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Irreducibility of Equisingular Families of Curves – Improved Conditions

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Abstract

In [7] we gave sufficient conditions for the irreducibility of the family $V_{|D|}^{irr}\big(\mathcal{S}_1,\ldots,\mathcal{S}_r\big)$ of irreducible curves in the linear system $|D|_l$ with precisely r singular points of topological respectively analytical types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ on several classes of smooth projective surfaces Σ. The conditions were of the form

$$\sum_{i=1}^{r}\big(\tau^*(\mathcal{S}_i)+2\big)^2<\gamma\cdot(D-K_\Sigma)^2,$$

where τ^* is some invariant of singularity types, K_Σ is the canonical divisor of Σ and γ is some constant. In the present paper we improve this condition, that is the constant γ, by a factor 9.

Key Words: Algebraic geometry, singularity theory, equisingular families of curves

I. Introduction

If we fix a linear system $|D|_l$ on a smooth projective surface Σ over the complex numbers $\mathbb{C}$ and singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ we denote by $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ the variety of irreducible curves in $|D|_l$ with precisely r singular points of the given types. We would like to give numerical conditions, depending on the singularity types, the linear system and the surface, which ensure that the family V^{irr} is irreducible, once it is non-empty.

In order to keep the presentation as short as possible we refer the reader to [7] for an introduction to the significance of the question and for most of the notation we are going to use. Moreover, we will apply many of the technical results shown there. The proof runs along the same lines as the original one by showing that some irreducible "regular" subscheme of V^{irr} is dense in V^{irr}. We do this again by considering a morphism Φ on a certain subscheme of V^{irr} and comparing dimensions. However, the subscheme which we consider and the morphism are completely different.

In [7] the morphism associated to each curve C in V^{irr} its singularity scheme $X(C)$, and the regular part $V^{irr,reg}$ consisted of the curves belonging to smooth fibres of expected dimension. Since $V^{irr,reg}$ is irreducible by default, only conditions were needed which ensured the vanishing of $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(C)\big)$ generically. In the present paper we consider the morphism which associates to each curve C the set of singular points, and again the regular part $V^{irr,fix}$ of V^{irr} consists of the smooth fibres of expected dimension. Since $V^{irr,fix}$ will not be irreducible in general, we have to find conditions which ensure that it is dense in V^{irr} and that it shares a dense subset with $V^{irr,reg}$. The first part is the hard part and comes down a generic h^1-vanishing for the ideal sheaves of the schemes $X^*_{fix}(C)$. For this we use the technique of Bogomolov unstable vector bundles first applied in [2].

The general ideas for this approach in the case of planar curves can be found in [4], which has not been published since in the plane case better conditions could be derived replacing the Bogomolov arguments by the Castelnuovo function (see [5]), which can not so easily be generalised to other surfaces. We combine these ideas with the technical lemmata developed for surfaces with Picard number one, products of curves and geometrically ruled surfaces in [7].

The techniques applied in [7] and in this paper give conditions which depend on the square of the local degree of the involved zero-dimensional scheme, i. e. they depend on $\deg\big(X(C),z\big)^2$ respectively on $\deg\big(X^*_{fix}(C),z\big)^2 = \big(\tau^*(C,z)+2\big)^2$. The degree of $X(C)$ at z is not precisely known in general, so in [7] it is replaced by the best possible general upper bound $3\cdot\tau^*(C,z)+2$. The conditions this way are improved by a factor 9.

We now introduce the new objects. In Section II we formulate the main results, and they are proved in Section III. Lemma III.1 is the most important technical adjustment which leads to the improved coefficient.

