm{in}_w\left(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\right) = \left\langle\, \mathrm{in}_w(G)\right\rangle \lhd K\left[t^{\frac{1}{N\cdot M}},\underline{x}\right]$$

and

$$\text{t-in}_w\left(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\right) = \left\langle\, \text{t-in}_w(G)\right\rangle = \text{t-in}_w(I) \lhd K[\underline{x}]$$

Proof: If we consider $J = \langle I\rangle_{L[\underline{x}]}$ then by Remark 6.12 $J \cap R_N[\underline{x}] = I$, and moreover, by Lemma 6.16 also $\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, so that applying Remark 6.12 once again we also find $J\cap R_{N\cdot M}[\underline{x}] = \langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. The result therefore follows from Corollary 6.17. □

Corollary 6.19
Let $J \trianglelefteq L[\underline{x}]$ be an ideal such that $J = \langle J\cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, let $w = (-1,0,\ldots,0)$ and let $M \geq 1$. Then

$$1 \in \mathrm{in}_\omega\left(J\cap R_N[\underline{x}]\right) \quad\Longleftrightarrow\quad 1 \in \mathrm{in}_\omega\left(J\cap R_{N\cdot M}[\underline{x}]\right).$$

Proof: Suppose that $f \in J\cap R_{N\cdot M}[\underline{x}]$ with $\mathrm{in}_\omega(f) = 1$, and let $G = (g_1,\ldots,g_k)$ be standard basis of $J \cap R_N[\underline{x}]$ with respect to $>_w$. By Corollary 6.17

$$1 = \mathrm{in}_\omega(f) \in \left\langle\, \mathrm{in}_\omega(g_1),\ldots,\mathrm{in}_\omega(g_k)\right\rangle \lhd K\left[t^{\frac{1}{N\cdot M}},\underline{x}\right],$$

and since this ideal and 1 are w-quasihomogeneous, there exist w-quasihomogeneous elements $h_1, \ldots, h_k \in K\left[t^{\frac{1}{N\cdot M}}, \underline{x}\right]$ such that

$$1 = \sum_{i=1}^{k} h_i \cdot \mathrm{in}_\omega(g_i),$$

where each summand on the right hand side (possibly zero) is w-quasihomogeneous of w-degree zero. Since $w = (-1, 0, \ldots, 0)$ this forces $h_i \in K[\underline{x}]$ for all $i = 1, \ldots, k$ and thus $1 \in \mathrm{in}_\omega(J \cap R_N[\underline{x}])$. The converse is clear anyhow. $\square$

We want to conclude the section by a remark on the saturation.

Proposition 6.20
If $f_1, \ldots, f_k \in K[t, \underline{x}]$ *and* $I = \langle f_1, \ldots, f_k \rangle \trianglelefteq K[t]_{\langle t \rangle}[\underline{x}]$ *then*

$$\langle I \rangle_{R_1[\underline{x}]} : \langle t \rangle^\infty = \big\langle I : \langle t \rangle^\infty \big\rangle_{R_1[\underline{x}]}.$$

Proof: Let $>_1$ be any global monomial ordering on $\mathrm{Mon}(\underline{x})$ and define a t-local monomial ordering on $\mathrm{Mon}\left(t, \underline{x}\right)$ by

$$t^\alpha \cdot \underline{x}^\beta \ > \ t^{\alpha'} \cdot \underline{x}^{\beta'}$$

if and only if

$$\underline{x}^\alpha \ >_1 \ \underline{x}^{\alpha'} \ \text{ or } \ \big(\underline{x}^\alpha = \underline{x}^{\alpha'} \ \text{ and } \ \alpha < \alpha'\big).$$

Then

$$\{f \in R_1[\underline{x}] \mid \mathrm{lt}_>(f) = 1\} = \{1 + t \cdot p \mid p \in K[t]\},$$

and thus

$$R_1[\underline{x}]_> = R_1[\underline{x}] \quad \text{and} \quad K[t, \underline{x}]_> = K[t]_{\langle t \rangle}[\underline{x}].$$

Using Remark 4.9 we can compute at the same time a standard basis of $\langle I \rangle_{R_1[\underline{x}]} : \langle t \rangle^\infty$ and of $\langle I \rangle_{K[t]_{\langle t \rangle}[\underline{x}]} : \langle t \rangle^\infty$ with respect to $>$. Since a standard basis is a generating set in the localised ring the result follows. $\square$

PAPER IX

An Algorithm for Lifting Points in a Tropical Variety

Abstract: The aim of this paper is to give a constructive proof of one of the basic theorems of tropical geometry: given a point on a tropical variety (defined using initial ideals), there exists a Puiseux-valued "lift" of this point in the algebraic variety. This theorem is so fundamental because it justifies why a tropical variety (defined combinatorially using initial ideals) carries information about algebraic varieties: it is the image of an algebraic variety over the Puiseux series under the valuation map. We have implemented the "lifting algorithm" using SINGULAR and Gfan if the base field is $\mathbb{Q}$. As a byproduct we get an algorithm to compute the Puiseux expansion of a space curve singularity in $(K^{n+1}, 0)$.

This paper is a joint work with Anders Nedergaard Jensen, Berlin, and Hannah Markwig, Minneapolis, [**JMM07**].

1. Introduction

In tropical geometry, algebraic varieties are replaced by certain piecewise linear objects called tropical varieties. Many algebraic geometry theorems have been "translated" to the tropical world (see for example [**Mik05**], [**Vig04**], [**SpS04a**], [**GaM07a**] and many more). Because new methods can be used in the tropical world — for example, combinatorial methods — and because the objects seem easier to deal with due to their piecewise linearity, tropical geometry is a promising tool for deriving new results in algebraic geometry. (For example, the Welschinger invariant can be computed tropically, see [**Mik05**]).

There are two ways to define the tropical variety $\mathrm{Trop}(J)$ for an ideal J in the polynomial ring $K\{\{t\}\}[x_1, \ldots, x_n]$ over the field of Puiseux series (see Definition 2.1). One way is to define the tropical variety combinatorially using t-initial ideals (see Definition 2.4 and Definition 2.10) — this definition is more helpful when computing and it is the definition we use in this paper. The other way to define tropical varieties is as (the closure of) the image of the algebraic variety $V(J)$ of J in $K\{\{t\}\}^n$ under the negative of the valuation map (see Remark 2.2) — this gives more insight why tropical varieties carry information about algebraic varieties.

It is our main aim in this paper to give a constructive proof that these two concepts coincide (see Theorem 2.13), and to derive that way an algorithm which allows to lift a given point $\omega \in \operatorname{Trop}(J)$ to a point in $V(J)$ up to given order (see Algorithms 3.8 and 4.8). The algorithm has been implemented using the commutative algebra system SINGULAR (see **[GPS05]**) and the programme `Gfan` (see **[Jen07]**), which computes Gröbner fans and tropical varieties.

Theorem 2.13 has been proved in the case of a principal ideal by **[EKL04]**, Theorem 2.1.1. There is also a constructive proof for a principal ideal in **[Tab05]**, Theorem 2.4. For the general case, there is a proof in **[SpS04b]**, Theorem 2.1, which has a gap however. Furthermore, there is a proof in **[Dra06]**, Theorem 4.2, using affinoid algebras, and in **[Kat06]**, Lemma 5.2.2, using flat schemes. A more general statement is proved in **[Pay07]**, Theorem 4.2. (Note that what we call a tropical variety is called a Speyer-Sturmfels set in Payne's paper.) Our proof has the advantage that it is constructive and works for an arbitrary ideal J.

We describe our algorithm first in the case where the ideal is 0-dimensional. This algorithm can be viewed as a variant of an algorithm presented by Joseph Maurer in **[Mau80]**, a paper from 1980. In fact, he uses the term "critical tropism" for a point in the tropical variety, even though tropical varieties were not defined by that time. Apparently, the notion goes back to Monique Lejeune-Jalabert and Bernard Teissier[1] (see **[LJT73]**).

This paper is organised as follows: In Section 2 we recall basic definitions and state the main result. In Section 3 we give a constructive proof of the main result in the 0-dimensional case and deduce an algorithm. In Section 4 we reduce the arbitrary case algorithmically to the 0-dimensional case, and in Section 5 we gather some simple results from commutative algebra for the lack of a better reference. The proofs of both cases heavily rely on a good understanding of the relation of the dimension of an ideal J over the Puiseux

[1] Asked about this coincidence in the two notions Bernard Teissier sent us the following kind and interesting explanation: *As far as I know the term did not exist before. We tried to convey the idea that giving different weights to some variables made the space "anisotropic", and we were intrigued by the structure, for example, of anisotropic projective spaces (which are nowadays called weighted projective spaces). From there to "tropismes critiques" was a quite natural linguistic movement. Of course there was no "tropical" idea around, but as you say, it is an amusing coincidence. The Greek "Tropos" usually designates change, so that "tropisme critique" is well adapted to denote the values where the change of weights becomes critical. The term "Isotropic", apparently due to Cauchy, refers to the property of presenting the same (physical) characters in all directions. Anisotropic is, of course, its negation. The name of Tropical geometry originates, as you probably know, from tropical algebra which honours a Brazilian computer scientist living close to the tropics, where the course of the sun changes back to the equator. In a way the tropics of Capricorn and Cancer represent, for the sun, critical tropisms.*

series with its t-initial ideal respectively with its restriction to the rings $R_N[\underline{x}]$ introduced below (see Definition 2.1). This will be studied in Section 6. Some of the theoretical as well as the computational results use Theorem 2.8 which was proved in [**Mar07**] using standard bases in the mixed power series polynomial ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

We would like to thank Bernd Sturmfels for suggesting the project and for many helpful discussions, and Michael Brickenstein, Gerhard Pfister and Hans Schönemann for answering many questions concerning SINGULAR. Also we would like to thank Sam Payne for helpful remarks and for pointing out a mistake in an earlier version of this paper.

Our programme can be downloaded from the web page

www.mathematik.uni-kl.de/~keilen/en/tropical.html.

2. Basic Notations and the Main Theorem

In this section we will introduce the basic notations used throughout the paper.

Definition 2.1
Let K be an arbitrary field. We consider for $N \in \mathbb{N}_{>0}$ the discrete valuation ring

$$R_N = K\big[\big[t^{\frac{1}{N}}\big]\big] = \left\{ \sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \;\middle|\; a_\alpha \in K \right\}$$

of formal power series in the unknown $t^{\frac{1}{N}}$ with *discrete valuation*

$$\operatorname{val}\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}}\right) = \operatorname{ord}_t\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}}\right) = \min\left\{\frac{\alpha}{N} \;\middle|\; a_\alpha \neq 0\right\} \in \frac{1}{N}\cdot\mathbb{Z}\cup\{\infty\},$$

and we denote by $L_N = \operatorname{Quot}(R_N)$ its quotient field. If $N \mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L = K\{\{t\}\} = \varinjlim L_N = \bigcup_{N>0} L_N$$

the *field of (formal) Puiseux series* over K.

Remark 2.2
If $0 \neq N \in \mathbb{N}$ then $S_N = \left\{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, t^{\frac{3}{N}}, \ldots\right\}$ is a multiplicatively closed subset of R_N, and obviously

$$L_N = S_N^{-1} R_N = \left\{ t^{\frac{-\alpha}{N}} \cdot f \;\middle|\; f \in R_N, \alpha \in \mathbb{N} \right\}.$$

The valuation of R_N extends to L_N, and thus L, by $\operatorname{val}\left(\frac{f}{g}\right) = \operatorname{val}(f) - \operatorname{val}(g)$ for $f, g \in R_N$ with $g \neq 0$. In particular, $\operatorname{val}(0) = \infty$.

Notation 2.3
Since an ideal $J \trianglelefteq L[\underline{x}]$ is generated by finitely many elements, the set

$$\mathcal{N}(J) = \big\{N \in \mathbb{N}_{>0} \;\big|\; \langle J \cap R_N[\underline{x}]\rangle_{L[\underline{x}]} = J\big\}$$

is non-empty, and if $N \in \mathcal{N}(J)$ then $N \cdot \mathbb{Z} \subseteq \mathcal{N}(J)$. We also introduce the notation $J_{R_N} = J \cap R_N[\underline{x}]$.

Remark and Definition 2.4
Let $N \in \mathbb{N}_{>0}$, $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.

We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta}$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta}.$$

As a K-vector space the formal power series ring $K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]$ is just

$$K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big] = \prod_{q \in \mathbb{R}} V_{q,w,N},$$

and we can thus write any power series $f \in K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]$ in a unique way as

$$f = \sum_{q \in \mathbb{R}} f_{q,w} \quad \text{with} \quad f_{q,w} \in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f \in K\big[\big[t^{\frac{1}{N'}}, \underline{x}\big]\big]$ for some other $N' \in \mathbb{N}_{>0}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.

Moreover, if $0 \neq f \in R_N[\underline{x}] \subset K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]$, then there is a *maximal* $\hat{q} \in \mathbb{R}$ such that $f_{\hat{q},w} \neq 0$ and $f_{q,w} \in W_{q,w,N}$ for all $q \in \mathbb{R}$, since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w}) = q \in \mathbb{R}$,

$$\mathrm{in}_w(f) := f_{\hat{q},w} \in K\big[t^{\frac{1}{N}}, \underline{x}\big]$$

the *w-initial form* of f, and

$$\mathrm{ord}_w(f) := \hat{q} = \max\{\deg_w(f_{q,w}) \mid f_{q,w} \neq 0\}$$

the *w-order* of f. Set $\in_\omega(0) = 0$. If $t^{\beta}x^{\alpha} \neq t^{\beta'}x^{\alpha'}$ are both monomials of $\mathrm{in}_w(f)$, then $\alpha \neq \alpha'$.

For $I \subseteq R_N[\underline{x}]$ we call

$$\mathrm{in}_w(I) = \big\langle \mathrm{in}_w(f) \;\big|\; f \in I\big\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big]$$

the w-*initial ideal* of I. Note that its definition depends on N.

Moreover, we call for $f \in R_N[\underline{x}]$

$$\text{t-in}_w(f) = \text{in}_w(f)(1, \underline{x}) = \text{in}_w(f)_{|t=1} \in K[\underline{x}]$$

the t-*initial form of* f *w.r.t.* w, and if $f = t^{\frac{-\alpha}{N}} \cdot g \in L[\underline{x}]$ with $g \in R_N[\underline{x}]$ we set

$$\text{t-in}_w(f) := \text{t-in}_w(g).$$

This definition does not depend on the particular representation of f.

If $J \subseteq L[\underline{x}]$ is a subset of $L[\underline{x}]$, then

$$\text{t-in}_w(J) = \langle \text{t-in}_w(f) \mid f \in J \rangle \trianglelefteq K[\underline{x}]$$

is the t-*initial ideal* of J, which does not depend on any N.

For two w-quasihomogeneous elements $f_{q,w} \in W_{q,w,N}$ and $f_{q',w} \in W_{q',w,N}$ we have $f_{q,w} \cdot f_{q',w} \in W_{q+q',w,N}$. In particular, $\text{in}_w(f \cdot g) = \text{in}_w(f) \cdot \text{in}_w(g)$ for $f, g \in R_N[\underline{x}]$, and $\text{t-in}_w(f \cdot g) = \text{t-in}_w(f) \cdot \text{t-in}_w(g)$ for $f, g \in L[\underline{x}]$.

Example 2.5
Let $w = (-1, -2, -1)$ and

$$f = \left(2t + t^{\frac{3}{2}} + t^2\right) \cdot x^2 + (-3t^3 + 2t^4) \cdot y^2 + t^5 xy^2 + \left(t + 3t^2\right) \cdot x^7 y^2.$$

Then $\text{ord}_w(f) = -5$, $\text{in}_w(f) = 2tx^2 - 3t^3y^2$, and $\text{t-in}_w(f) = 2x^2 - 3y^2$.

Notation 2.6
Throughout this paper we will mostly use the weight -1 for the variable t, and in order to simplify the notation we will then usually write for $\omega \in \mathbb{R}^n$

$$\text{in}_\omega \quad \text{instead of} \quad \text{in}_{(-1,\omega)}$$

and

$$\text{t-in}_\omega \quad \text{instead of} \quad \text{t-in}_{(-1,\omega)}\,.$$

The case that $\omega = (0, \ldots, 0)$ is of particular interest, and we will simply write

$$\text{in}_0 \quad \text{respectively} \quad \text{t-in}_0\,.$$

This should not lead to any ambiguity.

In general, the t-initial ideal of an ideal J is not generated by the t-initial forms of the given generators of J.

Example 2.7
Let $J = \langle tx + y, x + t \rangle \lhd L[x, y]$ and $\omega = (1, -1)$. Then $y - t^2 \in J$, but

$$y = \text{t-in}_\omega(y - t^2) \notin \langle \text{t-in}_\omega(tx + y), \text{t-in}_\omega(x + t) \rangle = \langle x \rangle.$$

We can compute the t-initial ideal using standard bases by **[Mar07]**, Corollary 6.11.

Theorem 2.8
Let $J = \langle I \rangle_{L[\underline{x}]}$ *with* $I \trianglelefteq K\left[t^{\frac{1}{N}}, \underline{x}\right]$, $\omega \in \mathbb{Q}^n$ *and* G *be a standard basis of* I *with respect to* $>_\omega$ *(see Remark 3.7 for the definition of* $>_\omega$*).*

Then $\text{t-in}_\omega(J) = \text{t-in}_\omega(I) = \big\langle \text{t-in}_\omega(G) \big\rangle \trianglelefteq K[\underline{x}]$.

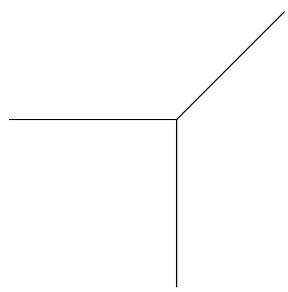
The proof of this theorem uses standard basis techniques in the ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

Example 2.9
In Example 2.7, $G = (tx + y, x + t, y - t^2)$ is a suitable standard basis and thus $\text{t-in}_\omega(J) = \langle x, y \rangle$.

Definition 2.10
Let $J \trianglelefteq L[\underline{x}]$ be an ideal then the *tropical variety* of J is defined as

$$\text{Trop}(J) = \{\omega \in \mathbb{R}^n \mid \text{t-in}_\omega(J) \text{ is monomial free}\}.$$

Example 2.11
Let $J = \langle x + y + 1 \rangle \subset L[x, y]$. As J is generated by one polynomial f which then automatically is a standard basis, the t-initial ideal $\text{t-in}_\omega(J)$ will be generated by $\text{t-in}_\omega(f)$ for any ω. Hence $\text{t-in}_\omega(J)$ contains no monomial if and only if $\text{t-in}_\omega(f)$ is not a monomial. This is the case for all ω such that $\omega_1 = \omega_2 \geq 0$, or $\omega_1 = 0 \geq \omega_2$, or $\omega_2 = 0 \geq \omega_1$. Hence the tropical variety $\text{Trop}(J)$ looks as follows:



We need the following basic results about tropical varieties, which are easy to prove.

Lemma 2.12
Let $J, J_1, \ldots, J_k \trianglelefteq L[\underline{x}]$ *be ideals. Then:*

(a) $J_1 \subseteq J_2 \implies \text{Trop}(J_1) \supseteq \text{Trop}(J_2)$,

(b) $\text{Trop}(J_1 \cap \ldots \cap J_k) = \text{Trop}(J_1) \cup \ldots \cup \text{Trop}(J_k)$,

(c) $\text{Trop}(J) = \text{Trop}\left(\sqrt{J}\right) = \bigcup_{P \in \text{minAss}(J)} \text{Trop}(P)$, *and*

(d) $\text{Trop}(J_1 + J_2) \subseteq \text{Trop}(J_1) \cap \text{Trop}(J_2)$.