A. The Deformation Determinacy

If $\mathcal{S}$ is a topological (respectively analytical) singularity type with representative (C,z) then

$$\nu^s(\mathcal{S}) = \nu^s(C,z) = \min\big\{m\geq 0 \;\big|\; \mathfrak{m}_{\Sigma,z}^{m+1}\subseteq I^s(C,z)\big\}$$

respectively

$$\nu^a(\mathcal{S}) = \nu^a(C,z) = \min\big\{m\geq 0 \;\big|\; \mathfrak{m}_{\Sigma,z}^{m+1}\subseteq I^a(C,z)\big\},$$

where $I^s(C,z) = \mathcal{J}_{X^s(C)/\Sigma,z}$ is the singularity ideal of the topological singularity type (C,z) and $I^a(C,z) = \mathcal{J}_{X^a(C)/\Sigma,z}$ is the analytical singularity ideal of (C,z) respectively (cf. [7] Section 1.3). These are invariants of the topological (respectively analytical) singularity type satisfying (cf. [5] Section 1.2 and 1.3)

$$\nu^s(\mathcal{S})\leq\tau^{es}(\mathcal{S}) \quad \text{respectively} \quad \nu^a(\mathcal{S})\leq\tau(\mathcal{S}),$$

and they are called *topological deformation determinacy* (respectively *analytical deformation determinacy*)

B. Singularity Schemes

For a reduced curve $C \subset \Sigma$ we recall the definition of the zero-dimensional schemes $X^{es}_{fix}(C)$ and $X^{ea}_{fix}(C)$ from [5] Section 1.1. They are defined by the ideal sheaves $\mathcal{J}_{X^{es}_{fix}(C)/\Sigma}$ and $\mathcal{J}_{X^{ea}_{fix}(C)/\Sigma}$ respectively, given by the following stalks

- $\mathcal{J}_{X^{es}_{fix}(C)/\Sigma,z} = I^{es}_{fix}(C,z) = \Big\{ g \in \mathcal{O}_{\Sigma,z} \;\Big|\; {\scriptstyle f+\varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \atop \text{along the trivial section}} \Big\}$, where $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of C at z.
- $\mathcal{J}_{X^{ea}_{fix}(C)/\Sigma,z} = I^{ea}_{fix}(C,z) = \langle f \rangle + \mathfrak{m} \cdot \big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \big\rangle \subseteq \mathcal{O}_{\Sigma,z}$, where x, y denote local coordinates of Σ at z and $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of C.

So by definition we have

$$\deg\big(X^{es}_{fix}(C), z\big) = \tau^{es}(C,z) + 2$$

and

$$\deg\big(X^{ea}_{fix}(C), z\big) = \tau(C,z) + 2.$$

Throughout this article we will frequently treat topological and analytical singularities at the same time. Whenever we do so, we will write $X^{*}_{fix}(C)$ for $X^{es}_{fix}(C)$ respectively for $X^{ea}_{fix}(C)$, we will write $\nu^*(\mathcal{S})$ for $\nu^s(\mathcal{S})$ respectively $\nu^a(\mathcal{S})$, and we will write $\tau^*(\mathcal{S})$ for $\tau^{es}(\mathcal{S})$ respectively $\tau(\mathcal{S})$. For the schemes borrowed from [7] we stick to the analogous convention made there.

C. Equisingular Families

Given a divisor $D \in \mathrm{Div}(\Sigma)$ and topological (respectively analytical) singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$.

We denote by $V^{irr,fix} = V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ the open subscheme of V^{irr} given as

$$V^{irr,fix} = \big\{ C \in V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r) \;\big|\; h^1\big(\Sigma, \mathcal{J}_{X^{*}_{fix}(C)/\Sigma}(D)\big) = 0 \big\}.$$

We define the map $\Phi = \Phi_D(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ by

$$\Phi : V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r) \longrightarrow \operatorname{Sym}^r(\Sigma) : C \longmapsto \operatorname{Sing}(C),$$

sending a curve C to the unordered tuple of its singular points.

Note that $H^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big)/H^0(\mathcal{O}_\Sigma)$ is the tangent space of the fibre $\Phi^{-1}\big(\Phi(C)\big)$ at $C \in V^{irr}$, so that

$$\dim\Big(\Phi^{-1}\big(\Phi(C)\big)\Big) \leq h^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big) - 1. \tag{I.1}$$

Moreover, suppose that $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) = 0$, then the germ of the fibration at C

$$(\Phi, C) : \big(V, C\big) \to \big(\operatorname{Sym}^r(\Sigma), \operatorname{Sing}(C)\big)$$

is smooth of fibre dimension $h^0\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)}(D)\big) - 1$, i. e. locally at C the morphism Φ is a projection of the product of the smooth base space with the smooth fibre. This implies in particular, that close to C there is a curve having its singularities in very general position. (Cf. [9] Proposition 2.1 (e).)

II. The Main Results

In this section we give sufficient conditions for the irreducibility of equisingular families of curves on certain surfaces with Picard number one – including the projective plane, general surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and general K3-surfaces –, on products of curves, and on a subclass of geometrically ruled surfaces.

A. Surfaces with Picard Number One

Theorem II.1
Let Σ be a surface such that

(i) $\operatorname{NS}(\Sigma) = L \cdot \mathbb{Z}$ *with L ample, and*
(ii) $h^1(\Sigma, C) = 0$*, whenever C is effective.*

Let $D \in \mathrm{Div}(\Sigma)$*, let* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological (respectively analytical) singularity types.*

Suppose that

(II.1) $D - K_\Sigma$ *is big and nef,*

(II.2) $D + K_\Sigma$ *is nef,*

(II.3) $\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big) < \beta \cdot (D - K_\Sigma)^2$ *for some* $0 < \beta \leq \frac{1}{4}$*, and*

(II.4) $\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \gamma \cdot (D - K_\Sigma)^2$,

$$\textit{where } \gamma = \frac{\big(1+\sqrt{1-4\beta}\big)^2 \cdot L^2}{4 \cdot \chi(\mathcal{O}_\Sigma) + \max\{0, 2 \cdot K_\Sigma.L\} + 6 \cdot L^2}.$$

Then either $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is empty, or it is irreducible of the expected dimension.* □

Remark II.2
If we set

$$\gamma = \frac{36\alpha}{(3\alpha+4)^2} \quad \text{with} \quad \alpha = \frac{4 \cdot \chi(\mathcal{O}_\Sigma) + \max\{0, 2 \cdot K_\Sigma.L\} + 6 \cdot L^2}{L^2},$$

then a simple calculation shows that (II.3) becomes redundant. For this we have to take into account that $\tau^*(\mathcal{S}) \geq 1$ for any singularity type $\mathcal{S}$. The claim then follows with $\beta = \frac{1}{3} \cdot \gamma \leq \frac{1}{4}$. □

We now apply the result in several special cases, combining the above theorem with the existence results in [8] and the T-smoothness results in [3].

Corollary II.3
Let $d \geq 3$*,* $L \subset \mathbb{P}^2_{\mathbb{C}}$ *be a line, and* $\mathcal{S}_1, \ldots, \mathcal{S}_r$ *be topological or analytical singularity types.*

Suppose that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 < \tfrac{90}{289} \cdot (d+3)^2.$$

Then $V_{|dL|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ *is non-empty, irreducible and T-smooth.* □

The best general results in this case can still be found in [5] (see also [9] Corollary 6.1), where the coefficient on the right hand side is $\frac{9}{10}$.

A smooth complete intersection surface with Picard number one satisfies the assumptions of Theorem II.1. Thus by the Theorem of Noether the result applies in particular to general surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

Corollary II.4
Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$, let $H \subset \Sigma$ be a hyperplane section, and suppose that the Picard number of Σ is one. Let $d \geq n+6$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological (respectively analytical) singularity types.

Suppose that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big)^2 < \frac{6\cdot\big(n^3-3n^2+8n-6\big)\cdot n^2}{\big(n^3-3n^2+10n-6\big)^2} \cdot (d+4-n)^2,$$

Then $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty and irreducible of the expected dimension.

Moreover, if we assume additionally $d \geq n \cdot \big(\tau^*(\mathcal{S}_i)+1\big)$ for all $i = 1, \ldots, r$, then $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is also T-smooth.

A general K3-surface has Picard number one and in this situation, by the Kodaira Vanishing Theorem Σ also satisfies the assumption (ii) in Theorem II.1.