We are now able to state our main theorem.

Theorem 2.13
If K *is algebraically closed of characteristic zero and* $J \trianglelefteq K\{\{t\}\}[\underline{x}]$ *is an ideal*

then

$$\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n \qquad \Longleftrightarrow \qquad \exists\, p \in V(J) :\; -\mathrm{val}(p) = \omega \in \mathbb{Q}^n,$$

where val *is the coordinate-wise valuation.*

The proof of one direction is straight forward and it does not require that K is algebraically closed.

Proposition 2.14
If $J \trianglelefteq L[\underline{x}]$ *is an ideal and* $p \in V(J) \cap (L^*)^n$*, then* $-\mathrm{val}(p) \in \mathrm{Trop}(J)$.

Proof: Let $p = (p_1, \ldots, p_n)$, and let $\omega = -\mathrm{val}(p) \in \mathbb{Q}^n$. If $f \in J$, we have to show that $\text{t-in}_\omega(f)$ is not a monomial, but since this property is preserved when multiplying with some $t^{\frac{\alpha}{N}}$ we may as well assume that $f \in J_{R_N}$. As $p \in V(J)$, we know that $f(p) = 0$. In particular the terms of lowest t-order in $f(p)$ have to cancel. But the terms of lowest order in $f(p)$ are $\mathrm{in}_\omega(f)(a_1 \cdot t^{\omega_1}, \ldots, a_n \cdot t^{\omega_n})$, where $p_i = a_i \cdot t^{\omega_i} + h.o.t.$. Hence $\mathrm{in}_\omega(f)(a_1 t^{\omega_1}, \ldots, a_n t^{\omega_n}) = 0$, which is only possible if $\mathrm{in}_\omega(f)$, and thus $\text{t-in}_\omega(f)$, is not a monomial. $\square$

Remark 2.15
If the base field K in Theorem 2.13 is not algebraically closed or not of characteristic zero, then the Puiseux series field is not algebraically closed (see e.g. [**Ked01**]). We therefore cannot expect to be able to lift each point in the tropical variety of an ideal $J \lhd K\{\{t\}\}[\underline{x}]$ to a point in $V(J) \subseteq K\{\{t\}\}^n$. However, if we replace $V(J)$ by the vanishing set, say W, of J over the algebraic closure $\overline{L}$ of $K\{\{t\}\}$ then it is still true that each point ω in the tropical variety of J can be lifted to a point $p \in W$ such that $\mathrm{val}(p) = -\omega$. For this we note first that if $\dim(J) = 0$ then the non-constructive proof of Theorem 3.1 works by passing from J to $\langle J \rangle_{\overline{L}[\underline{x}]}$, taking into account that the non-archemdian valuation of a field in a natural way extends to its algebraic closure. And if $\dim(J) > 0$ then we can add generators to J by Proposition 4.6 and Remark 4.5 so as to reduce to the zero dimensional case before passing to the algebraic closure of $K\{\{t\}\}$.

Note, it is even possible to apply Algorithm 3.8 in the case of positive characteristic. However, due to the weird nature of the algebraic closure of the Puiseux series field in that case we cannot guarantee that the result will coincide with a solution of J up to the order up to which it is computed. It may very well be the case that some intermediate terms are missing (see [**Ked01**] Section 5).

3. Zero-Dimensional Lifting Lemma

In this section we want to give a constructive proof of the Lifting Lemma 3.1.

Theorem 3.1 (Lifting Lemma)
Let K *is an algebraically closed field of characteristic zero and* $L = K\{\{t\}\}$*. If*

$J \lhd L[\underline{x}]$ is a zero dimensional ideal and $\omega \in \operatorname{Trop}(J) \cap \mathbb{Q}^n$ then there is a point $p \in V(J)$ such that $-\operatorname{val}(p) = \omega$.

Non-Constructive Proof: If $\omega \in \operatorname{Trop}(J)$ then by Lemma 2.12 there is an associated prime $P \in \operatorname{minAss}(J)$ such that $\omega \in \operatorname{Trop}(P)$. But since $\dim(J) = 0$ the ideal P is necessarily a maximal ideal, and since L is algebraically closed it is of the form

$$P = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$$

with $p_1, \ldots, p_n \in L$. Since $\omega \in \operatorname{Trop}(P)$ the ideal $\text{t-in}_\omega(P)$ does not contain any monomial, and therefore necessarily $\operatorname{ord}_t(p_i) = -\omega_i$ for all $i = 1, \ldots, n$. This shows that $p = (p_1, \ldots, p_n) \in V(P) \subseteq V(J)$ and $\operatorname{val}(p) = -\omega$. $\square$

The drawback of this proof is that in order to find p one would have to be able to find the associated primes of J which would amount to something close to primary decomposition over L. This is of course not feasible. We will instead adapt the constructive proof that L is algebraically closed, i.e. the Newton-Puiseux Algorithm for plane curves, which has already been generalised to space curves (see [**Mau80**], [**AMNR92**]) to our situation in order to compute the point p up to any given order.

The idea behind this is very simple and the first recursion step was basically already explained in the proof of Proposition 2.14. Suppose we have a polynomial $f \in R_N[\underline{x}]$ and a point

$$p = \big(u_1 \cdot t^{\alpha_1} + v_1 \cdot t^{\beta_1} + \ldots, \ldots, u_n \cdot t^{\alpha_n} + v_n \cdot t^{\beta_n} + \ldots\big) \in L[\underline{x}].$$

Then, a priori, the term of lowest t-order in $f(p)$ will be $\operatorname{in}_{-\alpha}(f)(u_1 \cdot t^{\alpha_1}, \ldots, u_n \cdot t^{\alpha_n})$. Thus, in order for $f(p)$ to be zero it is necessary that $\text{t-in}_\omega(f)(u_1, \ldots, u_n) = 0$. Let p' denote the tail of p, that is $p_i = u_i \cdot t^{\alpha_i} + t^{\alpha_i} \cdot p_i'$. Then p' is a zero of

$$f' = f\big(t^{\alpha_1} \cdot (u_1 + x_1), \ldots, t^{\alpha_n} \cdot (u_n + x_n)\big).$$

The same arguments then show that $\text{t-in}_{\alpha-\beta}(f')(v_1, \ldots, v_n) = 0$, and assuming now that none of the v_i is zero we find $\text{t-in}_{\alpha-\beta}(f')$ must be monomial free, that is $\alpha - \beta$ is a point in the tropical variety and all its components are strictly negative.

The basic idea for the algorithm which computes a suitable p is thus straight forward. Given $\omega = -\alpha$ in the tropical variety of an ideal J, compute a point $u \in \text{t-in}_\omega(J)$ apply the above transformation to J and compute a negative-valued point in the tropical variety of the transformed ideal. Then go on recursively.

It may happen that the solution that we are about to construct this way has some component with only finitely many terms. Then after a finite number of steps there might be no suitable ω in the tropical variety. However, in that situation we can simply eliminate the corresponding variable for the further computations.

Example 3.2

Consider the ideal $J = \langle f_1, \ldots, f_4 \rangle \lhd L[x,y]$ with

$$\begin{aligned} f_1 = &\ y^2 + 4t^2y + (-t^3 + 2t^4 - t^5), \\ f_2 = &\ (1+t)\cdot x - y + (-t - 3t^2), \\ f_3 = &\ xy + (-t + t^2)\cdot x + (t^2 - t^4), \\ f_4 = &\ x^2 - 2tx + (t^2 - t^3). \end{aligned}$$

The t-initial ideal of J with respect to $\omega = \big(-1, -\frac{3}{2}\big)$ is

$$\text{t-in}_\omega(J) = \langle y^2 - 1, x - 1 \rangle,$$

so that $\omega \in \operatorname{Trop}(J)$ and $u = (1,1)$ is a suitable choice. Applying the transformation $\gamma_{\omega,u} : (x,y) \mapsto \big(t\cdot(1+x), t^{\frac{3}{2}}\cdot(1+y)\big)$ to J we get $J' = \langle f_1', \ldots, f_4' \rangle$ with

$$\begin{aligned} f_1' &= t^3y^2 + \big(2t^3 + 4t^{\frac{7}{2}}\big)\cdot y + \big(4t^{\frac{7}{2}} + 2t^4 - t^5\big), \\ f_2' &= (t + t^2)\cdot x - t^{\frac{3}{2}}\cdot y + \big(-t^{\frac{3}{2}} - 2t^2\big), \\ f_3' &= t^{\frac{5}{2}}\cdot xy + \big(-t^2 + t^3 + t^{\frac{5}{2}}\big)\cdot x + t^{\frac{5}{2}}\cdot y + \big(t^{\frac{5}{2}} + t^3 - t^4\big), \\ f_4' &= t^2x^2 - t^3. \end{aligned}$$

This shows that the x-coordinate of a solution of J' necessarily is $x = \pm t^{\frac{1}{2}}$, and we could substitute this for x in the other equations in order to reduce by one variable. We will instead see what happens when we go on with our algorithm.

The t-initial ideal of J' with respect to $\omega' = \big(-\frac{1}{2}, -\frac{1}{2}\big)$ is

$$\text{t-in}_{\omega'}(J') = \langle y + 2, x - 1 \rangle,$$

so that $\omega' \in \operatorname{Trop}(J')$ and $u' = (1,-2)$ is our only choice. Applying the transformation $\gamma_{\omega',u'} : (x,y) \mapsto \big(t^{\frac{1}{2}}\cdot(1+x), t^{\frac{1}{2}}\cdot(-2+y)\big)$ to J' we get the ideal $J'' = \langle f_1'', \ldots, f_4'' \rangle$ with

$$\begin{aligned} f_1'' &= t^4y^2 + 2t^{\frac{7}{2}}y + \big(-2t^4 - t^5\big), \\ f_2'' &= \big(t^{\frac{3}{2}} + t^{\frac{5}{2}}\big)\cdot x - t^2\cdot y + t^{\frac{5}{2}}, \\ f_3'' &= t^{\frac{7}{2}}\cdot xy + \big(-t^{\frac{5}{2}} + t^3 - t^{\frac{7}{2}}\big)\cdot x + \big(t^3 + t^{\frac{7}{2}}\big)\cdot y + \big(-t^{\frac{7}{2}} - t^4\big), \\ f_4'' &= t^3x^2 + 2t^3x. \end{aligned}$$

If we are to find an $\omega'' \in \operatorname{Trop}(J'')$ then f_4'' implies that necessarily $\omega_1'' = 0$. But we are looking for an ω'' all of whose entries are strictly negative. The reason why this does not exist is that there is a solution of J'' with $x = 0$. We thus have to eliminate the variable x, and replace J'' by the ideal $J''' = \langle f''' \rangle$ with

$$f''' = y - t^{\frac{1}{2}}.$$

Then $\omega''' = -\frac{1}{2} \in \text{Trop}(J''')$ and $\text{t-in}_{\omega'''}(f''') = y - 1$. Thus $u''' = 1$ is our only choice, and since $f'''(u''' \cdot t^{-\omega'''}) = f'''(t^{\frac{1}{2}}) = 0$ we are done.

Backwards substitution gives

$$\begin{aligned} p &= \left(t^{\omega_1} \cdot \left(u_1 + t^{\omega_1'} \cdot (u_1' + 0)\right), t^{\omega_2} \cdot \left(u_2 + t^{\omega_2'} \cdot \left(u_2' + t^{\omega_2'''} \cdot u'''\right)\right)\right) \\ &= \left(t \cdot \left(1 + t^{\frac{1}{2}}\right), t^{\frac{3}{2}} \cdot \left(1 + t^{\frac{1}{2}} \cdot \left(-2 + t^{\frac{1}{2}}\right)\right)\right) \\ &= \left(t + t^{\frac{3}{2}}, t^{\frac{3}{2}} - 2t^2 + t^{\frac{5}{2}}\right) \end{aligned}$$

as a point in $V(J)$ with $\text{val}(p) = \left(1, \frac{3}{2}\right) = -\omega$. Note that in general the procedure will not terminate.

For the proof that this algorithm works we need two types of transformations which we are now going to introduce and study.

Definition and Remark 3.3
For $\omega' \in \mathbb{Q}^n$ let us consider the L-algebra isomorphism

$$\Phi_{\omega'} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i'} \cdot x_i,$$

and the isomorphism which it induces on L^n

$$\phi_{\omega'} : L^n \to L^n : (p_1', \ldots, p_n') \mapsto \left(t^{-\omega_1'} \cdot p_1', \ldots, t^{-\omega_n'} \cdot p_n'\right).$$

Suppose we have found a $p' \in V\left(\Phi_{\omega'}(J)\right)$, then $p = \phi_{\omega'}(p') \in V(J)$ and $\text{val}(p) = \text{val}(p') - \omega'$.

Thus choosing ω' appropriately we may in Theorem 3.1 assume that $\omega \in \mathbb{Q}^n_{<0}$, which due to Corollary 6.15 implies that the dimension of J behaves well when contracting to the power series ring $R_N[\underline{x}]$ for a suitable N.

Note also the following properties of $\Phi_{\omega'}$, which we will refer to quite frequently. If $J \trianglelefteq L[\underline{x}]$ is an ideal, then

$$\dim(J) = \dim\left(\Phi_{\omega'}(J)\right) \text{ and } \text{t-in}_{\omega'}(J) = \text{t-in}_0\left(\Phi_{\omega'}(J)\right),$$

where the latter is due to the fact that

$$\deg_w\left(t^\alpha \cdot \underline{x}^\beta\right) = -\alpha + \omega' \cdot \beta = \deg_v\left(t^{\alpha - \omega' \cdot \beta} \cdot \underline{x}^\beta\right) = \deg_v\left(\Phi_{\omega'}(t^\alpha \cdot \underline{x}^\beta)\right)$$

with $w = (-1, \omega')$ and $v = (-1, 0, \ldots, 0)$.

Definition and Remark 3.4
For $u = (u_1, \ldots, u_n) \in K^n$, $\omega \in \mathbb{Q}^n$ and $w = (-1, \omega)$ we consider the L-algebra isomorphism

$$\gamma_{\omega,u} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i} \cdot (u_i + x_i),$$

and its effect on a w-quasihomogeneous element

$$f_{q,w} = \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{n+1} \\ -\frac{\alpha}{N}+\omega\cdot\beta=q}} a_{\alpha,\beta}\cdot t^{\frac{\alpha}{N}}\cdot \underline{x}^{\beta}.$$

If we set

$$p_\beta := \prod_{i=1}^{n}(u_i+x_i)^{\beta_i} - u^\beta \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}]$$

then

$$\begin{aligned}
\gamma_{\omega,u}(f_{q,w}) &= \sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot t^{\frac{\alpha}{N}}\cdot\prod_{i=1}^{n} t^{-\omega_i\cdot\beta_i}\cdot(u_i+x_i)^{\beta_i}\\
&= t^{-q}\cdot\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot\left(u^\beta+p_\beta\right)\\
&= t^{-q}\cdot\left(f_{q,w}(1,u)+\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta\right)\\
&= t^{-q}\cdot f_{q,w}(1,u)+t^{-q}\cdot p_{f_{q,w},u},
\end{aligned} \tag{3.1}$$

with

$$p_{f_{q,w},u} := \sum_{-\frac{\alpha}{N}+w\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}].$$

In particular, if $\omega\in\frac{1}{N}\cdot\mathbb{Z}^n$ and $f=\sum_{q\le\hat{q}} f_{q,w}\in R_N[\underline{x}]$ with $\hat{q}=\mathrm{ord}_\omega(f)$ then

$$\gamma_{\omega,u}(f) = t^{-\hat{q}}\cdot g$$

where

$$g=\sum_{q\le\hat{q}}\left(t^{\hat{q}-q}\cdot f_{q,w}(1,u)+t^{\hat{q}-q}\cdot p_{f_{q,w},u}\right)\in R_N[\underline{x}].$$

□

The following lemma shows that if we consider the transformed ideal $\gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ in the power series ring $K\left[\left[t^{\frac{1}{N}},\underline{x}\right]\right]$ then it defines the germ of a space curve through the origin. This allows us then in Corollary 3.6 to apply normalisation to find a negative-valued point in the tropical variety of $\gamma_{\omega,u}(J)$.

Lemma 3.5
Let $J\lhd L[\underline{x}]$, let $\omega\in\mathrm{Trop}(J)\cap\frac{1}{N}\cdot\mathbb{Z}^n$, and $u\in V\big(\text{t-in}_\omega(J)\big)\subset K^n$. Then

$$\gamma_{\omega,u}(J)\cap R_N[\underline{x}]\subseteq\left\langle t^{\frac{1}{N}},x_1,\ldots,x_n\right\rangle\lhd R_N[\underline{x}].$$

Proof: Let $w=(-1,\omega)$ and $0\neq f=\gamma_{\omega,u}(h)\in\gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ with $h\in J$. Since f is a polynomial in $\underline{x}$ we have

$$h=\gamma_{\omega,u}^{-1}(f)=f(t^{\omega_1}\cdot x_1-u_1,\ldots,t^{\omega_n}\cdot x_n-u_n)\in t^m\cdot R_N[\underline{x}]$$

for some $m\in\frac{1}{N}\cdot\mathbb{Z}$. We can thus decompose $g:=t^{-m}\cdot h\in J_{R_N}$ into its w-quasihomogeneous parts, say

$$t^{-m}\cdot h=g=\sum_{q\le\hat{q}} g_{q,w},$$

where $\hat{q} = \mathrm{ord}_\omega(g)$ and thus $g_{\hat{q},w} = \mathrm{in}_\omega(g)$ is the w-initial form of g. As we have seen in Remark 3.4 there are polynomials $p_{g_{q,w},u} \in \langle x_1, \ldots, x_n\rangle \lhd K[\underline{x}]$ such that

$$\gamma_{\omega,u}(g_{q,w}) = t^{-q} \cdot g_{q,w}(1,u) + t^{-q} \cdot p_{g_{q,w},u}.$$

But then

$$\begin{aligned} f = \gamma_{\omega,u}(h) = \gamma_{\omega,u}(t^m \cdot g) = t^m \cdot \gamma_{\omega,u}(g) = t^m \cdot \gamma_{\omega,u}\left(\sum_{q\leq\hat{q}} g_{q,\omega}\right) \\ = t^m \cdot \sum_{q\leq\hat{q}} \big(t^{-q} \cdot g_{q,w}(1,u) + t^{-q} \cdot p_{g_{q,w},u}\big) \\ = t^{m-\hat{q}} \cdot g_{\hat{q},w}(1,u) + t^{m-\hat{q}} \cdot p_{g_{\hat{q},w},u} + \sum_{q<\hat{q}} t^{m-q} \cdot \big(g_{q,w}(1,u) + p_{g_{q,w},u}\big). \end{aligned}$$

However, since $g \in J$ and $u \in V\big(\text{t-in}_\omega(J)\big)$ we have

$$g_{\hat{q},w}(1,u) = \text{t-in}_\omega(g)(u) = 0$$

and thus using (3.1) we get

$$p_{g_{\hat{q},w},u} = t^{\hat{q}} \cdot \big(\gamma_{\omega,u}(g_{\hat{q},w}) - t^{-\hat{q}} \cdot g_{\hat{q},w}(1,u)\big) = t^{\hat{q}} \cdot \gamma_{\omega,u}(g_{\hat{q},w}) \neq 0,$$

since $g_{\hat{q},w} = \mathrm{in}_\omega(g) \neq 0$ and $\gamma_{\omega,u}$ is an isomorphism. We see in particular, that $m - \hat{q} \geq 0$ since $f \in R_N[\underline{x}]$ and $p_{g_{\hat{q},w},u} \in \langle x_1, \ldots, x_n\rangle \lhd K[\underline{x}]$, and hence

$$f = t^{m-\hat{q}} \cdot p_{g_{\hat{q},w},u} + \sum_{q<\hat{q}} t^{m-q} \cdot \big(g_{q,w}(1,u) + p_{g_{q,w},u}\big) \in \big\langle t^{\frac{1}{N}}, x_1, \ldots, x_n\big\rangle.$$

□

The following corollary assures the existence of a negative-valued point in the tropical variety of the transformed ideal – after possibly eliminating those variables for which the components of the solution will be zero.