Corollary II.5
Let Σ be a smooth K3-surface with $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$ with L ample and set $n = L^2$. Let $d > 0$, $D \sim_a dL$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological (respectively analytical) singularity types.

Suppose that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i)+2\big)^2 < \tfrac{54n^2+72n}{(11n+12)^2} \cdot d^2 \cdot n.$$

Then $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is irreducible and T-smooth, once it is non-empty.

Moreover, if $d \geq 19$, then $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty. $\square$

B. Products of Curves

If Σ is the product of smooth projective curves, then for a general choice of the curves the Néron–Severi group $\mathrm{NS}(\Sigma)$ will be generated by two fibres of the canonical projections.If both curves are elliptic, then "general" just means that the two curves are non-isogenous.

Theorem II.6
Let C_1 and C_2 be smooth projective curves of genera g_1 respectively g_2 with $g_1 \geq g_2 \geq 0$, such that $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$ for $\Sigma = C_1 \times C_2$.

Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_1 + bC_2$ with

$$a \geq \begin{cases} \max\left\{2, \nu^*(\mathcal{S}_i) \;\middle|\; i = 1, \ldots, r\right\}, & \text{if } g_2 = 0, \\ 2g_2 - 1, & \text{else,} \end{cases}$$

and

$$b \geq \begin{cases} \max\left\{2, \nu^*(\mathcal{S}_i) \;\middle|\; i = 1, \ldots, r\right\}, & \text{if } g_1 = 0, \\ 2g_1 - 1, & \text{else.} \end{cases}$$

Suppose that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{II.5}$$

where γ may be taken from the following table with $\alpha = \frac{a-2g_2+2}{b-2g_1+2} > 0$.

g_1	g_2	γ
0	0	$\frac{1}{24}$
1	0	$\frac{1}{\max\{32,2\alpha\}}$
≥ 2	0	$\frac{1}{\max\{24+16g_1,4g_1\alpha\}}$
1	1	$\frac{1}{\max\left\{32,2\alpha,\frac{2}{\alpha}\right\}}$
≥ 2	≥ 1	$\frac{1}{\max\left\{24+16g_1+16g_2,4g_1\alpha,\frac{4g_2}{\alpha}\right\}}$

Then either $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty, or it is irreducible of the expected dimension. $\square$

C. Geometrically Ruled Surfaces

Let $\pi : \Sigma = \mathbb{P}_{\mathbb{C}}(\mathcal{E}) \to C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [6] V.2.8.1). The Néron–Severi group of Σ is $\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z}$ with intersection matrix $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where $F \cong \mathbb{P}^1_{\mathbb{C}}$ is a fibre of π, C_0 a section of π with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}(\mathcal{E})}(1)$, $g = g(C)$ the genus of C, $\mathfrak{e} = \Lambda^2\mathcal{E}$ and $e = -\deg(\mathfrak{e}) \geq -g$. For the canonical divisor we have $K_\Sigma \sim_a -2C_0 + (2g-2-e) \cdot F$.

Theorem II.7
Let $\pi : \Sigma \to C$ be a geometrically ruled surface with $e \leq 0$. Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_0 + bF$ with $a \geq \max\big\{2, \nu^*(\mathcal{S}_i) \,\big|\, i = 1, \ldots, r\big\}$, and,

$$b > \begin{cases} \max\big\{1, \nu^*(\mathcal{S}_i) - 1 \,\big|\, i = 1, \ldots, r\big\}, & \text{if } g = 0, \\ 2g - 2 + \frac{ae}{2}, & \text{if } g > 0. \end{cases}$$

Suppose that

$$\sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{II.6}$$

where γ may be taken from the table below with $\alpha = \frac{a+2}{b+2-2g-\frac{ae}{2}} > 0$.