Corollary 3.6
Suppose that K is an algebraically closed field of characteristic zero. Let $J \lhd L[\underline{x}]$ be a zero-dimensional ideal, let $\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n$, and $u \in V\big(\text{t-in}_\omega(J)\big) \subset K^n$. Then

$$\exists\, p = (p_1, \ldots, p_n) \in V\big(\gamma_{\omega,u}(J)\big) \;:\; \forall i : \; \mathrm{val}(p_i) \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

In particular, if $n_p = \#\{p_i \mid p_i \neq 0\} > 0$ and $\underline{x}_p = (x_i \mid p_i \neq 0)$, then

$$\mathrm{Trop}\,\big(\gamma_{\omega,u}(J) \cap L[\underline{x}_p]\big) \cap \mathbb{Q}^{n_p}_{<0} \neq \emptyset.$$

Proof: We may choose an $N \in \mathcal{N}(\gamma_{\omega,u}(J))$ and such that $\omega \in \frac{1}{N} \cdot \mathbb{Z}^n_{\leq 0}$. Let $I = \gamma_{\omega,u}(J) \cap R_N[\underline{x}]$.

Since $\gamma_{\omega,u}$ is an isomorphism we know that

$$0 = \dim(J) = \dim\big(\gamma_{\omega,u}(J)\big),$$

and by Proposition 5.3 we know that

$$\mathrm{Ass}(I) = \big\{P_{R_N} \;\big|\; P \in \mathrm{Ass}\,\big(\gamma_{\omega,u}(J)\big)\big\}.$$

Since the maximal ideal

$$\mathfrak{m} = \big\langle t^{\frac{1}{N}}, x_1, \ldots, x_n\big\rangle_{R_N[\underline{x}]} \lhd R_N[\underline{x}]$$

contains the element $t^{\frac{1}{N}}$, which is a unit in $L[\underline{x}]$, it cannot be the contraction of a prime ideal in $L[\underline{x}]$. In particular, $\mathfrak{m} \not\in \mathrm{Ass}(I)$. Thus there must be a $P \in \mathrm{Ass}(I)$ such that $P \subsetneqq \mathfrak{m}$, since by Lemma 3.5 $I \subset \mathfrak{m}$ and since otherwise $\mathfrak{m}$ would be minimal over I and hence associated to I.

The strict inclusion implies that $\dim(P) \geq 1$, while Theorem 6.10 shows that

$$\dim(P) \leq \dim(I) \leq \dim\big(\gamma_{\omega,u}(J)\big) + 1 = 1.$$

Hence the ideal P is a 1-dimensional prime ideal in $R_N[\underline{x}] \subset K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]$, where the latter is the completion of the former with respect to $\mathfrak{m}$. Since $P \subset \mathfrak{m}$, the completion $\hat{P}$ of P with respect to $\mathfrak{m}$ is also 1-dimensional and the normalisation

$$\psi : K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]/\hat{P} \hookrightarrow \widetilde{R}$$

gives a parametrisation where we may assume that $\psi\big(t^{\frac{1}{N}}\big) = s^M$ since K is algebraically closed and of characteristic zero (see e.g. [**DeP00**] Cor. 4.4.10 for $K = \mathbb{C}$). Let now $s_i = \psi(x_i) \in K[[s]]$ then necessarily $a_i = \mathrm{ord}_s(s_i) > 0$, since ψ is a local K-algebra homomorphism, and $f(s^M, s_1, \ldots, s_n) = \psi(f) = 0$ for all $f \in \hat{P}$. Taking $I \subseteq P \subset \hat{P}$ and $\gamma_{\omega,u}(J) = \langle I\rangle$ into account and replacing s by $t^{\frac{1}{N\cdot M}}$ we get

$$f\big(t^{\frac{1}{N}}, p\big) = 0 \quad \text{for all } \; f \in \gamma_{\omega,u}(J)$$

where

$$p = \Big(s_1\big(t^{\frac{1}{N\cdot M}}\big), \ldots, s_n\big(t^{\frac{1}{N\cdot M}}\big)\Big) \in R_{N\cdot M}^n \subseteq L^n.$$

Moreover,

$$\mathrm{val}(p_i) = \frac{a_i}{N \cdot M} \in \mathbb{Q}_{>0} \cup \{\infty\},$$

and every $f \in \pi_{\underline{x}_p} \circ \gamma_{\omega,u}(J)$ vanishes at the point $p' = (p_i \mid p_i \neq 0)$. By Proposition 2.14

$$-\,\mathrm{val}(p') \in \mathrm{Trop}\,\big(\gamma_{\omega,u}(J) \cap L[\underline{x}_p]\big) \cap \mathbb{Q}^n_{<0}.$$

$\square$

Constructive Proof of Theorem 3.1: Recall that by Remark 3.3 we may assume that $\omega \in \mathbb{Q}^n_{<0}$. It is our first aim to construct recursively sequences of the following objects for $\nu \in \mathbb{N}$:

- natural numbers $1 \leq n_\nu \leq n$,
- natural numbers $1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$,
- subsets of variables $\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}})$,
- ideals $J'_\nu \lhd L[\underline{x}_{\nu-1}]$,

- ideals $J_\nu \lhd L[\underline{x}_\nu]$,
- vectors $\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \operatorname{Trop}(J_\nu) \cap (\mathbb{Q}_{<0})^{n_\nu}$, and
- vectors $u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\text{t-in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}$.

We set $n_0 = n$, $\underline{x}_{-1} = \underline{x}_0 = \underline{x}$, $J_0 = J_0' = J$, and $\omega_0 = \omega$, and since $\text{t-in}_\omega(J)$ is monomial free by assumption and K is algebraically closed we may choose a $u_0 \in V\big(\text{t-in}_{\omega_0}(J_0)\big) \cap (K^*)^{n_0}$. We then define recursively for $\nu \geq 1$

$$J_\nu' = \gamma_{\omega_{\nu-1},u_{\nu-1}}(J_{\nu-1}).$$

By Corollary 3.6 we may choose a point $q \in V(J_\nu') \subset L^{n_{\nu-1}}$ such that $\operatorname{val}(q_i) = \operatorname{ord}_t(q_i) > 0$ for all $i = 1, \ldots, n_{\nu-1}$. As in Corollary 3.6 we set

$$n_\nu = \#\{q_i \mid q_i \neq 0\} \in \{0, \ldots, n_{\nu-1}\},$$

and we denote by

$$1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$$

the indexes i such that $q_i \neq 0$.

If $n_\nu = 0$ we simply stop the process, while if $n_\nu \neq 0$ we set

$$\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}}) \subseteq \underline{x}_{\nu-1}.$$

We then set

$$J_\nu = \big(J_\nu' + \langle \underline{x}_{\nu-1} \setminus \underline{x}_\nu \rangle\big) \cap L[\underline{x}_\nu],$$

and by Corollary 3.6 we can choose

$$\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \operatorname{Trop}(J_\nu) \cap \mathbb{Q}^{n_\nu}_{<0}.$$

Then $\text{t-in}_{\omega_\nu}(J_\nu)$ is monomial free, so that we can choose a

$$u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\text{t-in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}.$$

Next we define

$$\varepsilon_i = \sup\big\{\nu \;\big|\; i \in \{i_{\nu,1}, \ldots, i_{\nu,n_\nu}\}\big\} \in \mathbb{N} \cup \{\infty\} \text{ and}$$

$$p_{\mu,i} = \sum_{\nu=0}^{\min\{\varepsilon_i,\mu\}} u_{\nu,i} \cdot t^{-\sum_{j=0}^{\nu} \omega_{j,i}}$$

for $i = 1, \ldots, n$. All $\omega_{\nu,i}$ are strictly negative, which is necessary to see that the $p_{\mu,i}$ converge to a Puiseux series. Note that in the case $n = 1$ the described procedure is just the classical Puiseux expansion (see e.g. [**DeP00**] Thm. 5.1.1 for the case $K = \mathbb{C}$). To see that the $p_{\mu,i}$ converge to a Puiseux series (i.e. that there exists a common denominator N for the exponents as μ goes to infinity), the general case can easily be reduced to the case $n = 1$ by projecting the variety to all coordinate lines, analogously to the proof in section 3 of [**Mau80**]. The ideal of the projection to one coordinate line is principal. Transformation and intersection commute.

It is also easy to see that at $p = (p_1, \ldots, p_n) \in L^n$ all polynomials in J vanish, where

$$p_i = \lim_{\mu \to \infty} p_{\mu,i} = \sum_{\nu=0}^{\infty} u_{\nu,i} \cdot t^{-\sum_{j=0}^{\nu} \omega_{j,i}} \in R_N \subset L.$$

$\square$

Remark 3.7

The proof is basically an algorithm which allows to compute a point $p \in V(J)$ such that $\mathrm{val}(p) = -\omega$. However, if we want to use a computer algebra system like SINGULAR for the computations, then we have to restrict to generators of J which are polynomials in $t^{\frac{1}{N}}$ as well as in $\underline{x}$. Moreover, we should pass from $t^{\frac{1}{N}}$ to t, which can be easily done by the K-algebra isomorphism

$$\Psi_N : L[\underline{x}] \longrightarrow L[\underline{x}] : t \mapsto t^N, x_i \mapsto x_i.$$

Whenever we do a transformation which involves rational exponents we will clear the denominators using this map with an appropriate N.

We will in the course of the algorithm have to compute the t-initial ideal of J with respect to some $\omega \in \mathbb{Q}^n$, and we will do so by a standard basis computation using the monomial ordering $>_\omega$, given by

$$t^\alpha \cdot \underline{x}^\beta >_\omega t^{\alpha'} \cdot \underline{x}^{\beta'} \iff$$
$$-\alpha + \omega \cdot \beta > -\alpha' + \omega \cdot \beta' \text{ or } (-\alpha + \omega \cdot \beta = -\alpha' + \omega \cdot \beta' \text{ and } \underline{x}^\beta > \underline{x}^{\beta'}),$$

where $>$ is some fixed global monomial ordering on the monomials in $\underline{x}$.

Algorithm 3.8 (ZDL – Zero Dimensional Lifting Algorithm)

INPUT: $(m, f_1, \ldots, f_k, \omega) \in \mathbb{N}_> \times K[t, \underline{x}]^k \times \mathbb{Q}^n$ such that $\dim(J) = 0$ and $\omega \in \mathrm{Trop}(J)$ for $J = \langle f_1, \ldots, f_k \rangle_{L[\underline{x}]}$.

OUTPUT: $(N, p) \in \mathbb{N} \times K[t, t^{-1}]^n$ such that $p(t^{\frac{1}{N}})$ coincides with the first m terms of a solution of $V(J)$ and such that $\mathrm{val}(p) = -\omega$.

INSTRUCTIONS:

- Choose $N \geq 1$ such that $N \cdot \omega \in \mathbb{Z}^n$.
- FOR $i = 1, \ldots, k$ DO $f_i := \Psi_N(f_i)$.
- $\omega := N \cdot \omega$
- IF some $\omega_i > 0$ THEN
  - FOR $i = 1, \ldots, k$ DO $f_i := \Phi_\omega(f_i) \cdot t^{-\mathrm{ord}_t\left(\Phi_\omega(f_i)\right)}$.
  - $\tilde{\omega} := \omega$.
  - $\omega := (0, \ldots, 0)$.
- Compute a standard basis $(g_1, \ldots, g_l)$ of $\langle f_1, \ldots, f_k \rangle_{K[t,\underline{x}]}$ with respect to the ordering $>_\omega$.
- Compute a zero $u \in (K^*)^n$ of $\langle \text{t-in}_\omega(g_1), \ldots, \text{t-in}_\omega(g_l) \rangle_{K[\underline{x}]}$.
- IF $m = 1$ THEN $(N, p) := \left(N, u_1 \cdot t^{-\omega_1}, \ldots, u_n \cdot t^{-\omega_n}\right)$.

- ELSE
  - Set $G = \big(\gamma_{\omega,u}(f_i) \;\big|\; i = 1, \ldots, k\big)$.
  - FOR $i = 1, \ldots, n$ DO
    - Compute a generating set G' of $\langle G, x_i\rangle_{K[t,\underline{x}]} : \langle t\rangle^\infty$.
    - IF $G' \subseteq \langle t, \underline{x}\rangle$ THEN
      - $\underline{x} := \underline{x} \setminus \{x_i\}$
      - Replace G by a generating set of $\langle G'\rangle \cap K[t, \underline{x}]$.
  - IF $\underline{x} = \emptyset$ THEN $(N, p) := \big(N, u_1 \cdot t^{-\omega_1}, \ldots, u_n \cdot t^{-\omega_n}\big)$.
  - ELSE
    - Compute a point ω' in the negative orthant of the tropical variety of $\langle G\rangle_{L[\underline{x}]}$.
    - $(N', p') = ZDL(m - 1, G, \omega')$.
    - $N := N \cdot N'$.
    - FOR $j = 1, \ldots, n$ DO
      - IF $x_i \in \underline{x}$ THEN $p_i := t^{-\omega_i \cdot N'} \cdot (u_i + p'_i)$.
      - ELSE $p_i := t^{-\omega_i \cdot N'} \cdot u_i$.
- IF some $\tilde{\omega}_i > 0$ THEN $p := \big(t^{-\tilde{\omega}_1} \cdot p_1, \ldots, t^{-\tilde{\omega}_n} \cdot p_n\big)$.

Proof: The algorithm which we describe here is basically one recursion step in the constructive proof of Theorem 3.1 given above, and thus the correctness follows once we have justified why our computations do what is required by the recursion step.

If we compute a standard basis $(g_1, \ldots, g_l)$ of $\langle f_1, \ldots, f_k\rangle_{K[t,\underline{x}]}$ with respect to $>_\omega$, then by Theorem 2.8 the t-initial forms of the g_i generate the t-initial ideal of $J = \langle f_1, \ldots, f_k\rangle_{L[\underline{x}]}$. We thus compute a zero u of the t-initial ideal as required.

Next the recursion in the proof of Theorem 3.1 requires to find an $\omega \in \big(\mathbb{Q}_{>0} \cup \{\infty\}\big)^n$, which is $-\operatorname{val}(q)$ for some $q \in V(J)$, and we have to eliminate those components which are zero. Note that the solutions with first component zero are the solutions of $J + \langle x_1\rangle$. Checking if there is a solution with strictly positive valuation amounts by the proof of Corollary 3.6 to checking if $\big(J + \langle x_1\rangle\big) \cap K[[t]][\underline{x}] \subseteq \langle t, \underline{x}\rangle$, and the latter is equivalent to $G' \subseteq \langle t, \underline{x}\rangle$ by Lemma 3.9. If so, we eliminate the variable x_1 from $\langle G'\rangle_{K[t,\underline{x}]}$, which amounts to projecting all solutions with first component zero to L^{n-1}. We then continue with the remaining variables. That way we find a set of variables $\{x_{i_1}, \ldots, x_{i_s}\}$ such that there is a solution of $V(J)$ with strictly positive valuation where precisely the other components are zero.

The rest follows from the constructive proof of Theorem 3.1. $\square$

Lemma 3.9

Let $f_1, \ldots, f_k \in K[t, \underline{x}]$, $J = \langle f_1, \ldots, f_k\rangle_{L[\underline{x}]}$, $I = \langle f_1, \ldots, f_k\rangle_{K[t,\underline{x}]} : \langle t\rangle^\infty$*, and let* G

be a generating set of I*. Then:*

$$J \cap K[[t]][\underline{x}] \subseteq \langle t, \underline{x}\rangle \iff I \subseteq \langle t, \underline{x}\rangle \iff G \subseteq \langle t, \underline{x}\rangle.$$

Proof: The last equivalence is clear since I is generated by G, and for the first equivalence it suffices to show that $J \cap K[[t]][\underline{x}] = \langle I\rangle_{K[[t]][\underline{x}]}$.

For this let us consider the following two ideals $I' = \langle f_1, \ldots, f_k\rangle_{K[[t]][\underline{x}]} : \langle t\rangle^\infty$ and $I'' = \langle f_1, \ldots, f_k\rangle_{K[t]_{\langle t\rangle}[\underline{x}]} : \langle t\rangle^\infty$. By Lemma 6.6 we know that $J \cap K[[t]][\underline{x}] = I'$ and by **[Mar07]** Prop. 6.20 we know that $I' = \langle I''\rangle_{K[[t]][\underline{x}]}$. It thus suffice to show that $I'' = \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. Obviously $I \subseteq I''$, which proves one inclusion. Conversely, if $f \in I''$ then f satisfies a relation of the form

$$t^m \cdot f \cdot u = \sum_{i=1}^{k} g_i \cdot f_i,$$

with $m \geq 0$, $u \in K[t]$, $u(0) = 1$ and $g_1, \ldots, g_k \in K[t, \underline{x}]$. Thus $f \cdot u \in I$ and $f = \frac{f\cdot u}{u} \in \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. $\square$

Remark 3.10
In order to compute the point ω' we may want to compute the tropical variety of $\langle G\rangle_{L[\underline{x}]}$. The tropical variety can be computed as a subcomplex of a Gröbner fan or more efficiently by applying Algorithm 5 in **[BJS⁺07]** for computing tropical bases of tropical curves.

Remark 3.11
We have implemented the above algorithm in the computer algebra system SINGULAR (see **[GPS05]**) since nearly all of the necessary computations are reduced to standard basis computations over $K[t, \underline{x}]$ with respect to certain monomial orderings. In SINGULAR however we do not have an algebraically closed field K over which we can compute the zero u of an ideal. We get around this by first computing the absolute minimal associated primes of $\langle \text{t-in}_\omega(g_1), \ldots, \text{t-in}_\omega(g_k)\rangle_{K[t,\underline{x}]}$ all of which are maximal by Corollary 6.15, using the absolute primary decomposition in SINGULAR. Choosing one of these maximal ideals we only have to adjoin one new variable, say a, to realise the field extension over which the zero lives, and the minimal polynomial, say m, for this field extension is provided by the absolute primary decomposition. In subsequent steps we might have to enlarge the minimal polynomial, but we can always get away with only one new variable.

The field extension should be the coefficient field of our polynomial ring in subsequent computations. Unfortunately, the program `gfan` which we use in order to compute tropical varieties does not handle field extensions. (It would not be a problem to actually implement field extensions — we would not have to come up with new algorithms.) But we will see in Lemma 3.12 that we can get away with computing tropical varieties of ideals in the polynomial ring over the extension field of K by computing just over K. More precisely, we want to compute a negative-valued point ω' in the tropical variety of a transformed ideal $\gamma_{\omega,u}(J)$. Instead, we compute a point $(\omega',0)$ in the tropical variety of the ideal $\gamma_{\omega,u}(J)+\langle m\rangle$. So to justify this it is enough to show that ω is in the tropical variety of an ideal $J \trianglelefteq K[a]/\langle m\rangle\{\{t\}\}[\underline{x}]$ if and only if $(\omega,0)$ is in the tropical variety of the ideal $J+\langle m\rangle \trianglelefteq K\{\{t\}\}[\underline{x},a]$. Recall that $\omega \in \mathrm{Trop}(J)$ if and only if $\text{t-in}_\omega(J)$ contains no monomial, and by Theorem 2.8, $\text{t-in}_\omega(J)$ is equal to $\text{t-in}_\omega(J_{R_N})$, where $N \in \mathcal{N}(J)$.

Lemma 3.12
Let $m \in K[a]$ be an irreducible polynomial, let $\varphi : k[t^{\frac{1}{N}},\underline{x},a] \to (k[a]/\langle m\rangle)[t^{\frac{1}{N}},\underline{x}]$ take elements to their classes, and let $I \trianglelefteq (k[a]/\langle m\rangle)[t^{\frac{1}{N}},\underline{x}]$. Then $\mathrm{in}_\omega(I)$ contains no monomial if and only if $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains no monomial. In particular, the same holds for $\text{t-in}_\omega(I)$ and $\text{t-in}_{(\omega,0)}(\varphi^{-1}(I))$.

Proof: Suppose $\mathrm{in}_{(\omega,0)}\,\varphi^{-1}(I)$ contains a monomial. Then there exists an $f \in \varphi^{-1}(I)$ such that $\mathrm{in}_{(\omega,0)}(f)$ is a monomial. The polynomial $\varphi(f)$ is in I. When applying φ the monomial $\mathrm{in}_{(\omega,0)}(f)$ maps to a monomial whose coefficient in $k[a]/\langle m\rangle$ has a representative $h \in k[a]$ with just one term. The representative h cannot be 0 modulo $\langle m\rangle$ since $\langle m\rangle$ does not contain a monomial. Thus $\varphi\big(\mathrm{in}_{(\omega,0)(f)}\big) = \mathrm{in}_\omega(\varphi(f))$ is a monomial.

For the other direction, suppose $\mathrm{in}_\omega(I)$ contains a monomial. We must show that $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains a monomial. This is equivalent to showing that $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : ((t^{\frac{1}{N}}\cdot x_1\cdots x_n)^\infty)$ contains a monomial. By assumption there exists an $f \in I$ such that $\mathrm{in}_\omega(f)$ is a monomial. Let g be in $\varphi^{-1}(I)$ such that g maps to f under the surjection φ and with the further condition that the support of g projected to the $(t^{\frac{1}{N}},\underline{x})$-coordinates equals the support of f. The initial form $\mathrm{in}_{(\omega,0)}(g)$ is a polynomial with all exponent vectors having the same $(t^{\frac{1}{N}},\underline{x})$ parts as $\mathrm{in}_\omega(f)$ does. Let g' be $\mathrm{in}_{(\omega,0)}(g)$ with the common $(t^{\frac{1}{N}},\underline{x})$-part removed from the monomials, that is $g' \in k[a]$. Notice that $\varphi(g') \neq 0$. We now have $g' \notin \langle m\rangle$ and hence $\langle g',m\rangle = k[a]$ since $\langle m\rangle$ is maximal. Now m and g' are contained in $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : (t^{\frac{1}{N}}\cdot x_1\cdots x_n)^\infty)$, implying that $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : (t^{\frac{1}{N}}\cdot x_1\cdots x_n)^\infty) \supseteq k[a]$. This shows that $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains a monomial. □

Remark 3.13
In Algorithm 3.8 we choose zeros of the t-initial ideal and we choose points in the negative quadrant of the tropical variety. If we instead do the same computations for all zeros and points of the negative quadrant of the tropical variety, then we get Puiseux expansions of all branches of the space curve germ defined by the ideal $\langle f_1,\ldots,f_k\rangle_{K[[t,\underline{x}]]}$ in $(K^{n+1},0)$.

4. Reduction to the Zero Dimensional Case

In this section, we want to give a proof of the lifting Theorem 3.1 for any ideal J of dimension $\dim J = d > 0$, using our algorithm for the zero-dimensional case.

Given $\omega \in \mathrm{Trop}(J)$ we would like to intersect $\mathrm{Trop}(J)$ with another tropical variety $\mathrm{Trop}(J')$ containing ω, such that $\dim(J + J') = 0$ and apply the zero-dimensional algorithm to $J + J'$. However, we cannot conclude that $\omega \in \mathrm{Trop}(J + J')$ — we have $\mathrm{Trop}(J + J') \subseteq \mathrm{Trop}(J) \cap \mathrm{Trop}(J')$ but equality does not need to hold. For example, two plane tropical lines (given by two linear forms) which are not equal can intersect in a ray, even though the ideal generated by the two linear forms defines just a point.

So we have to find an ideal J' such that $J + J'$ is zero-dimensional and still $\omega \in \mathrm{Trop}(J + J')$ (see Proposition 4.6). We will use some ideas of **[Kat06]** Lemma 4.4.3 — the ideal J' will be generated by $\dim(J)$ sufficiently general linear forms. The proof of the proposition needs some technical preparations.

Notation 4.1
We denote by

$$V_\omega = \{a_0 + a_1 \cdot t^{\omega_1} \cdot x_1 + \ldots + a_n \cdot t^{\omega_n} \cdot x_n \mid a_i \in K\}$$

the $n+1$-dimensional K-vector space of *linear* polynomials over K, which in a sense are *scaled* by $\omega \in \mathbb{Q}^n$. Of most interest will be the case where $\omega = 0$.

The following lemma geometrically says that an affine variety of dimension at least one will intersect a generic hyperplane.

Lemma 4.2
Let K be an infinite field and $J \lhd L[\underline{x}]$ an equidimensional ideal of dimension $\dim(J) \geq 1$. Then there is a Zariski open dense subset U of V_0 such that $\langle f\rangle + Q \neq L[\underline{x}]$ for all $f \in U$ and $Q \in \mathrm{minAss}(J)$.

If V is an affine variety which meets $(K^*)^n$ in dimension at least 1, then a generic hyperplane section of V meets $(K^*)^n$ as well. The algebraic formulation of this geometric fact is the following lemma:

Lemma 4.3
Let K be an infinite field and $I \lhd K[\underline{x}]$ be an equidimensional ideal with $\dim(I) \geq 1$ and such that $x_1 \cdots x_n \notin \sqrt{I}$, then there is a Zariski open subset U of V_0 such that $x_1 \cdots x_n \notin \sqrt{I + \langle f\rangle}$ for $f \in U$.

The following lemma is an algebraic formulation of the geometric fact that given any affine variety none of its components will be contained in a generic hyperplane.

Lemma 4.4
Let K be an infinite field, let R be a ring containing K, and let $J \trianglelefteq R[\underline{x}]$ be

an ideal. Then there is a Zariski open dense subset U of V_0 such that $f \in U$ satisfies $f \notin P$ for $P \in \operatorname{minAss}(J)$.

Remark 4.5

If $\#K < \infty$ we can still find a suitable $f \in K[\underline{x}]$ which satisfies the conditions in Lemma 4.2, Lemma 4.3 and Lemma 4.4 due to Prime Avoidance. However, it may not be possible to choose a linear one.

With these preparations we can show that we can reduce to the zero dimensional case by cutting with generic hyperplanes.

Proposition 4.6

Suppose that K is an infinite field, and let $J \lhd L[\underline{x}]$ be an equidimensional ideal of dimension d and $\omega \in \operatorname{Trop}(J)$.

Then there exist Zariski open dense subsets $U_1, \ldots, U_d$ of V_ω such that $(f_1, \ldots, f_d) \in U_1 \times \ldots \times U_d$ and $J' = \langle f_1, \ldots, f_d \rangle_{L[\underline{x}]}$ satisfy:

- $\dim(J + J') = \dim\big(\text{t-in}_\omega(J) + \text{t-in}_\omega(J')\big) = 0$,
- $\dim\big(\text{t-in}_\omega(J')\big) = \dim(J') = n - d$,
- $x_1 \cdots x_n \notin \sqrt{\text{t-in}_\omega(J) + \text{t-in}_\omega(J')}$, *and*
- $\sqrt{\text{t-in}_\omega(J) + \text{t-in}_\omega(J')} = \sqrt{\text{t-in}_\omega(J + J')}$.

In particular, $\omega \in \operatorname{Trop}(J + J')$.

Proof: Applying Φ_ω to J first and then applying $\Phi_{-\omega}$ to J' later we may assume that $\omega = 0$. Moreover, we may choose an N such that $N \in \mathcal{N}(J)$ and $N \in \mathcal{N}(P)$ for all $P \in \operatorname{minAss}(J)$. By Lemma 6.7 then also $\text{t-in}_0(J) = \text{t-in}_0(J_{R_N})$ and $\text{t-in}_0(P) = \text{t-in}_0(P_{R_N})$ for $P \in \operatorname{minAss}(J)$.

By Lemma 6.16

$$\operatorname{minAss}(J_{R_N}) = \{P_{R_N} \mid P \in \operatorname{minAss}(J)\}. \tag{4.1}$$

In particular, all minimal associated primes P_{R_N} of J_{R_N} have codimension $n-d$ by Corollary 6.9.

Since $0 \in \operatorname{Trop}(J)$ there exists a $P \in \operatorname{minAss}(J)$ with $0 \in \operatorname{Trop}(P)$ by Lemma 2.12. Hence $1 \notin \text{t-in}_0(P)$ and we conclude by Corollary 6.17 that

$$\dim(J) = \dim\big(\text{t-in}_0(J)\big) = \dim(Q) \tag{4.2}$$

for all $Q \in \operatorname{minAss}\big(\text{t-in}_0(J)\big)$. In particular, all minimal associated prime ideals of $\text{t-in}_0(J)$ have codimension $n - d$.

Moreover, since $0 \in \operatorname{Trop}(J)$ we know that $\text{t-in}_0(J)$ is monomial free, and in particular

$$x_1 \cdots x_n \notin \sqrt{\text{t-in}_0(J)}. \tag{4.3}$$

If $d = 0$ then $J' = \langle \emptyset \rangle = \{0\}$ works due to (4.2) and (4.3). We may thus assume that $d > 0$.

Since K is infinite we can apply Lemma 4.2 to J, Lemma 4.4 to $J \lhd L[\underline{x}]$, to $J_{R_N} \lhd R_N[\underline{x}]$ and to $\text{t-in}_0(J) \lhd K[\underline{x}]$ and Lemma 4.3 to $\text{t-in}_0(J) \lhd K[\underline{x}]$ (take (4.3) into account), and thus there exist Zariski open dense subsets U, U', U'', U''' and U'''' in V_0 such that no $f_1 \in U_1 = U \cap U' \cap U'' \cap U''' \cap U''''$ is contained in any minimal associated prime of either J, J_{R_N} or $\text{t-in}_0(J)$, such that $1 \not\in J + \langle f_1 \rangle_{L[\underline{x}]}$ and such that $x_1 \cdots x_n \not\in \sqrt{\text{t-in}_0(J) + \langle f_1 \rangle}$. Since the intersection of four Zariski open and dense subsets is non-empty, there is such an f_1 and by Lemma 5.6 the minimal associated primes of the ideals $J + \langle f_1 \rangle_{L[\underline{x}]}$, $J_{R_N} + \langle f_1 \rangle_{R_N[\underline{x}]}$, and $\text{t-in}_0(J) + \langle f_1 \rangle_{K[\underline{x}]}$ all have the same codimension $n - d + 1$.

We claim that $t^{\frac{1}{N}} \not\in Q$ for any $Q \in \text{minAss}(J_{R_N} + \langle f_1 \rangle_{R_N[\underline{x}]})$. Suppose the contrary, then by Lemma 6.8 (b), (f) and (g)

$$\dim(Q) = n + 1 - \text{codim}(Q) = d.$$

Consider now the residue class map

$$\pi : R_N[\underline{x}] \longrightarrow R_N[\underline{x}]/\big\langle t^{\frac{1}{N}} \big\rangle = K[\underline{x}].$$

Then $\text{t-in}_0(J) = \pi\big(J_{R_N} + \big\langle t^{\frac{1}{N}} \big\rangle\big)$, and we have

$$\text{t-in}_0(J) + \langle f_1 \rangle_{K[\underline{x}]} \subseteq \pi\big(J_{R_N} + \langle t^{\frac{1}{N}}, f_1 \rangle_{R_N[\underline{x}]}\big) \subseteq \pi(Q).$$

Since $t^{\frac{1}{N}} \in Q$ the latter is again a prime ideal of dimension d. However, due to the choice of f_1 we know that every minimal associated prime of $\text{t-in}_0(J) + \langle f_1 \rangle_{K[\underline{x}]}$ has codimension $n - d + 1$ and hence the ideal itself has dimension $d - 1$. But then it cannot be contained in an ideal of dimension d.

Applying the same arguments another $d - 1$ times we find Zariski open dense subsets $U_2, \ldots, U_d$ of V_0 such that for all $(f_1, \ldots, f_d) \in U_1 \times \cdots \times U_d$ the minimal associated primes of the ideals

$$J + \langle f_1, \ldots, f_k \rangle_{L[\underline{x}]}$$

respectively

$$J_{R_N} + \langle f_1, \ldots, f_k \rangle_{R_N[\underline{x}]}$$

respectively

$$\text{t-in}_0(J) + \langle f_1, \ldots, f_k \rangle_{K[\underline{x}]}$$

all have codimension $n - d + k$ for each $k = 1, \ldots, d$, such that $1 \not\in J + \langle f_1, \ldots, f_k \rangle_{L[\underline{x}]}$, and such that

$$x_1 \cdots x_n \not\in \sqrt{\text{t-in}_0(J) + \langle f_1, \ldots, f_k \rangle_{K[\underline{x}]}}.$$

Moreover, none of the minimal associated primes of $J_{R_N} + \langle f_1, \ldots, f_k \rangle_{R_N[\underline{x}]}$ contains $t^{\frac{1}{N}}$.

In particular, since $f_i \in K[\underline{x}]$ we have (see Theorem 2.8)

$$\text{t-in}_0(J') = \text{t-in}_0\left(\langle f_1, \ldots, f_d \rangle_{K[t,\underline{x}]}\right) = \langle f_1, \ldots, f_d \rangle_{K[\underline{x}]},$$

and J' obviously satisfies the first three requirements of the proposition.

For the fourth requirement it suffices to show

$$\operatorname{minAss}\big(\text{t-in}_0(J)+\text{t-in}_0(J')\big)=\operatorname{minAss}\big(\text{t-in}_0(J+J')\big).$$

For this consider the ring extension

$$R_N[\underline{x}]\subseteq S_N^{-1}R_N[\underline{x}]=L_N[\underline{x}]$$

given by localisation and denote by $I^c=I\cap R_N[\underline{x}]$ the contraction of an ideal I in $L_N[\underline{x}]$ and by $I^e=\langle I\rangle_{L_N[\underline{x}]}$ the extension of an ideal I in $R_N[\underline{x}]$. Moreover, we set $J_0=J\cap L_N[\underline{x}]$ and $J_0'=J'\cap L_N[\underline{x}]$, so that $J_0^c=J_{R_N}$ and $J_0'^c=\langle f_1,\ldots,f_d\rangle_{R_N[\underline{x}]}$. Note then first that

$$(J_0^c+J_0'^c)^e=J_0^{ce}+J_0'^{ce}=J_0+J_0',$$

and therefore by the correspondence of primary decomposition under localisation (see [**AtM69**] Prop. 4.9)

$$\operatorname{minAss}\big((J_0+J_0')^c\big)=\big\{Q\in\operatorname{minAss}(J_0^c+J_0'^c)\;\big|\;t^{\frac{1}{N}}\not\in Q\big\}=\operatorname{minAss}\big(J_0^c+J_0'^c\big).$$

This then shows that

$$\sqrt{J_0^c+J_0'^c}=\sqrt{(J_0+J_0')^c},$$

and since $\pi(J_0^c)=\text{t-in}_0(J_{R_N})=\text{t-in}_0(J)$, $\pi(J_0'^c)=\text{t-in}_0(J')$ and $\pi\big((J_0+J_0')^c\big)=\text{t-in}_0(J+J')$ we get

$$\begin{aligned}\sqrt{\text{t-in}_0(J)+\text{t-in}_0(J')}&=\sqrt{\pi(J_0^c)+\pi(J_0'^c)}=\pi\left(\sqrt{J_0^c+J_0'^c}\right)\\&=\pi\left(\sqrt{(J_0+J_0')^c}\right)=\sqrt{\pi\big((J_0+J_0')^c\big)}=\sqrt{\text{t-in}_0(J+J')}.\end{aligned}$$

It remains to show the "in particular" part. However, since

$$x_1\cdots x_n\not\in\sqrt{\text{t-in}_\omega(J)+\text{t-in}_\omega(J')}=\sqrt{\text{t-in}_\omega(J+J')},$$

the ideal $\text{t-in}_\omega(J+J')$ is monomial free, or equivalently $\omega\in\operatorname{Trop}(J+J')$. □

Remark 4.7
Proposition 4.6 shows that the ideal J' can be found by choosing d linear forms $f_j=\sum_{i=1}^n a_{ji}\cdot t^{\omega_i}\cdot x_i+a_{j0}$ with random $a_{ji}\in K$, and we only need that K is infinite.

We are now in the position to finish the proof of Theorem 2.13.

Proof of Theorem 2.13: If $\omega\in\operatorname{Trop}(J)\cap\mathbb{Q}^n$ then there is a minimal associated prime ideal $P\in\operatorname{minAss}(J)$ such that $\omega\in\operatorname{Trop}(P)$ by Lemma 2.12. By assumption the field K is algebraically closed and therefore infinite, so that Proposition 4.6 applied to P shows that we can choose an ideal P' such that $\omega\in\operatorname{Trop}(P+P')$ and $\dim(P+P')=0$. By Theorem 3.1 there exists a point $p\in V(P+P')\subseteq V(J)$ such that $\operatorname{val}(p)=-\omega$. This finishes the proof in view of Proposition 2.14. □

Algorithm 4.8 (RDZ - Reduction to Dimension Zero)
INPUT: a prime ideal $P \in K(t)[\underline{x}]$ and $\omega \in \mathrm{Trop}(P)$.

OUTPUT: an ideal J such that $\dim(J) = 0$, $P \subset J$ and $\omega \in \mathrm{Trop}(J)$.

INSTRUCTIONS:

- $d := \dim(P)$
- $J := P$
- WHILE $\dim(J) \neq 0$ OR $\text{t-in}_\omega(J)$ not monomial-free DO
  - FOR $j = 0$ TO d pick random values $a_{0,j}, \ldots, a_{n,j} \in K$, and define $f_j := a_{0,j} + \sum a_{i,j} \cdot t^{\omega_i} x_i$.
  - $J := P + \langle f_1, \ldots, f_d \rangle$

Proof: We only have to show that the random choices will lead to a suitable ideal J with probability 1. To see this, we want to apply Proposition 4.6. For this we only have to see that $P^e = \langle P \rangle_{L[\underline{x}]}$ is equidimensional of dimension $d = \dim(P)$. By **[Mar07]** Corollary 6.13 the intersection of P^e with $K(t)[\underline{x}]$, P^{ec}, is equal to P. Using Proposition 5.3 we see that

$$\{P\} = \mathrm{minAss}(P^{ec}) \subseteq \{Q^c \mid Q \in \mathrm{minAss}(P^e)\} \subseteq \mathrm{Ass}(P^{ec}) = \{P\}.$$

By Lemma 5.4 we have $\dim Q = \dim(P) = d$ for every $Q \in \mathrm{minAss}(P^e)$, hence P^e is equidimensional of dimension d. $\square$

Remark 4.9
Note that we cannot perform primary decomposition over $L[\underline{x}]$ computationally. Given a d-dimensional ideal J and $\omega \in \mathrm{Trop}(J)$ in our implementation of the lifting algorithm, we perform primary decomposition over $K(t)[\underline{x}]$. By Lemma 2.12, there must be a minimal associated prime P of J such that $\omega \in \mathrm{Trop}(P)$. Its restriction to $K(t)[\underline{x}]$ is one of the minimal associated primes that we computed, and this prime is our input for algorithm 4.8.