g	e	γ
0	0	$\frac{1}{24}$
1	0	$\frac{1}{\max\{24, 2\alpha\}}$
1	-1	$\frac{1}{\max\left\{\min\left\{30+\frac{16}{\alpha}+4\alpha, 40+9\alpha\right\}, \frac{13}{2}\alpha\right\}}$
≥ 2	0	$\frac{1}{\max\{24+16g, 4g\alpha\}}$
≥ 2	< 0	$\frac{1}{\max\left\{\min\left\{24+16g-9e\alpha, 18+16g-9e\alpha-\frac{16}{e\alpha}\right\}, 4g\alpha-9e\alpha\right\}}$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty, or it is irreducible of the expected dimension. $\square$

For geometrically ruled surfaces as well as for products of curves, only when $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, we are in the lucky situation that the constant γ does not at all depend on the chosen divisor D, whereas otherwise the ratio of a and b is involved in γ. This means that an asymptotical behaviour can only be examined if the ratio remains unchanged.

If Σ is a product $C \times \mathbb{P}^1_{\mathbb{C}}$ the constant γ derived here is the same as in Section B.

III. The Proofs

Our approach to the problem proceeds along the lines of an unpublished result of Greuel, Lossen and Shustin (cf. [4]), which is based on ideas of Chiantini and Ciliberto (cf. [1]). It is a slight modification of the proof given in [7]. We tackle the problem in three steps:

Step 1: By [7] Theorem 3.1 we know that the open subvariety $V^{irr,reg}$ of curves in V^{irr} with $h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0$ is always irreducible, and hence so is its closure in V^{irr}.

Step 2: We find conditions which ensure that the open subvariety $V^{irr,fix}$ of curves in V^{irr} with $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) = 0$ is dense in V^{irr}.

Step 3: And finally, we combine these conditions with conditions which guarantee that $V^{irr,reg}$ is dense in $V^{irr,fix}$ by showing that they share some open dense subset V_U^{gen} of curves with singularities in very general position.

More precisely, taking Lemma III.2 into account, we deduce from Lemma III.3 conditions which ensure that there exists a very general subset $U \subset \Sigma^r$ such that the family $V_U^{gen} = V_{|D|,U}^{gen}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, as defined there, satisfies

(a) V_U^{gen} is dense in $V^{irr,fix}$, and
(b) $V_U^{gen} \subseteq V^{irr,reg}$.

But then $V^{irr,reg}$ is dense in V^{irr}, which is irreducible by Step 1. $\square$

The difficult part is Step 2. For this one we consider the restriction of the morphism (cf. Subsection C)

$$\Phi : V^{irr} \to \mathrm{Sym}^r(\Sigma) =: \mathcal{B}$$

to an irreducible component V^* of V^{irr} not contained in the closure $\overline{V^{irr,fix}}$ in V^{irr}. Knowing, that the dimension of V^* is at least the expected dimension $\dim\big(V^{irr,fix}\big)$ we deduce that the codimension of $\Phi\big(V^*\big)$ in $\mathcal{B}$ is at most $h^1\big(\Sigma, \mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$, where $C \in V^*$ (cf. Lemma III.1). It thus suffices to find conditions which contradict this inequality, that is, we have to get our hands on $\mathrm{codim}_{\mathcal{B}}\big(\Phi(V^*)\big)$. This is achieved by applying the results of [7] Lemma 4.1 to Lemma 4.6 to the zero-dimensional scheme $X_0 = X^*_{fix}(C)$.

These considerations lead to the following proofs.

Proof of Theorem II.1: We may assume that the family $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty. As indicated above it suffices to show that:

Step 2: $V^{irr} = \overline{V^{irr,fix}}$, where $V^{irr,fix} = V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, and

Step 3: the conditions of Lemma III.3 are fulfilled.

For Step 3 we note that $\nu^*(\mathcal{S}_i) \leq \tau^*(\mathcal{S}_i)$. Thus (II.4) implies that

$$\begin{aligned}\sum_{i=1}^{r} \big(\nu^*(\mathcal{S}_i) + 2\big)^2 \leq \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2 \\ \leq \gamma \cdot (D - K_\Sigma)^2 \leq \tfrac{1}{2} \cdot (D - K_\Sigma)^2,\end{aligned}$$

which gives the first condition in Lemma III.3. Since a surface with Picard number one has no curves of selfintersection zero, the second condition in Lemma III.3 is void, while the last condition is satisfied by (II.1).