Example 4.10
Assume $P = \langle x+y+t \rangle \trianglelefteq L[x,y]$, and $\omega = (-1,-2)$. Choose coefficients randomly and add for example the linear form $f = -2xt^{-1} + 2t^{-2}y - 1$. Then $J = \langle x+y+t, f \rangle$ has dimension 0 and ω is contained in $\mathrm{Trop}(J)$. Note that the intersection of $\mathrm{Trop}(P)$ with $\mathrm{Trop}(f)$ is not transversal, as the vertex of the tropical line $\mathrm{Trop}(f)$ is at ω.

5. Some Commutative Algebra

In this section we gather some simple results from commutative algebra for the lack of a better reference. They are primarily concerned with the dimension of an ideal under contraction respectively extension for certain ring extensions. The results in this section are independent of the previous sections

Notation 5.1
In this section we denote by $I^e = \langle I\rangle_{R'}$ the extension of $I \trianglelefteq R$ and by $J^c = \varphi^{-1}(J)$ the contraction of $J \trianglelefteq R'$, where $\varphi : R \to R'$ is a ring extension. If no ambiguity can arise we will not explicitly state the ring extension.

We first want to understand how primary decomposition behaves under restriction. The following lemma is an easy consequence of the definitions.

Lemma 5.2
If $\varphi : R \to R'$ is any ring extension and $Q \lhd R'$ a P-primary ideal, then Q^c is P^c-primary.

Proposition 5.3
Let $\varphi : R \to R'$ be any ring extension, let $J \trianglelefteq R'$ be an ideal such that $(J^c)^e = J$, and let $J = Q_1 \cap \ldots \cap Q_k$ be a minimal primary decomposition. Then

$$\mathrm{Ass}(J^c) = \{P^c \mid P \in \mathrm{Ass}(J)\} = \left\{\sqrt{Q_i}^c \;\middle|\; i = 1, \ldots, k\right\},$$

and $J^c = \bigcap_{P \in \mathrm{Ass}(J^c)} Q_P$ is a minimal primary decomposition, where

$$Q_P = \bigcap_{\sqrt{Q_i}^c = P} Q_i^c.$$

Moreover, we have $\mathrm{minAss}(J^c) \subseteq \{P^c \mid P \in \mathrm{minAss}(J)\}$.

Note that the $\sqrt{Q_i}^c$ are not necessarily pairwise different, and thus the cardinality of $\mathrm{Ass}(J^c)$ *may be strictly smaller than k.*

Proof: Let $\mathcal{P} = \left\{\sqrt{Q_i}^c \mid i = 1, \ldots, k\right\}$ and let Q_P be defined as above for $P \in \mathcal{P}$. Since contraction commutes with intersection we have

$$J^c = \bigcap_{P \in \mathcal{P}} Q_P. \tag{5.1}$$

By Lemma 5.2 the Q_i^c with $P = \sqrt{Q_i}^c$ are P-primary, and thus so is their intersection, so that (5.1) is a primary decomposition. Moreover, by construction the radicals of the Q_P are pairwise different. It thus remains to show that none of the Q_P is superfluous. Suppose that there is a $P = \sqrt{Q_i}^c \in \mathcal{P}$ such that

$$J^c = \bigcap_{P' \in \mathcal{P}\setminus\{P\}} Q_{P'} \subseteq \bigcap_{j \neq i} Q_j^c,$$

then

$$J = (J^c)^e \subseteq \bigcap_{j \neq i} (Q_j^c)^e \subseteq \bigcap_{j \neq i} Q_j$$

in contradiction to the minimality of the given primary decomposition of J. This shows that (5.1) is a minimal primary decomposition and that $\mathrm{Ass}(J^c) = \mathcal{P}$.

Finally, if $P \in \mathrm{Ass}(J)$ such that P^c is minimal over J^c then necessarily there is a $\tilde{P} \in \mathrm{minAss}(J)$ such that $P^c = \tilde{P}^c$.

□

We will use this result to show that dimension behaves well under extension for polynomial rings over a field extension.

Lemma 5.4
If $F \subseteq F'$ is a field extension, $I \trianglelefteq F[\underline{x}]$ is an ideal and $I^e = \langle I \rangle_{F'[\underline{x}]}$ then

$$\dim(I^e) = \dim(I).$$

Moreover, if I is prime then $\dim(P) = \dim(I)$ *for all* $P \in \operatorname{minAss}(I^e)$.

Proof: Choose any global degree ordering $>$ on the monomials in $\underline{x}$ and compute a standard basis G' of I with respect to $>$. Then G' is also a standard basis of I^e by Buchberger's Criterion. If M is the set of leading monomials of elements of G' with respect to $>$, then the dimension of the ideal generated by M does not depend on the base field but only on M (see e.g. **[GrP02]** Prop. 3.5.8). Thus we have (see e.g. **[GrP02]** Cor. 5.3.14)

$$\dim(I) = \dim\big(\langle M \rangle_{F[\underline{x}]}\big) = \dim\big(\langle M \rangle_{F'[\underline{x}]}\big) = \dim(I^e). \tag{5.2}$$

Let now I be prime. It remains to show that I^e is equidimensional.

If we choose a maximal independent set $\underline{x}' \subseteq \underline{x}$ of $L_>(I^e) = \langle M \rangle_{F'[\underline{x}]}$ then by definition (see **[GrP02]** Def. 3.5.3) $\langle M \rangle \cap F'[\underline{x}'] = \{0\}$, so that necessarily $\langle M \rangle_{F[\underline{x}]} \cap F[\underline{x}'] = \{0\}$. This shows that $\underline{x}'$ is an independent set of $L_>(I) = \langle M \rangle_{F[\underline{x}]}$, and it is maximal since its size is $\dim(I^e) = \dim(I)$ by (5.2). Moreover, by **[GrP02]** Ex. 3.5.1 $\underline{x}'$ is a maximal independent set of both I and I^e. Choose now a global monomial ordering $>'$ on the monomials in $\underline{x}'' = \underline{x} \setminus \underline{x}'$.

We claim that if $G = \{g_1, \ldots, g_k\} \subset F[\underline{x}]$ is a standard basis of $\langle I \rangle_{F(\underline{x}')[\underline{x}'']}$ with respect to $>'$ and if $0 \neq h = \operatorname{lcm}\big(\operatorname{lc}_{>'}(g_1), \ldots, \operatorname{lc}_{>'}(g_k)\big) \in F[\underline{x}']$, then $I^e : \langle h \rangle^\infty = I^e$. For this we consider a minimal primary decomposition $I^e = Q_1 \cap \ldots \cap Q_l$ of I^e. Since $I^{ece} = I^e$ we may apply Proposition 5.3 to get

$$\Big\{ \sqrt{Q_i}^{\,c} \;\Big|\; i = 1, \ldots, l \Big\} = \operatorname{Ass}(I^{ec}) = \{I\}, \tag{5.3}$$

where the latter equality is due to $I^{ec} = I$ (see e.g. **[Mar07]** Cor. 6.13) and to I being prime. Since $\underline{x}'$ is an independent set of I we know that $h \notin I$ and thus (5.3) shows that $h^m \notin \sqrt{Q_i}$ for any $i = 1, \ldots, l$ and any $m \in \mathbb{N}$. Let now $f \in I^e : \langle h \rangle^\infty$, then there is an $m \in \mathbb{N}$ such that $h^m \cdot f \in I^e \subseteq Q_i$ and since Q_i is primary and $h^m \notin \sqrt{Q_i}$ this forces $f \in Q_i$. But then $f \in Q_1 \cap \ldots \cap Q_l = I^e$, which proves the claim.

With the same argument as at the beginning of the proof we see that G is a standard basis of $\langle I^e \rangle_{F'(\underline{x}')[\underline{x}'']}$, and we may thus apply **[GrP02]** Prop. 4.3.1 to the ideal I^e which shows that $I^e : \langle h \rangle^\infty$ is equidimensional. We are thus done by the claim. □

If the field extension is algebraic then dimension also behaves well under restriction.

Lemma 5.5
Let $F \subseteq F'$ be an algebraic field extension and let $J \lhd F'[\underline{x}]$ be an ideal, then $\dim(J) = \dim(J \cap F[\underline{x}])$.

Proof: Since the field extension is algebraic the ring extension $F[\underline{x}] \subseteq F'[\underline{x}]$ is integral again. But then the ring extension $F[\underline{x}]/J \cap F[\underline{x}] \hookrightarrow F'[\underline{x}]/J$ is integral again (see [**AtM69**] Prop. 5.6), and in particular they have the same dimension (see [**Eis96**] Prop. 9.2). □

For Section 4 — where we want to intersect an ideal of arbitrary dimension to get a zero-dimensional ideal — we need to understand how dimension behaves when we intersect. The following result is concerned with that question. Geometrically it just means that intersecting an equidimensional variety with a hypersurface which does not contain any irreducible component leads again to an equidimensional variety of dimension one less. We need this result over R_N instead of a field K.

Lemma 5.6
Let R be a catenary integral domain, let $I \lhd R$ with $\operatorname{codim}(Q) = d$ *for all* $Q \in \operatorname{minAss}(I)$*, and let $f \in R$ such that $f \notin Q$ for all* $Q \in \operatorname{minAss}(I)$*. Then*

$$\operatorname{minAss}(I + \langle f \rangle) = \bigcup_{Q \in \operatorname{minAss}(I)} \operatorname{minAss}(Q + \langle f \rangle).$$

In particular, $\operatorname{codim}(Q') = d + 1$ *for all* $Q' \in \operatorname{minAss}(I + \langle f \rangle)$.

Proof: If $Q' \in \operatorname{minAss}(I + \langle f \rangle)$ then Q' is minimal among the prime ideals containing $I + \langle f \rangle$. Moreover, since $I \subseteq Q'$ there is a minimal associated prime $Q \in \operatorname{minAss}(I)$ of I which is contained in Q'. And, since $f \in Q'$ we have $Q + \langle f \rangle \subseteq Q'$ and Q' must be minimal with this property since it is minimal over $I + \langle f \rangle$. Hence $Q' \in \operatorname{minAss}(Q + \langle f \rangle)$.

Conversely, if $Q' \in \operatorname{minAss}(Q + \langle f \rangle)$ where $Q \in \operatorname{minAss}(I)$, then $I + \langle f \rangle \subseteq Q'$. Thus there exists a $Q'' \in \operatorname{minAss}(I + \langle f \rangle)$ such that $Q'' \subseteq Q'$. Then $I \subseteq Q''$ and therefore there exists a $\tilde{Q} \in \operatorname{minAss}(I)$ such that $\tilde{Q} \subseteq Q''$. Moreover, since $f \notin \tilde{Q}$ but $f \in Q''$ this inclusion is strict which implies

$$\operatorname{codim}(Q') \geq \operatorname{codim}(Q'') \geq \operatorname{codim}\left(\tilde{Q}\right) + 1 = \operatorname{codim}(Q) + 1,$$

where the first inequality comes from $Q'' \subseteq Q'$ and the last equality is due to our assumption on I. But by Krull's Principal Ideal Theorem (see [**AtM69**] Cor. 11.17) we have

$$\operatorname{codim}(Q'/Q) = 1,$$

since Q'/Q by assumption is minimal over f in R/Q where f is neither a unit (otherwise $Q + \langle f\rangle = R$ and no Q' exists) nor a zero divisor. Finally, since R is catenary and thus all maximal chains of prime ideals from $\langle 0\rangle$ to Q' have the same length this implies

$$\operatorname{codim}(Q') = \operatorname{codim}(Q) + 1. \tag{5.4}$$

This forces that $\operatorname{codim}(Q') = \operatorname{codim}(Q'')$ and thus $Q' = Q'' \in \operatorname{minAss}(I + \langle f\rangle)$.

The "in particular" part follows from (5.4). □

6. Good Behaviour of the Dimension

In this section we want to show (see Theorem 6.14) that for an ideal $J \trianglelefteq L[\underline{x}]$, $N \in \mathcal{N}(J)$ and a point $\omega \in \operatorname{Trop}(P) \cap \mathbb{Q}^n_{\leq 0}$ in the non-positive quadrant of the tropical variety of an associated prime P of maximal dimension we have

$$\dim(J_{R_N}) = \dim\big(\text{t-in}_\omega(J)\big) + 1 = \dim(J) + 1.$$

The results in this section are independent of Sections 2, 3 and 4.

Let us first give examples which show that the hypotheses on ω are necessary.

Example 6.1
Let $J = \langle 1 + tx\rangle \lhd L[x]$ and consider $\omega = 1 \in \operatorname{Trop}(J)$. Then $\text{t-in}_\omega(J) = \langle 1 + x\rangle$ has dimension zero in $K[x]$, and

$$I = J \cap R_1[x] = \langle 1 + tx\rangle_{R_1[x]}$$

has dimension zero as well by Lemma 6.8 (d).

Example 6.2
Let $J = \langle x - 1\rangle \lhd L[x]$ and $\omega = -1 \notin \operatorname{Trop}(J)$, then $\text{t-in}_\omega(J) = \langle 1\rangle$ has dimension -1, while $J \cap R_1[x] = \langle x - 1\rangle$ has dimension 1.

Example 6.3
Let $J = P\cdot Q = P\cap Q \lhd L[x, y, z]$ with $P = \langle tx-1\rangle$ and $Q = \langle x-1, y-1, z-1\rangle$, and let $\omega = (0, 0, 0) \in \operatorname{Trop}(Q) \cap \mathbb{Q}^3_{\leq 0}$. Then $\text{t-in}_\omega(J) = \langle x - 1, y - 1, z - 1\rangle \lhd K[x, y, z]$ has dimension zero, while

$$J \cap R_1[x, y, z] = (P \cap R_1[x, y, z]) \cap (Q \cap R_1[x, y, z])$$

has dimension two by Lemma 6.8 (d).

Before now starting with studying the behaviour of dimension we have to collect some technical results used throughout the proofs.

Lemma 6.4
Let $J \trianglelefteq L[\underline{x}]$ *be an ideal and* $\operatorname{Trop}(J) \cap \mathbb{Q}^n_{\leq 0} \neq \emptyset$*, then* $1 \notin \operatorname{in}_0(J_{R_N})$.

Proof: Let $\omega \in \operatorname{Trop}(J) \cap \mathbb{Q}^n_{\leq 0}$ and suppose that $f \in J_{R_N}$ with $\operatorname{in}_0(f) = 1$. If $t^\alpha \cdot \underline{x}^\beta$ is a monomial of f with $t^\alpha \cdot \underline{x}^\beta \neq 1$, then $\operatorname{in}_0(f) = 1$ implies $\alpha > 0$, and hence $-\alpha + \beta_1 \cdot \omega_1 + \ldots + \beta_n \cdot \omega_n < 0$, since $\omega_1, \ldots, \omega_n \leq 0$ and $\beta_1, \ldots, \beta_n \geq 0$. But this shows that $\operatorname{in}_\omega(f) = 1$, and therefore $1 \in \text{t-in}_\omega(J)$, in contradiction to our assumption that $\text{t-in}_\omega(J)$ is monomial free. $\square$

Lemma 6.5
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \langle t^{\frac{1}{N}} \rangle^\infty$ and let $P \in \operatorname{Ass}(I)$, then $P = P : \langle t^{\frac{1}{N}} \rangle^\infty$ and $t^{\frac{1}{N}} \notin P$.

Proof: Since $R_N[\underline{x}]$ is noetherian and P is an associated prime there is an $f \in R_N[\underline{x}]$ such that $P = I : \langle f \rangle$ (see **[AtM69]** Prop. 7.17).

Suppose that $t^{\frac{\alpha}{N}} \cdot g \in P$ for some $g \in R_N[\underline{x}]$ and $\alpha > 0$. Then $t^{\frac{\alpha}{N}} \cdot g \cdot f \in I$, and since I is saturated with respect to $t^{\frac{1}{N}}$ it follows that $g \cdot f \in I$. This, however, implies that $g \in P$. Thus P is saturated with respect to $t^{\frac{1}{N}}$. If $t^{\frac{1}{N}} \in P$ then $1 \in P$, which contradicts the fact that P is a prime ideal. $\square$

Contractions of ideals in $L[\underline{x}]$ to $R_N[\underline{x}]$ are always $t^{\frac{1}{N}}$-saturated.

Lemma 6.6
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal in $R_N[\underline{x}]$ and $J = \langle I \rangle_{L[\underline{x}]}$, then $J_{R_N} = I : \langle t^{\frac{1}{N}} \rangle^\infty$.

Proof: Since $L_N \subset L$ is a field extension **[Mar07]** Corollary 6.13 implies $J \cap L_N[\underline{x}] = \langle I \rangle_{L_N[\underline{x}]}$, and it suffices to see that $\langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}] = I : \langle t^{\frac{1}{N}} \rangle^\infty$. If $I \cap S_N \neq \emptyset$ then both sides of the equation coincide with $R_N[\underline{x}]$, so that we may assume that $I \cap S_N$ is empty. Recall that $L_N = S_N^{-1} R_N$, so that if $f \in R_N[\underline{x}]$ with $t^{\frac{\alpha}{N}} \cdot f \in I$ for some α, then

$$f = \frac{t^{\frac{\alpha}{N}} \cdot f}{t^{\frac{\alpha}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}].$$

Conversely, if

$$f = \frac{g}{t^{\frac{\alpha}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}]$$

with $g \in I$, then $g = t^{\frac{\alpha}{N}} \cdot f \in I$ and thus f is in the right hand side. $\square$

Lemma 6.7
Let $J \trianglelefteq L[\underline{x}]$ and $N \in \mathcal{N}(J)$. Then $\text{t-in}_0(J) = \text{t-in}_0(J_{R_N})$, and

$$1 \notin \text{t-in}_0(J) \quad \Longleftrightarrow \quad 1 \notin \operatorname{in}_0(J_{R_N}).$$

Proof: Suppose that $f \in J_{R_N} \subset J$ then $\text{t-in}_0(f) \in \text{t-in}_0(J)$, and if in addition $\operatorname{in}_0(f) = 1$, then by definition $1 = \text{t-in}_0(f) \in \text{t-in}_0(J)$.

Let now $f \in J$, then by assumption there are $f_1, \ldots, f_k \in R_{N \cdot M}[\underline{x}]$ for some $M \geq 1$, $g_1, \ldots, g_k \in J_{R_N}$ and some $\alpha \geq 0$ such that

$$t^{\frac{\alpha}{M \cdot N}} \cdot f = f_1 \cdot g_1 + \ldots + f_k \cdot g_k \in R_{N \cdot M}[\underline{x}].$$

By [**Mar07**] Corollary 6.17 we thus get

$$\text{t-in}_0(f) = \text{t-in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right) \in \text{t-in}_0(J_{R_{N\cdot M}}) = \text{t-in}_0(J_{R_N}).$$

Moreover, if we assume that $1 = \text{t-in}_0(f) = \text{t-in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right)$ then there is an $\alpha' \geq 0$ such that

$$t^{\frac{\alpha'}{M\cdot N}} \cdot \text{t-in}_0(f) = \text{in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right) \in \text{in}_0(J_{R_{N\cdot M}}).$$

This necessarily implies that each monomial in $t^{\frac{\alpha}{N\cdot M}} \cdot f$ is divisible by $t^{\frac{\alpha'}{N\cdot M}}$, or by Lemma 6.5 equivalently that $t^{\frac{\alpha-\alpha'}{N\cdot M}} \cdot f \in J_{R_{N\cdot M}}$. But then

$$1 = \text{in}_0\left(t^{\frac{\alpha-\alpha'}{N\cdot M}} \cdot f\right) \in \text{in}_0(J_{R_{N\cdot M}}),$$

and thus by [**Mar07**] Corollary 6.19 also $1 \in \text{in}_0(J_{R_N})$. □

In the following lemma we gather the basic information on the ring $R_N[\underline{x}]$ which is necessary to understand how the dimension of an ideal in $L[\underline{x}]$ behaves when restricting to $R_N[\underline{x}]$.

Lemma 6.8

Consider the ring extension $R_N[\underline{x}] \subset L_N[\underline{x}]$. Then:

(a) *R_N is universally catenary, and thus $R_N[\underline{x}]$ is catenary.*

(b) *If $I \trianglelefteq R_N[\underline{x}]$, then the following are equivalent:*
  (1) $1 \not\in \text{in}_0(I)$.
  (2) $\forall\, p \in R_N[\underline{x}] \;:\; 1 + t^{\frac{1}{N}} \cdot p \not\in I$.
  (3) $I + \langle t^{\frac{1}{N}} \rangle \subsetneqq R_N[\underline{x}]$.
  (4) $\exists\, P \lhd R_N[\underline{x}]$ *maximal such that* $I \subseteq P$ *and* $t^{\frac{1}{N}} \in P$.
  (5) $\exists\, P \lhd R_N[\underline{x}]$ *maximal such that* $I \subseteq P$ *and* $1 \not\in \text{in}_0(P)$.

  In particular, if $P \lhd R_N[\underline{x}]$ is a maximal ideal, then

$$1 \not\in \text{in}_0(P) \quad \Longleftrightarrow \quad t^{\frac{1}{N}} \in P.$$

(c) *If $P \lhd R_N[\underline{x}]$ is a maximal ideal such that $1 \not\in \text{in}_0(P)$, then every maximal chain of prime ideals contained in P has length $n+2$.*

(d) *If $I \trianglelefteq R_N[\underline{x}]$ is any ideal with $1 \in \text{in}_0(I)$, then $R_N[\underline{x}]/I \cong L_N[\underline{x}]/\langle I\rangle$, and $I \cap S_N = \emptyset$ unless $I = R_N[\underline{x}]$. In particular,* $\dim(I) = \dim\left(\langle I\rangle_{L_N[\underline{x}]}\right)$.

(e) *If $P \lhd R_N[\underline{x}]$ is a maximal ideal such that $1 \in \text{in}_0(P)$, then every maximal chain of prime ideals contained in P has length $n+1$.*

(f) $\dim(R_N[\underline{x}]) = n+1$.

(g) *If $P \lhd R_N[\underline{x}]$ is a prime ideal such that $1 \not\in \text{in}_0(P)$, then*

$$\dim(P) + \text{codim}(P) = \dim(R_N[\underline{x}]) = n+1.$$

(h) *If* $P \lhd R_N[\underline{x}]$ *is a prime ideal such that* $1 \in \mathrm{in}_0(P)$*, then*

$$\dim(P) + \mathrm{codim}(P) = n.$$

Proof: For (a), see **[Mat86]** Thm. 29.4.

In (b), the equivalence of (1) and (2) is obvious from the definitions. Let us now use this to show that for a maximal ideal $P \lhd R_N[\underline{x}]$

$$1 \notin \mathrm{in}_0(P) \quad \Longleftrightarrow \quad t^{\frac{1}{N}} \in P.$$

If $t^{\frac{1}{N}} \notin P$ then $t^{\frac{1}{N}}$ is a unit in the field $R_N[\underline{x}]/P$ and thus there is a $p \in R_N[\underline{x}]$ such that $1 \equiv t^{\frac{1}{N}} \cdot p \pmod P$, or equivalently that $1 - t^{\frac{1}{N}} \cdot p \in P$. If on the other hand $t^{\frac{1}{N}} \in P$ then $1 + t^{\frac{1}{N}} \cdot p \in P$ would imply that $1 = (1 + t^{\frac{1}{N}} \cdot p) - t^{\frac{1}{N}} \cdot p \in P$.

This proves the claim and shows at the same time the equivalence of (4) and (5).

If there is a maximal ideal P containing I and such that $1 \notin \mathrm{in}_0(P)$, then of course also $1 \notin \mathrm{in}_0(I)$. Therefore (5) implies (1).

Let now I be an ideal such that $1 \notin \mathrm{in}_0(I)$. Suppose that $I + \langle t^{\frac{1}{N}} \rangle = R_N[\underline{x}]$. Then $1 = q + t^{\frac{1}{N}} \cdot p$ with $q \in I$ and $p \in R_N[\underline{x}]$, and thus $q = 1 - t^{\frac{1}{N}} \cdot p \in I$, which contradicts our assumption. Thus $I + \langle t^{\frac{1}{N}} \rangle \neq R_N[\underline{x}]$, and (1) implies (3).

Finally, if $I + \langle t^{\frac{1}{N}} \rangle \neq R_N[\underline{x}]$, then there exists a maximal ideal P such that $I + \langle t^{\frac{1}{N}} \rangle \subseteq P$. This shows that (3) implies (4), and we are done.

To see (c), note that if $1 \notin \mathrm{in}_0(P)$, then $t^{\frac{1}{N}} \in P$ by (b), and we may consider the surjection $\psi : R_N[\underline{x}] \longrightarrow R_N[\underline{x}]/\langle t^{\frac{1}{N}} \rangle = K[\underline{x}]$. The prime ideals of $K[\underline{x}]$ are in $1:1$-correspondence with those prime ideals of $R_N[\underline{x}]$ which contain $t^{\frac{1}{N}}$. In particular, $P/\langle t^{\frac{1}{N}} \rangle = \psi(P)$ is a maximal ideal of $K[\underline{x}]$ and thus any maximal chain of prime ideals in P which starts with $\langle t^{\frac{1}{N}} \rangle$, say $\langle t^{\frac{1}{N}} \rangle = P_0 \subset \ldots \subset P_n = P$ has precisely $n+1$ terms since every maximal chain of prime ideals in $K[\underline{x}]$ has that many terms. Moreover, by Krull's Principal Ideal Theorem (see e.g. **[AtM69]** Cor. 11.17) the prime ideal $\langle t^{\frac{1}{N}} \rangle$ has codimension 1, so that the chain of prime ideals

$$\langle 0 \rangle \subset \langle t^{\frac{1}{N}} \rangle = P_0 \subset \ldots \subset P_n = P$$

is maximal. Since by (a) the ring $R_N[\underline{x}]$ is catenary every maximal chain of prime ideals in between $\langle 0 \rangle$ and P has the same length $n+2$.

For (d), we assume that there exists an element $1 + t^{\frac{1}{N}} \cdot p \in I$ due to (b). But then $t^{\frac{1}{N}} \cdot (-p) \equiv 1 \pmod I$. Thus the elements of $S_N = \left\{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, \ldots\right\}$ are invertible modulo I. Therefore

$$R_N[\underline{x}]/I \cong S_N^{-1}(R_N[\underline{x}]/I) \cong S_N^{-1}R_N[\underline{x}]/S_N^{-1}I = L_N[\underline{x}]/\langle I \rangle.$$

In particular, if $I \neq R_N[\underline{x}]$ then $\langle I \rangle \neq L_N[\underline{x}]$ and thus $I \cap S_N = \emptyset$.

To show (e), note that by assumption there is an element $1+t^{\frac{1}{N}}\cdot p\in P$ due to (b), and since P is maximal $p\not\in R_N$. Choose a prime ideal Q contained in P which is minimal w.r.t. the property that it contains $1+t^{\frac{1}{N}}\cdot p$. Since $1+t^{\frac{1}{N}}\cdot p$ is neither a unit nor a zero divisor Krull's Principal Ideal Theorem implies that $\operatorname{codim}(Q)=1$. Moreover, since $Q\cap S_N=\emptyset$ by Part (d) the ideal $\langle Q\rangle_{L_N[\underline{x}]}$ is a prime ideal which is minimal over $1+t^{\frac{1}{N}}\cdot p$ by the one-to-one correspondence of prime ideals under localisation. Since every maximal chain of primes in $L_N[\underline{x}]$ has length n , and by Part (d) we have $\dim(Q)=\dim\big(\langle Q\rangle_{L_N[\underline{x}]}\big)=n-1$. Hence there is a maximal chain of prime ideals of length n from $\langle Q\rangle_{L_N[\underline{x}]}$ to $\langle P\rangle_{L_N[\underline{x}]}$. Since $\operatorname{codim}(Q)=1$ it follows that there is a chain of prime ideals of length $n+1$ starting at $\langle 0\rangle$ and ending at P which cannot be prolonged. But by (a) the ring $R_N[\underline{x}]$ is catenary, and thus every maximal chain of prime ideals in P has length $n+1$.

Claim (f) follows from (c) and (e).

To see (g), note that by (b) there exists a maximal ideal Q containing P and $t^{\frac{1}{N}}$. If $k=\operatorname{codim}(P)$ then we may choose a maximal chain of prime ideals of length $k+1$ in P, and we may prolong it by at most $\dim(P)$ prime ideal to a maximal chain of prime ideals in Q, which by (b) and (c) has length $n+2$. Taking (f) into account this shows that

$$\dim(P)\geq(n+2)-(k+1)=\dim(R_N[\underline{x}])-\operatorname{codim}(P).$$

However, the converse inequality always holds, which finishes the proof.

For (h) note that by (b) there is no maximal ideal which contains $t^{\frac{1}{N}}$ so that every maximal ideal containing P has codimension n. The result then follows as in (g). □

Corollary 6.9
Let $P\lhd L[\underline{x}]$ be a prime ideal and $N\geq 1$, then

$$\dim(P_{R_N})=\dim(P)+1 \iff 1\not\in\operatorname{in}_0(P_{R_N}),\quad \textit{and}$$
$$\dim(P_{R_N})=\dim(P) \iff 1\in\operatorname{in}_0(P_{R_N}).$$

In any case

$$\operatorname{codim}(P_{R_N})=\operatorname{codim}(P).$$

Proof: Since the field extension $L_N\subset L$ is algebraic by Lemma 5.5 we have

$$\dim(P)=\dim\big(P\cap L_N[\underline{x}]\big) \tag{6.1}$$

in any case. If $1\in\operatorname{in}_0\big(P_{R_N}\big)$, then Lemma 6.8(d) implies

$$\dim\big(P_{R_N}\big)=\dim\big(\langle P_{R_N}\rangle_{L_N[\underline{x}]}\big)=\dim\big(P\cap L_N[\underline{x}]\big),$$

since $L_N[\underline{x}]$ is a localisation of $R_N[\underline{x}]$.

It thus suffices to show that $\dim\big(P_{R_N}\big)=\dim(P)+1$ if $1\not\in\operatorname{in}_0\big(P_{R_N}\big)$.

Since $P \neq L[\underline{x}]$ we know that $S_N \cap P = \emptyset$. The $1:1$-correspondence of prime ideals under localisation thus shows that

$$l := \operatorname{codim}\big(P \cap L_N[\underline{x}]\big) = \operatorname{codim}\big(P_{R_N}\big).$$

Hence there exists a maximal chain of prime ideals

$$\langle 0\rangle = Q_0 \subsetneqq \ldots \subsetneqq Q_l = P_{R_N}$$

of length $l+1$ in $R_N[\underline{x}]$. Note also that by (6.1)

$$l = \operatorname{codim}\big(P \cap L_N[\underline{x}]\big) = n - \dim\big(P \cap L_N[\underline{x}]\big) = n - \dim(P), \tag{6.2}$$

since $L_N[\underline{x}]$ is a polynomial ring over a field.

Moreover, since $1 \notin \operatorname{in}_0\big(P_{R_N}\big)$ by Lemma 6.8(b), there exists a maximal ideal $Q \lhd R_N[\underline{x}]$ containing P_{R_N} such that $1 \notin \operatorname{in}_0(Q)$. Choose a maximal chain of prime ideals

$$P_{R_N} = Q_l \subsetneqq Q_{l+1} \subsetneqq \ldots \subsetneqq Q_k = Q$$

in $R_N[\underline{x}]$ from P_{R_N} to Q, so that taking (6.2) into account

$$\dim(P_{R_N}) \geq k - l = k - n + \dim(P). \tag{6.3}$$

Finally, since the sequence

$$\langle 0\rangle = Q_0 \subsetneqq Q_1 \subsetneqq \ldots \subsetneqq Q_l \subsetneqq \ldots \subsetneqq Q_k = Q$$

cannot be prolonged and since $1 \notin \operatorname{in}_0(Q)$, Lemma 6.8(c) implies that $k = n+1$. But since we always have

$$\dim\big(P_{R_N}\big) \leq \dim\big(R_N[\underline{x}]\big) - \operatorname{codim}\big(P_{R_N}\big) = n + 1 - l,$$

it follows from (6.2) and (6.3)

$$\dim(P) + 1 \leq \dim\big(P_{R_N}\big) \leq n + 1 - l = \dim(P) + 1.$$

The claim for the codimensions then follows from Lemma 6.8 (g) and (h). □

As an immediate corollary we get one of the main results of this section.

Theorem 6.10
Let $J \unlhd L[\underline{x}]$ *and* $N \in \mathcal{N}(J)$. *Then* $\dim\big(J_{R_N}\big) = \dim(J) + 1$ *if and only if* $\exists\, P \in \operatorname{Ass}(J)$ *s.t.* $\dim(P) = \dim(J)$ *and* $1 \notin \operatorname{in}_0\big(P_{R_N}\big)$. *Otherwise* $\dim\big(J_{R_N}\big) = \dim(J)$.

Proof: If there is such a $P \in \operatorname{Ass}(J)$ then Corollary 6.9 implies

$$\begin{aligned} \dim\big(P_{R_N}\big) &= \dim(P) + 1 = \dim(J) + 1 \text{ and} \\ \dim\big(P'_{R_N}\big) &\leq \dim(P') + 1 \leq \dim(J) + 1 \end{aligned}$$

for any other $P' \in \operatorname{Ass}(J)$. This shows that

$$\dim\big(J_{R_N}\big) = \max\{\dim\big(P'_{R_N}\big) \mid P' \in \operatorname{Ass}(J)\} = \dim(J) + 1,$$

due to Proposition 5.3.

If on the other hand $1 \in \operatorname{in}_0\big(P_{R_N}\big)$ for all $P \in \operatorname{Ass}(J)$ with $\dim(P) = \dim(J)$, then again by Corollary 6.9 $\dim(P_{R_N}) \leq \dim(J)$ for all associated primes with equality for some, and we are done with Proposition 5.3. $\square$

It remains to show that also the dimension of the t-initial ideal behaves well.

Proposition 6.11
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \langle t^{\frac{1}{N}}\rangle^\infty$ and such that $1 \notin \operatorname{in}_0(P)$ for some $P \in \operatorname{Ass}(I)$ with $\dim(P) = \dim(I)$. Then

$$\dim(I) = \dim\big(\text{t-in}_0(I)\big) + 1.$$

More precisely, $\dim(Q') = \dim(P) - 1$ *for all* $Q' \in \operatorname{minAss}\big(\text{t-in}_0(P)\big)$.

Proof: We first want to show that

$$\text{t-in}_0(I) = \big(I + \langle t^{\frac{1}{N}}\rangle\big) \cap K[\underline{x}].$$

Any element $f \in \langle t^{\frac{1}{N}}\rangle + I$ can be written as $f = t^{\frac{1}{N}} \cdot g + h$ with $g \in R_N[\underline{x}]$ and $h \in I$ such that $\operatorname{in}_0(h) \in K[\underline{x}]$, and if in addition $f \in K[\underline{x}]$ then obviously $f = \operatorname{in}_0(h) = \text{t-in}_0(h) \in \text{t-in}_0(I)$. If, on the other hand, $g = \text{t-in}_0(f) \in \text{t-in}_0(I)$ for some $f \in I$, then $t^{\frac{\alpha}{N}} \cdot g = \operatorname{in}_0(f) \in \operatorname{in}_0(I)$ for some $\alpha \geq 0$, and every monomial in f is necessarily divisible by $t^{\frac{\alpha}{N}}$. Thus $f = t^{\frac{\alpha}{N}} \cdot h$ for some $h \in R_N[\underline{x}]$ and $g = \operatorname{in}_0(h) \equiv h \pmod{\langle t^{\frac{1}{N}}\rangle}$. But since I is saturated with respect to $t^{\frac{1}{N}}$ it follows that $h \in I$, and thus g is in the right hand side. This proves the claim.

Therefore, the inclusion $K[\underline{x}] \hookrightarrow R_N[\underline{x}]$ induces an isomorphism

$$K[\underline{x}]/\,\text{t-in}_0(I) \cong R_N[\underline{x}]/\big(\langle t^{\frac{1}{N}}\rangle + I\big) \tag{6.4}$$

which shows that

$$\dim\big(K[\underline{x}]/\,\text{t-in}_0(I)\big) = \dim\Big(R_N[\underline{x}]/\Big(I + \big\langle t^{\frac{1}{N}}\big\rangle\Big)\Big). \tag{6.5}$$

Next, we want to show that

$$\dim\Big(P + \big\langle t^{\frac{1}{N}}\big\rangle\Big) = \dim(P) - 1 = \dim(I) - 1. \tag{6.6}$$

For this we consider an arbitrary $P' \in \operatorname{minAss}\Big(P + \big\langle t^{\frac{1}{N}}\big\rangle\Big)$. By Lemma 6.8 (b), $1 \notin \operatorname{in}_0(P')$. Applying Lemma 6.8 (g) to P and P' we get

$$\dim(R_N[\underline{x}]) = \dim(P) + \operatorname{codim}(P) \text{ and } \dim(R_N[\underline{x}]) = \dim(P') + \operatorname{codim}(P').$$

Moreover, since I is saturated with respect to $t^{\frac{1}{N}}$ by Lemma 6.5 P does not contain $t^{\frac{1}{N}}$. Thus $t^{\frac{1}{N}}$ is neither a zero divisor nor a unit in $R_N[\underline{x}]/P$, and by Krull's Principal Ideal Theorem (see **[AtM69]** Cor. 11.17) we thus get $\operatorname{codim}(P') = \operatorname{codim}(P) + 1$, since by assumption P' is minimal over $t^{\frac{1}{N}}$ in $R_N[\underline{x}]/P$. Plugging the two previous equations in we get

$$\dim(P') = \dim(P) - 1. \tag{6.7}$$

This proves (6.6), since P' was an arbitrary minimal associated prime of $P + \langle t^{\frac{1}{N}}\rangle$.

We now claim that

$$\dim\left(P + \langle t^{\frac{1}{N}}\rangle\right) = \dim\left(I + \langle t^{\frac{1}{N}}\rangle\right). \tag{6.8}$$

Suppose this is not the case, then there is a $P' \in \operatorname{Ass}\left(I + \langle t^{\frac{1}{N}}\rangle\right)$ such that

$$\dim(P') > \dim\left(P + \langle t^{\frac{1}{N}}\rangle\right) = \dim(I) - 1,$$

and since $I \subset P'$ it follows that

$$\dim(P') = \dim(I).$$

But then P' is necessarily a minimal associated prime of I in contradiction to Lemma 6.5, since P' contains $t^{\frac{1}{N}}$. This proves (6.8).