It remains to show Step 2, i. e. $V^{irr} = \overline{V^{irr,fix}}$. Suppose the contrary, that is, there is an irreducible curve $C_0 \in V^{irr} \setminus \overline{V^{irr,fix}}$, in particular $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$ for $X_0 = X^*_{fix}(C_0)$. Since $\deg(X_0) = \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)$ and $\sum_{z\in\Sigma} \big(\deg(X_{0,z})\big)^2 = \sum_{i=1}^{r} \big(\tau^*(\mathcal{S}_i) + 2\big)^2$ the assumptions (0)-(3) of [7] Lemma 4.1 and (4) of [7] Lemma 4.3 are fulfilled. Thus [7] Lemma 4.3 implies that C_0 satisfies Condition (III.1) in Lemma III.1 below, which it cannot satisfy by the same Lemma. Thus we have derived a contradiction. $\square$

Proof of Theorem II.6: The assumptions on a and b ensure that $D - K_\Sigma$ is big and nef and that $D + K_\Sigma$ is nef. Thus, once we know that (II.5) implies Condition (3) in [7] Lemma 4.1 we can do the same proof as in Theorem II.1, just replacing [7] Lemma 4.3 by [7] Lemma 4.4.

For Condition (3) we note that

$$\sum_{i=1}^r \deg\big(X^*_{fix}(\mathcal{S}_i)\big) \leq \sum_{i=1}^r \big(\tau^*(\mathcal{S}_i)+2\big)^2$$
$$\leq \tfrac{1}{24}\cdot(D-K_\Sigma)^2 < \tfrac{1}{4}\cdot(D-K_\Sigma)^2.$$

$\square$

Proof of Theorem II.7: The proof is identical to that of Theorem II.6, just replacing [7] Lemma 4.4 by [7] Lemma 4.6. $\square$

A. Some Technical Lemmata

We have applied [7] Lemma 4.1 to Lemma 4.6 to the zero-dimensional scheme $X_0 = X^*_{fix}(C)$, for a curve $C \in V^{irr} \setminus \overline{V^{irr,fix}}$, in order to find with the aid of Bogomolov instability curves Δ_i and subschemes $X_i^0 \subseteq X_i$, where $X_i = X_{i-1} : \Delta_i$, such that for $X_S = \bigcup_{i=1}^m X_i^0$

$$h^1\big(\Sigma, \mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma, \mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) < \#X_S.$$

And we are now going to show that this simply is not possible.

Lemma III.1
Let $D \in \operatorname{Div}(\Sigma)$, $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be pairwise distinct topological (respectively analytical) singularity types. Suppose that $V^{irr,fix}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is non-empty.

Then there is no curve[*] $C \in V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r) \setminus \overline{V_{|D|}^{irr,fix}(\mathcal{S}_1,\ldots,\mathcal{S}_r)}$ *such that for the zero-dimensional scheme* $X_0 = X^*_{fix}(C)$ *there exist curves* $\Delta_1,\ldots,\Delta_m \subset \Sigma$ *and zero-dimensional locally complete intersections* $X_i^0 \subseteq X_{i-1}$ *for* $i=1,\ldots,m$*, where* $X_i = X_{i-1} : \Delta_i$ *for* $i=1,\ldots,m$ *such that* $X_S = \bigcup_{i=1}^m X_i^0$ *satisfies*

$$h^1\big(\Sigma,\mathcal{J}_{X_0}(D)\big) + \sum_{i=1}^m \Big(h^0\big(\Sigma,\mathcal{O}_\Sigma(\Delta_i)\big) - 1\Big) < \#X_S. \qquad \text{(III.1)}$$

Proof: Throughout the proof we set $V^{irr} = V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ and $V^{irr,fix} = V_{|D|}^{irr,fix}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$.