Equations (6.5), (6.6) and (6.8) finish the proof of the first claim. For the "more precisely" part notice that replacing I by P in (6.4) we see that there is a dimension preserving $1:1$-correspondence between $\operatorname{minAss}\left(P + \langle t^{\frac{1}{N}}\rangle\right)$ and $\operatorname{minAss}\left(\text{t-in}_0(P)\right)$. The result then follows from (6.7). $\square$

Remark 6.12

The condition that I is saturated with respect to $t^{\frac{1}{N}}$ in Proposition 6.11 is equivalent to the fact that I is the contraction of the ideal $\langle I\rangle_{L_N[\underline{x}]}$. Moreover, it implies that $R_N[\underline{x}]/I$ is a flat R_N-module, or alternatively that the family

$$\iota^* : \operatorname{Spec}\left(R_N[\underline{x}]/I\right) \longrightarrow \operatorname{Spec}(R_N)$$

is flat, where the generic fibre is just $\operatorname{Spec}\left(L_N[\underline{x}]/\langle I\rangle\right)$ and the special fibre is $\operatorname{Spec}\left(K[\underline{x}]/\,\text{t-in}_0(I)\right)$. The condition $1 \not\in \text{in}_0(P)$ implies that the component of $\operatorname{Spec}\left(R_N[\underline{x}]/I\right)$ defined by P surjects onto $\operatorname{Spec}(R_N)$. With this interpretation the proof of Proposition 6.11 is basically exploiting the dimension formula for local flat extensions.

Corollary 6.13

Let $J \lhd L[\underline{x}]$ and $\omega \in \mathbb{Q}^n$, then

$$\dim\left(\text{t-in}_\omega(J)\right) = \max\left\{\dim(P) \;\middle|\; P \in \operatorname{Ass}(J) \;:\; 1 \not\in \text{t-in}_\omega(P)\right\}.$$

Moreover, if J is prime, $1 \not\in \text{t-in}_\omega(J)$ and $Q' \in \operatorname{minAss}\left(\text{t-in}_\omega(J)\right)$ then

$$\dim(Q') = \dim(J).$$

Proof: Let $J = Q_1 \cap \ldots \cap Q_k$ be a minimal primary decomposition of J, and

$$\Phi_\omega(J) = \Phi_\omega(Q_1) \cap \ldots \cap \Phi_\omega(Q_k)$$

the corresponding minimal primary decomposition of $\Phi_\omega(J)$. If we define a new ideal

$$J' = \bigcap_{1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right)} \Phi_\omega(Q_i),$$

then this representation is already a minimal primary decomposition of J'. Choose an N such that $N \in \mathcal{N}(J)$, $N \in \mathcal{N}(J')$ and $N \in \mathcal{N}\big(\Phi_\omega(Q_i)\big)$ for all $i = 1, \ldots, k$. By Lemma 6.7 we have

$$1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right) \iff 1 \not\in \text{in}_0\left(\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]\right). \tag{6.9}$$

Proposition 5.3 implies

$$\text{Ass}(J_{R_N}) = \left\{ \sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}] \;\middle|\; i = 1, \ldots, k \right\}$$

where the $\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]$ are not necessarily pairwise different, and

$$\text{Ass}(J'_{R_N}) = \left\{ \sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}] \;\middle|\; 1 \not\in \text{in}_0\left(\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]\right) \right\},$$

for which we have to take (6.9) into account.

Moreover, by Lemma 6.6 J'_{R_N} is saturated with respect to $t^{\frac{1}{N}}$. Thus we can apply Proposition 6.11 to J'_{R_N} to deduce $\dim(J'_{R_N}) = \dim\big(\text{t-in}_0(J'_{R_N})\big) + 1$.

Taking (6.9) into account we can apply Theorem 6.10 to J' and deduce that then $\dim(J'_{R_N}) = \dim(J') + 1$, but

$$\begin{aligned} \dim(J') &= \max\left\{ \dim\left(\sqrt{\Phi_\omega(Q_i)}\right) \mid 1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right) \right\} \\ &= \max\left\{ \dim\left(\sqrt{Q_i}\right) \mid 1 \not\in \text{t-in}_\omega\left(\sqrt{Q_i}\right) \right\}. \end{aligned}$$

It remains to show that $\text{t-in}_0(J'_{R_N}) = \text{t-in}_\omega(J)$. By Lemma 6.7 and Definition 3.3 we have $\text{t-in}_0(J'_{R_N}) = \text{t-in}_0(J')$ and

$$\text{t-in}_\omega(J) = \text{t-in}_0\big(\Phi_\omega(J)\big) \subseteq \text{t-in}_0(J'),$$

since $J \subseteq J'$. By assumption for any $\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')$ there is an $f_i \in \sqrt{\Phi_\omega(Q_i)}$ such that $\text{t-in}_0(f_i) = 1$ and there is some m_i such that $f_i^{m_i} \in \Phi_\omega(Q_i)$. If $f \in J'$ is any element, then for

$$g := f \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} f_i^{m_i} \in \Big(J' \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} \Phi_\omega(Q_i)\Big) \subseteq J$$

we have

$$\text{t-in}_0(f) = \text{t-in}_0(f) \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} \text{t-in}_0(f_i)^{m_i} = \text{t-in}_0(g) \in \text{t-in}_0(J).$$

This finishes the proof of the first claim.

For the "moreover" part note that by Lemma 6.7

$$\text{t-in}_\omega(J) = \text{t-in}_0\big(\Phi_\omega(J)\big) = \text{t-in}_0\big(\Phi_\omega(J) \cap R_N[\underline{x}]\big)$$

and $\Phi_\omega(J) \cap R_N[\underline{x}]$ is saturated and prime. Applying Proposition 6.11 to

$$Q' \in \operatorname{minAss}\Big(\text{t-in}_0\left(\Phi_\omega(J) \cap R_N[\underline{x}]\right)\Big) = \operatorname{minAss}\big(\text{t-in}_\omega(J)\big)$$

we get

$$\dim(Q') = \dim\big(\Phi_\omega(J) \cap R_N[\underline{x}]\big) - 1 = \dim(J),$$

where the latter equality is due to Corollary 6.9. □

Theorem 6.14
Let $J \lhd L[\underline{x}]$, $N \in \mathcal{N}(J)$ *and* $\omega \in \mathbb{Q}^n_{\leq 0}$.

If there is a $P \in \operatorname{Ass}(J)$ *with* $\dim(P) = \dim(J)$ *and* $\omega \in \operatorname{Trop}(P)$, *then*

$$\dim(J_{R_N}) = \dim(J) + 1 = \dim\big(\text{t-in}_\omega(J)\big) + 1.$$

Proof: By Lemma 6.4 the condition $\omega \in \operatorname{Trop}(P) \cap \mathbb{Q}^n_{\leq 0}$ implies that $1 \notin \operatorname{in}_0(P_{R_N})$. The result then follows from Theorem 6.10 and Corollary 6.13. □

Corollary 6.15
If $J \unlhd L[\underline{x}]$ *is zero dimensional and* $\omega \in \operatorname{Trop}(J)$, *then* $\dim\big(\text{t-in}_\omega(J)\big) = \dim(J) = 0$. *If in addition* $\operatorname{Trop}(J) \cap \mathbb{Q}^n_{\leq 0} \neq \emptyset$ *and* $N \in \mathcal{N}(J)$ $\dim\big(J_{R_N}\big) = 1$.

Proof: Since $\dim(J) = 0$ also $\dim(P) = 0$ for every associated prime P. By 2.12 there exists a P with $\omega \in \operatorname{Trop}(P)$. The first assertion thus follows from Corollary 6.13. The second assertion follows from Theorem 6.14. □

When cutting down the dimension we need to understand how the minimal associated primes of J and J_{R_N} relate to each other.

Lemma 6.16
Let $J \lhd L[\underline{x}]$ *be equidimensional and* $N \in \mathcal{N}(J)$. *Then*

$$\operatorname{minAss}(J_{R_N}) = \{P_{R_N} \mid P \in \operatorname{minAss}(J)\}.$$

Proof: The left hand side is contained in the right hand side by default (see Proposition 5.3). Let therefore $P \in \operatorname{minAss}(J)$ be given. By Proposition 5.3 $P_{R_N} \in \operatorname{Ass}(J)$, and it suffices to show that it is minimal among the associated primes. Suppose therefore we have $Q \in \operatorname{Ass}(J)$ such that $Q_{R_N} \subseteq P_{R_N}$. By Corollary 6.9 and the assumption we have

$$\operatorname{codim}(P_{R_N}) = \operatorname{codim}(P) \leq \operatorname{codim}(Q) = \operatorname{codim}(Q_{R_N}),$$

so that indeed $P_{R_N} = Q_{R_N}$. □

Another consequence is that the t-initial ideal of an equidimensional ideal is again equidimensional.

Corollary 6.17
Let $J \lhd L[\underline{x}]$ be an equidimensional ideal and $\omega \in \mathbb{Q}^n$, then

$$\operatorname{minAss}\big(\text{t-in}_\omega(J)\big) = \bigcup_{P\in\operatorname{minAss}(J)} \operatorname{minAss}\big(\text{t-in}_\omega(P)\big).$$

In particular, if there is a $P \in \operatorname{minAss}(J)$ such that $1 \notin \text{t-in}_\omega(P)$ then $\text{t-in}_\omega(J)$ is equidimensional of dimension $\dim(J)$.

Proof: Applying Φ_ω we may assume that $\omega = 0$, and we then may choose an $N \in \mathcal{N}(J)$ and $N \in \mathcal{N}(P)$ for all $P \in \operatorname{minAss}(J)$.

Denoting by

$$\pi : R_N[\underline{x}] \longrightarrow R_N[\underline{x}]/\big\langle t^{\frac{1}{N}}\big\rangle = K[\underline{x}]$$

the residue class map we get

$$\text{t-in}_0(J) = \text{t-in}_0(J_{R_N}) = \pi\big(J_{R_N} + \langle t^{\frac{1}{N}}\rangle\big) \text{ and}$$
$$\text{t-in}_0(P) = \text{t-in}_0(P_{R_N}) = \pi\big(P_{R_N} + \langle t^{\frac{1}{N}}\rangle\big)$$

for all $P \in \operatorname{minAss}(J)$, where the first equality in both cases is due to Lemma 6.7 and where the last equality uses Lemma 6.6. Since there is a one-to-one correspondence between prime ideals in $K[\underline{x}]$ and prime ideals in $R_N[\underline{x}]$ which contain $t^{\frac{1}{N}}$, it suffices to show that

$$\operatorname{minAss}\big(J_{R_N} + \langle t^{\frac{1}{N}}\rangle\big) = \bigcup_{P\in\operatorname{minAss}(J)} \operatorname{minAss}\big(P_{R_N} + \langle t^{\frac{1}{N}}\rangle\big).$$

However, since the P_{R_N} are saturated with respect to $t^{\frac{1}{N}}$ by Lemma 6.6 they do not contain $t^{\frac{1}{N}}$. By Corollary 6.9 all P_{R_N} have the same codimension, since the P do by assumption. By Lemma 6.16,

$$\operatorname{minAss}(J_{R_N}) = \{P_{R_N} \mid P \in \operatorname{minAss}(J)\}.$$

Hence the result follows by Lemma 5.6.

The "in particular" part follows from Corollary 6.13. □

7. Computing t-Initial Ideals

This section is devoted to an alternative proof of Theorem 2.8 which does not need standard basis in the mixed power series polynomial ring $K[[t]][\underline{x}]$.

The following lemma is easy to show.

Lemma 7.1
Let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, $0 \neq f = \sum_{i=1}^k g_i \cdot h_i$ with $f, g_i, h_i \in R_N[\underline{x}]$ and $\operatorname{ord}_w(f) \geq \operatorname{ord}_w(g_i \cdot h_i)$ for all $i = 1, \ldots, k$. Then

$$\operatorname{in}_w(f) \in \big\langle \operatorname{in}_w(g_1), \ldots, \operatorname{in}_w(g_k)\big\rangle \lhd K\big[t^{\frac{1}{N}}, \underline{x}\big].$$

Proposition 7.2
Let $I \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big]$, $\omega \in \mathbb{Q}^n$ and G be a standard basis of I with respect to the monomial ordering $>_\omega$ introduced in Remark 3.7. Then

$$\operatorname{in}_\omega(I) = \big\langle \operatorname{in}_\omega(G)\big\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big] \ \textit{ and } \ \text{t-in}_\omega(I) = \big\langle \text{t-in}_\omega(G)\big\rangle \trianglelefteq K[\underline{x}].$$

Proof: It suffices to show that $\operatorname{in}_\omega(f) \in \langle \operatorname{in}_\omega(G)\rangle$ for every $f \in I$. Since $f \in I$ and G is a standard basis of I there exists a weak standard representation

$$u \cdot f = \sum_{g \in G} q_g \cdot g \tag{7.1}$$

of f where the leading term of u with respect to $>_\omega$ is $\operatorname{lt}_{>_\omega}(u) = 1$. But then the definition of $>_\omega$ implies that automatically $\operatorname{in}_\omega(u) = 1$. Since (7.1) is a standard representation we have $\operatorname{lm}_{>_\omega}(u \cdot f) \geq \operatorname{lm}_{>_\omega}(q_g \cdot g)$ for all g. But this necessarily implies that $\operatorname{ord}_w(f) \geq \operatorname{ord}_w(q_g \cdot g)$ where $w = (-1, \omega)$. Since $K\big[t^{\frac{1}{N}}, \underline{x}\big] \subset R_N[\underline{x}]$ we can use Lemma 7.1 to show

$$\operatorname{in}_w(f) = \operatorname{in}_w(u \cdot f) \in \langle \operatorname{in}_w(g) \mid g \in G\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big].$$

□

Proposition 7.3
Let $I \subseteq K[t,x]$ be an ideal, $J = \langle I \rangle_{L[x]}$ and $\omega \in \mathbb{R}^n$. Then $\text{t-in}_\omega(I) = \text{t-in}_\omega(J)$.

Proof: We need to prove the inclusion $\text{t-in}_\omega(I) \supseteq \text{t-in}_\omega(J)$. The other inclusion is clear since $I \subseteq J$. The right hand side is generated by elements of the form $f = \text{t-in}_\omega(g)$ where $g \in J$. Consider such f and g. The polynomial g must be of the form $g = \sum_i c_i \cdot g_i$ where $g_i \in I$ and $c_i \in L$. Let d be the $(-1,\omega)$-degree of $\operatorname{in}_\omega(g)$. The degrees of terms in g_i are bounded. Terms $a \cdot t^\beta$ in c_i of large enough t-degree will make the $(-1,\omega)$-degree of $a \cdot t^\beta \cdot g_i$ drop below d since the degree of t is negative. Consequently, these terms can simply be ignored since they cannot affect the initial form of $g = \sum_i c_i \cdot g_i$. Renaming and possibly repeating some g_i's we may write g as a finite sum $g = \sum_i c_i' \cdot g_i$ where $c_i' = a_i \cdot t^{\beta_i}$ and $g_i \in I$ with $a_i \in K$ and $\beta_i \in \mathbb{Q}$. We will split the sum into subsums grouping together the c_i''s that have the same t-exponent modulo $\mathbb{Z}$. For suitable index sets A_j we let $g = \sum_j G_j$ where $G_j = \sum_{i \in A_j} c_i' \cdot g_i$. Notice that all t-exponents in a G_j are congruent modulo $\mathbb{Z}$ while t-exponents from different G_j's are not. In particular there is no cancellation in the sum $g = \sum_j G_j$. As a consequence $\operatorname{in}_\omega(g) = \sum_{j \in S} \operatorname{in}_\omega(G_j)$ for a suitable subset S. We also have $\text{t-in}_\omega(g) = \sum_{j \in S} \text{t-in}_\omega(G_j)$. We wish to show that each $\text{t-in}_\omega(G_j)$ is in $\text{t-in}(I)$. We can write $t^{\gamma_j} \cdot G_j = \sum_{i \in A_j} t^{\gamma_j} \cdot c_i' \cdot g_i$ for suitable $\gamma_j \in \mathbb{Q}$ such that $t^{\gamma_j} \cdot c_i' \in K[t]$ for all $i \in A_j$. Observe that

$$\text{t-in}_\omega(G_j) = \text{t-in}_\omega(t^{\gamma_j} \cdot G_j) = \text{t-in}_\omega\Big(\sum_{i \in A_j} t^{\gamma_j} \cdot c_i' \cdot g_i\Big) \in \text{t-in}_\omega(I).$$

Applying $\text{t-in}_\omega(g) = \sum_{j \in S} \text{t-in}_\omega(G_j)$ we see that $f = \text{t-in}_\omega(g) \in \text{t-in}_\omega(I)$. □

By substituting $t := t^{\frac{1}{n}}$ and scaling ω we get Theorem 2.8 as a corollary.

Bibliography

[Abo06] Mohammed Abouzaid, *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*, Preprint, Math.SG/0610004, 2006.

[ACH05] Maria-Emilia Alonso, Francisco Castro, and Herwig Hauser, *Effective algebraic power series*, Preprint, 2005.

[AlH00] James Alexander and André Hirschowitz, *An asymptotic vanishing theorem for generic unions of multiple points*, Inventiones Math. **140** (2000), no. 2, 303–325.

[AMNR92] Maria-Emilia Alonso, Teo Mora, Gianfranco Niesi, and Mario Raimondo, *An algorithm for computing analytic branches of space curves at singular points*, Proceedings of the 1992 International Workshop on Mathematics Mechanization, International Academic Publishers, 1992, pp. 135–166.

[AMR77] Maria-Emilia Alonso, Teo Mora, and Mario Raimondo, *A computational model for algebraic power series*, J. Pure Appl. Alg. **77** (1977), 1–38.

[ArC81] Enrico Arbarello and Maurizio Cornalba, *Footnotes to a paper of Beniamino Segre*, Math. Ann. **256** (1981), 341–362.

[AtM69] Michael F. Atiyah and Ian G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

[Bay82] Dave Bayer, *The division algorithm and the Hilbert scheme*, Ph.D. thesis, Harvard University, 1982.

[Bec90] Thomas Becker, *Stability and Buchberger criterion for standard bases in power series rings*, J. Pure Appl. Algebra **66** (1990), 219–227.

[BHPV04] Wolf Barth, Klaus Hulek, Christian Peters, and Antonius Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 4, Springer, 2004.

[BiG84] Robert Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195.

[BJS$^+$07] Tristram Bogart, Anders Jensen, David Speyer, Bernd Sturmfels, and Rekha Thomas, *Computing tropical varieties*, J. Symbolic Computation **42** (2007), 54–73, math.AG/0507563.

[BoC05] Cristiano Bocci and Luca Chiantini, *Triple points imposing triple divisors and the defective hierarchy*, Projective Varieties with Unexpected Properties, De Gruyter, 2005, pp. 35–49.

[BoM04] Cristiano Bocci and Rick Miranda, *Topics on interpolation problems in algebraic geometry*, Rend. Semin. Mat. Torino **62** (2004), no. 4, 279–334.

[Bri77] Joel Briancon, *Description of* $\mathrm{Hilb}^n\,\mathbb{C}\{x,y\}$, Inv. math. **41** (1977), 45–89.

[BrK86] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser, 1986.

[Cas22] Maria Castellani, *Sulle superficie i cui spazi osculatori sono biosculatori*, Rend. Accad. Lincei **XXXI** (1922), 347–350.

[ChC99] Luca Chiantini and Ciro Ciliberto, *On the severi varieties of surfaces in* $\mathbb{P}^3$, J. Algebraic Geom. **8** (1999), no. 1, 67–83.

[ChC02] Luca Chiantini and Ciro Ciliberto, *Weakly defective varieties*, Trans. Amer. Math. Soc. **354** (2002), 151–178.

[ChM07a] Luca Chiantini and Thomas Markwig, *Triple-point defective regular surfaces*, Preprint arXive:0705.3912 (2007).

[ChM07b] Luca Chiantini and Thomas Markwig, *Triple-point defective ruled surfaces*, Preprint arXive:0705.3913 (2007).

[ChS97] Luca Chiantini and Edoardo Sernesi, *Nodal curves on surfaces of general type*, Math. Ann. **307** (1997), 41–56.

[Cil01] Ciro Ciliberto, *Linear systems with multiple assigned base points*, To appear in: Proceedings of the Conference Zero Dimensional Schemes and Applications, Feb. 2000, Naples, 2001.

[CiM98] Ciro Ciliberto and Rick Miranda, *Degenerations of planar linear systems*, J. Reine Angew. Math. **501** (1998), 191–220.

[CiM01] Ciro Ciliberto and Rick Miranda, *The Segre and Harbourne–Hirschowitz conjectures*, Applications of Algebraic Geometry to Coding Theory, Physics and Computation, NATO Science Series II (Mathematics, Physics and Chemistry), vol. 36, 2001, pp. 37–52.

[DeP00] Theo De Jong and Gerhard Pfister, *Local analytic geometry*, Vieweg, 2000.

[DeS07] Wolfram Decker and Frank-Olaf Schreyer, *Varieties, Gröbner bases and algebraic curves*, Saarbrücken, 2007.

[DVoL05a] Cindy De Volder and Antonio Laface, *Linear systems on generic K3 surfaces*, Bull. Belg. Math. Soc. **12** (2005), no. 4, 481–489.

[DVoL05b] Cindy De Volder and Antonio Laface, *Recent results on linear systems on generic K3 surfaces*, Rend. Semin. Mat. Torino **63** (2005), no. 1, 91–94.

[DVoL07] Cindy De Volder and Antonio Laface, *Linear systems on a class of anticanonical rational threefolds*, arXiv:0705.0659, 2007.

[DFS07] Alicia Dickenstein, Eva Maria Feichtner, and Bernd Sturmfels, *Tropical discriminants*, J. Amer. Math. Soc. **20** (2007), 1111-1133.

[Dra06] Jan Draisma, *A tropical approach to secant dimensions*, preprint math.AG/0605345, 2006.

[Eis96] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, no. 150, Springer, 1996.

[EKL04] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-archimedean amoebas and tropical varieties*, Preprint, math.AG/0408311, 2004.

[Eva98] Laurent Evain, *Une minoration du degré des courbes planes à singularités imposées*, Bulletin de la SMF **126** (1998), 525–543.

[Eva99] Laurent Evain, *La fonction de Hilbert de la réunion de 4^h gros points génériques de $\mathbb{P}^2_{\mathbb{C}}$*, J. Alg. Geom. **8** (1999), 787–796.

[Fla01] Flaminio Flamini, *Moduli of nodal curves on smooth surfaces of general type*, J. Algebraic Geom. **11** (2001), no. 4, 725–760.

[FlM01] Flaminio Flamini and Carlo Madonna, *Geometric linear normality for nodal curves on some projective surfaces*, Bollettino UMI **4-B** (2001), no. 8, 269–283.

[Fri98] Robert Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer, 1998.

[FrI01] Davide Franco and Giovanna Ilardi, *On multiosculating spaces*, Comm. in Alg. **29** (2001), no. 7, 2961–2976.

[FuP00] Luis Fuentes and Manuel Pedreira, *The projective theory of ruled surfaces*, math.AG/0006204, June 2000.

[GaH05] Sebastian Gann and Herwig Hauser, *Perfect bases for differential equations*, J. Symbolic Computation **40** (2005), 979–997.

[GaM05] Andreas Gathmann and Hannah Markwig, *Kontsevich's formula and the WDVV equations in tropical geometry*, Preprint, math.AG/0509628, 2005.

[GaM07a] Andreas Gathmann and Hannah Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, To appear in Math. Ann., math.AG/0504392, 2007.

[GaM07b] Andreas Gathmann and Hannah Markwig, *The numbers of tropical plane curves through points in general position*, J. Reine Angew. Math. **602** (2007), 155–177.

[GGM$^+$94] Hubert Grassmann, Gert-Martin Greuel, Bernd Martin, Wolfgang Neumann, Gerhard Pfister, Wilfred Pohl, Hans Sch"onemann, and Thomas Siebert, *Standard bases, syzygies and their implementation in* SINGULAR, Beitr"age zur angewandten Analysis und Informatik, Shaker, Aachen, 1994, pp. 69–96.

[Gim87] Alessandro Gimigliano, *On linear systems of plane curves*, Ph.D. thesis, Queen's University, Kingston, Ontario, 1987.

[GLS97] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices **13** (1997), 595–611.

[GLS98a] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Geometry of families of nodal curves on the blown-up projective plane*, Trans. Amer. Math. Soc. **350** (1998), 251–274.

[GLS98b] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Plane curves of minimal degree with prescribed singularities*, Inv. math. **133** (1998), 539–580.

[GLS00] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Castelnuovo function, zero-dimensional schemes, and singular plane curves*, J. Algebraic Geom. **9** (2000), no. 4, 663–710.

[GLS01] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *The variety of plane curves with ordinary singularities is not irreducible*, Intern. Math. Res. Notes **11** (2001), 542–550.

[GLS05] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Singular algebraic curves*, Springer, 2005.

[GPS05] G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR *3.0*, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2005, `http://www.singular.uni-kl.de`.

[Grä94] Hans-Gert Gr"abe, *The tangent cone algorithm and homogenization*, J. Pure Appl. Algebra **97** (1994), no. 3, 303–312.

[Gra72] Hans Grauert, *Über die Deformationen isolierter Singularitäten analytischer Mengen*, Inv. math. **15** (1972), 171–198.

[GrH85] Philipp Griffiths and Joe Harris, *On the Noether–Lefschetz theorem and some remarks on codimension-two cycles*, Math. Ann. **271** (1985), no. 1, 109–205.

[GrK89] Gert-Martin Greuel and Ulrich Karras, *Families of varieties with prescribed singularities*, Comp. math. **69** (1989), 83–110.

[GrL96] Gert-Martin Greuel and Christoph Lossen, *Equianalytic and equisingular families of curves on surfaces*, Manuscripta Math. **91** (1996), 323–342.

[GrP96] Gert-Martin Greuel and Gerhard Pfister, *Advances and improvements in the theory of standard bases and syzygies*, Arch. Math. **66** (1996), 163–176.

[GrP02] Gert-Martin Greuel and Gerhard Pfister, *A* SINGULAR *introduction to commutative algebra*, Springer, 2002.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer, 1977.

[Har85] Joe Harris, *On the Severi problem*, Inventiones Math. **84** (1985), 445–461.

[Har86] Brian Harbourne, *The geometry of rational surfaces and Hilbert functions of points in the plane*, Can. Math. Soc. Conf. Proc. **6** (1986), 95–111.

[Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Annals of Math. **79** (1964), no. 1, 109–203.

[Hir77] Heisuke Hironaka, *Idealistic exponents of singularity*, J. J. Sylvester Sympos.,Johns Hopkins Univ., Baltimore, Md., 1976, Johns Hopkins Univ. Press, 1977, pp. 52–125.

[Hir85] Alexandre Hirschowitz, *La méthode d'Horace pour l'interpolation à plusieurs variables*, Manuscripta Math. **50** (1985), 337–388.

[Hir89] Alexandre Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques*, J. Reine Angew. Math. **397** (1989), 208–213.

[Hir92] Atsuko Hirano, *Constructions of plane curves with cusps*, Saitama Math. J. **10** (1992), 21–24.

[Iar77] Anthony Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **10** (1977), no. 188, 1–97.

[IKS03] Ilia Itenberg, Viatcheslav Khalarmov, and Eugenii Shustin, *Welschinger invariants and enumeration of real rational curves*, Inter. Math. Res. Notices **49** (2003), 2639–2653.

[IKS04] Ilia Itenberg, Viatcheslav Khalarmov, and Eugenii Shustin, *Logarithmic equivalence of Welschinger invariants and Gromov-Witten invariants*, Preprint, math.AG/0407188, 2004.

[IKS06] Ilia Itenberg, Viatcheslav Khalarmov, and Eugenii Shustin, *A Caporaso-Harris type formula for Welschinger invariants of real toric Del Pezzo surfaces*, Preprint, math.AG/0608549, 2006.

[Jen07] Anders Nedergaard Jensen, *Gfan, a software system for Gröbner fans*, Available at `http://home.imf.au.dk/ajensen/software/gfan/gfan.html`, 2007.

[JMM07] Anders Needergard Jensen, Hannah Markwig, and Thomas Markwig, *An algorithm for lifting points in a tropical variety*, Preprint IMA, Minneapolis, arXiv:0705.2441, 2007.

[Kat06] Eric Katz, *A Tropical Toolkit*, preprint math.AG/0610878, 2006.

[Ked01] Kiran S. Kedlaya, *The algebraic closure of the power series field in positive characteristic*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3461–3470.

[Kei01] Thomas Keilen, *Families of curves with prescribed singularities*, Ph.D. thesis, Universität Kaiserslautern, 2001.

[Kei03] Thomas Keilen, *Irreducibility of equsingular families of curves*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3485–3512.

[Kei05a] Thomas Keilen, *Irreducibility of equsingular families of curves – improved conditions*, Comm. in Algebra **33** (2005), no. 2, 455–466.

[Kei05b] Thomas Keilen, *Smoothness of equisingular families of curves*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2467–2481.

[Kei06] Thomas Keilen, *Reducible families of curves with ordinary multiple points on surfaces in* $\mathbb{P}^3$, Comm. in Alg. **34** (2006), no. 5, 1921–1926.

[KeL05] Thomas Keilen and Christoph Lossen, *A new invariant for plane curve singularities*, Rend. Semin. Mat. Torino **63** (2005), no. 1, 15–42.

[KeT02] Thomas Keilen and Ilya Tyomkin, *Existence of curves with prescribed singularities*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1837–1860.

[KeM06] Michael Kerber and Hannah Markwig, *Counting tropical elliptic curves with fixed* j*-invariant*, Preprint, math.AG/0608472, 2006.

[LaB92] Herbert Lange and Christina Birkenhake, *Complex abelian varieties*, Grundlehren der mathematischen Wissenschaften, no. 302, Springer, 1992.

[Laf02] Antonio Laface, *On linear systems of curves on rational scrolls*, Geometriae Dedicata **90** (2002), no. 1, 127–144.

[Laf06] Antonio Laface, *Special systems through double points on an algebraic surface*, math.AG/0605108, 2006.

[LaM02] Antonio Lanteri and Raquel Mallavibarrena, *Jets of antimulticanonical bundles on Del Pezzo surfaces of degree* ≤ 2, Algebraic Geometry, A Volume in Memory of Paolo Francia, De Gruyter, 2002, pp. 258–276.

[LaU03a] Antonio Laface and Luca Ugaglia, *Quasi-homogeneous linear systems on* $\mathbb{P}^2$ *with base points of multiplicity* 5, Canad. J. of Math. **55** (2003), no. 3, 561–575.

[LaU03b] Antonio Laface and Luca Ugaglia, *Special linear systems on toric varieties*, math.AG/0310043, 2003.

[Laz97] Robert Lazarsfeld, *Lectures on linear series*, Complex Algebraic Geometry (János Kollár, ed.), IAS/Park City Mathematics Series, no. 3, Amer. Math. Soc., 1997, pp. 161–219.

[LJT73] Monique Lejeune-Jalabert and Bernard Teissier, *Tranversalité, polygone de Newton et installations*, Singularités à Cargeèse (Rencontre Singularités Géom. Anal., Inst. Études Sci. de Cargèse, 1972), Astérisque, no. 7+8, Soc. Math. France, 1973, pp. 75–119.

[Los98] Christoph Lossen, *The geometry of equisingular and equianalytic families of curves on a surface*, Phd thesis, FB Mathematik, Universität Kaiserslautern, Aug. 1998, http:// www. mathematik. uni-kl. de/ ˜lossen/ download/ Lossen002/ Lossen002.ps.gz.

[Lue87a] Ignacio Luengo, *The* μ*-constant stratum is not smooth*, Inventiones Math. **90** (1987), 139–152.

[Lue87b] Ignacio Luengo, *On the existence of complete families of plane curves, which are obstructed*, J. LMS **36** (1987), 33–43.

[Mar05] Thomas Markwig, *Some obstructed equisingular families of curves on surfaces in* $\mathbb{P}^3$, To appear in the Proceedings of the Conference "Singularities", 24th January–25th February, 2005, C.I.R.M. Luminy. arXiv:0706.2441, 2005.

[Mar06] Thomas Markwig, *A note on equimultiple deformations*, Preprint Kaiserslautern, arXiv:0705.3911, 2006.

[Mar07] Thomas Markwig, *Standard bases in* $k[[t_1,\ldots,t_m]][x_1,\ldots,x_n]$, Preprint IMA, Minneapolis, arXiv:0705.2436, 2007.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, no. 8, CUP, 1986.

[Mau80] Joseph Maurer, *Puiseux expansions for space curves*, Manuscripta Math. **32** (1980), 91–100.

[Mig00] Thierry Mignon, *Systèmes des courbes planes à singularités imposées: le cas des multplicités inférieures ou égales á quatre*, J. Pure Appl. Alg. **151** (2000), no. 2, 173–195.

[Mig01] Thierry Mignon, *An asymptotic existence theorem for plane curves with prescribed singularities*, J. Algebraic Geom. **10** (2001), no. 2, 281–297.

[Mik05] Grigory Mikhalkin, *Enumerative tropical geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), 313–377, Preprint, math.AG/0312530.

[Mir00] Rick Miranda, *A deformation theory approach to linear systems with general triple points*, Le Matematiche **LV** (2000), no. Fasciolo II, 259–270.

[Mor82] Teo Mora, *An algorithm to compute the equations of tangent cones*, Proceedings EUROCAM 82, Lecture Notes in Comp. Sci., 1982.

[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, PUP, 1966.

[Nag59] Masayoshi Nagata, *On the 14th problem of Hilbert*, Amer. J. Math. **33** (1959), 766–772.

[Nor83] Madhav V. Nori, *Zariski's conjecture and related problems*, Ann. scient. Éc. Norm. Sup. **16** (1983), 305–344.

[PaS04] Lior Pachter and Bernd Sturmfels, *Tropical geometry of statistical models*, Proceedings of the National Academy of SCiences **101** (2004), 16132–16137.

[Pay07] Sam Payne, *Fibers of tropicalization*, Preprint math.AG/0705.1732v1, 2007.

[Ran89] Ziv Ran, *Families of plane curves and their limits: Enriques' conjecture and beyond*, Annals of Math. **130** (1989), no. 1, 121–157.

[Rei97] Miles Reid, *Chapters on algebraic surfaces*, Complex Algebraic Geometry (János Kollár, ed.), IAS/Park City Mathematics Series, no. 3, Amer. Math. Soc., 1997, pp. 3–159.

[Roé01] Joaquim Roé, *On the existence of plane curves with imposed multiple points*, J. Pure Appl. Alg. **156** (2001), 115–126.

[Roé03] Joaquim Roé, *On the Nagata conjeture*, math.AG/0304124, 2003.

[Seg29] Beniamino Segre, *Estistenza e dimensione di sistemi continui di curve piane algebiche con dati caraterri*, Atti Acad. Naz. Lincei Rendiconti, Ser. 6, **10** (1929), 31–38.

[Seg62] Beniamino Segre, *Alcune questioni su insiemi finiti di punti in geometria algebrica*, Atti del Convegno Internazionale di Geometria Algebrica, Torino, 1961, 1962, pp. 15–33.

[Sev21] Francesco Severi, *Vorlesungen über Algebraische Geometrie*, Bibliotheca Mathematica Teubneriana, no. 32, Teubner, 1921.

[Shu87] Eugenii Shustin, *Versal deformation in the space of plane curves of fixed degree*, Funct. An. Appl. **21** (1987), 82–84.

[Shu91] Eugenii Shustin, *On manifolds of singular algebraic curves*, Selecta Math. Sov. **10** (1991), 27–37.

[Shu94] Eugenii Shustin, *Smoothness and irreducibility of varieties of algebraic curves with nodes and cusps*, Bull. SMF **122** (1994), 235–253.

[Shu96] Eugenii Shustin, *Smoothness and irreducibility of varieties of algebraic curves with ordinary singularities*, Israel Math. Conf. Proc., no. 9, Amer. Math. Soc., 1996, pp. 393–416.

[Shu97] Eugenii Shustin, *Smoothness of equisingular families of plane algebraic curves*, Math. Res. Not. **2** (1997), 67–82.

[Shu99] Eugenii Shustin, *Lower deformations of isolated hypersurface singularities*, Algebra i Analiz **10** (1999), no. 5, 221–249.

[Shu03] Eugenii Shustin, *Analytic order of singular and critical points*, Trans. Amer. Math. Soc. **356** (2003), no. 3, 953–985.

[Shu04] Eugenii Shustin, *A tropical calculation of Welschinger invariants of real toric Del Pezzo srufaces*, Preprint, math.AG/0406099, 2004.

[Spe05] David Speyer, *Tropical geometry*, Ph.D. thesis, University of California, Berkeley, 2005.

[SpS04a] David Speyer and Bernd Sturmfels, *The tropical grassmannian*, Adv. Geom. **4** (2004), 389–411.

[SpS04b] David Speyer and Bernd Sturmfels, *Tropical mathematics*, Preprint, math.CO/0408099, 2004.

[Stu02] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series, no. 97, Amer. Math. Soc., 2002.

[StY07] Bernd Sturmfels and Josephine Yu, *Tropical implicitisation and mixed fiber polytopes*, Preprint, arXive:0706.0564, 2007.

[Sze01] Tomasz Szemberg, *Global and local positvity of line bundles*, Ph.D. thesis, Habilitation, 2001.

[Tab05] Luis Felipe Tabera, *Tropical plane geometric constructions*, preprint math.AG/0511713, 2005.

[Tan80] Allen Tannenbaum, *Families of algebraic curves with nodes*, Compositio Math. **41** (1980), 107–126.

[Tan82] Allen Tannenbaum, *Families of curves with nodes on K3-surfaces*, Math. Ann. **260** (1982), 239–253.

[Tan04] Sheng-Li Tan, *Effective behaviour of multiple linear systems*, Asian J. Math. **8** (2004), no. 2, 287–304.

[Ter22] Alessandro Terracini, *Su due problemi concernenti la determinazione di alcune classi di superficie, considerate da G. Scorza e F. Platini*, Atti Soc. Natur. e Matem. Modena **6** (1921–22), 3–16.

[Vas90] Victor A. Vassiliev, *Stable cohomology of complements to the discriminants of deformations of singularities of smooth functions*, J. Sov. Math. **52** (1990), 3217–3230.

[Vig04] Magnus Dehli Vigeland, *The group law on a tropical elliptic curve*, Preprint, math.AG/0411485, 2004.

[Wah74a] Jonathan M. Wahl, *Deformations of plane curves with nodes and cusps*, Amer. J. Math. **96** (1974), 529–577.

[Wah74b] Jonathan M. Wahl, *Equisingular deformations of plane algebroid curves*, Trans. Amer. Math. Soc. **193** (1974), 143–170.

[Wes04] Eric Westenberger, *Families of hypersurfaces with many prescribed singularities*, Ph.D. thesis, TU Kaiserslautern, 2004, http://www.mathematik.uni-kl.de/˜westenb/files/Westenberger_Dissertation.ps.gz.

[Xu94] Geng Xu, *Curves in $\mathbb{P}^2$ and symplectic packings*, Math. Ann. **299** (1994), 609–613.

[Xu95] Geng Xu, *Ample line bundles on smooth surfaces*, J. reine angew. Math. **469** (1995), 199–209.

[Yan04] Stephanie Yang, *Linear systems in $\mathbb{P}^3$ with base points of bounded multiplicity*, math.AG0406591, 2004.

[Zar35] Oscar Zariski, *Algebraic surfaces*, vol. III, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 5, Springer, 1935.

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