Suppose there exists a curve $C \in V^{irr} \setminus \overline{V^{irr,fix}}$ satisfying the assumption of the Lemma, and let V^* be the irreducible component of V^{irr} containing C. Moreover, let $C_0 \in V^{irr,fix}$.

We consider in the following the morphism from Subsection C

$$\Phi = \Phi_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r) : V_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r) \to \mathrm{Sym}^r(\Sigma) =: \mathcal{B}.$$

Step 1: $h^0\big(\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) = h^0\big(\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) - h^1\big(\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)$.

By the choice of C_0 we have the exact sequence

$$H^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) \to H^1\big(\Sigma,\mathcal{O}_\Sigma(D)\big) \to H^1\big(\Sigma,\mathcal{O}_{X^*_{fix}(C_0)}(D)\big)$$

where the groups on the left and right side vanish, and thus D is non-special, i. e. $h^1\big(\Sigma,\mathcal{O}_\Sigma(D)\big) = 0$. But then

$$\begin{aligned} h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big) &= h^0\big(\Sigma,\mathcal{O}_\Sigma(D)\big) - \deg\big(X^*_{fix}(C_0)\big) \\ &= h^0\big(\Sigma,\mathcal{O}_\Sigma(D)\big) - \deg\big(X^*_{fix}(C)\big) \\ &= h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big). \end{aligned}$$

Step 2: $h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)}(D)\big) \geq \mathrm{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big)$.

Suppose the contrary, that is

$$\dim\Big(\Phi\big(V^*\big)\Big) < \dim(\mathcal{B}) - h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big).$$

---

[*] For a subset $U \subseteq V$ of a topological space V we denote by $\overline{U}$ the closure of U in V.

The vanishing of $h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big)$ implies that V^{irr} is smooth of the expected dimension $\dim\big(V^{irr,fix}\big)$ at C_0. Therefore, and by Step 1 and Equation (I.1) we have

$$\begin{aligned}\dim\big(V^*\big) &\leq \dim\Big(\Phi\big(V^*\big)\Big)+\dim\Big(\Phi^{-1}\big(\Phi(C)\big)\Big)\\ &< \dim(\mathcal{B})-h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)+h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)/\Sigma}(D)\big)-1\\ &= 2r+h^0\big(\Sigma,\mathcal{J}_{X^*_{fix}(C_0)/\Sigma}(D)\big)-1=\dim\big(V^{irr,fix}\big).\end{aligned}$$

However, any irreducible component of V^{irr} has at least the expected dimension $\dim\big(V^{irr,fix}\big)$, which gives a contradiction.

Step 3: $\operatorname{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big)\geq \#X_S-\sum_{i=1}^m\dim|\Delta_i|_l$.

The existence of the subschemes $X_i^0\subseteq X^*_{fix}(C)\cap\Delta_i$ imposes at least $\#X_i^0-\dim|\Delta_i|_l$ conditions on $X^*_{fix}(C)$ and increases thus the codimension of $\Phi\big(V^*\big)$ by the same number.

Step 4: Derive a contradiction.

Collecting the results we derive the following contradiction:

$$\begin{aligned}h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)}(D)\big) &\geq_{\text{Step 2}} \operatorname{codim}_{\mathcal{B}}\Big(\Phi\big(V^*\big)\Big)\\ &\geq_{\text{Step 3}} \#X_S-\textstyle\sum_{i=1}^m\dim|\Delta_i|_l >_{\text{(III.1)}} h^1\big(\Sigma,\mathcal{J}_{X^*_{fix}(C)}(D)\big).\end{aligned}$$

□

The next two lemmata provide conditions which ensure that $V^{irr,reg}$ and $V^{irr,fix}$ share some dense subset V_U^{gen}, and thus that $V^{irr,reg}$ is dense in $V^{irr,fix}$.

Lemma III.2
Let $\mathcal{S}_1,\ldots,\mathcal{S}_r$ be topological (respectively analytical) singularity types, let $D\in\operatorname{Div}(\Sigma)$ and let $V^{irr}=V^{irr}_{|D|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$.

There exists a very general subset $U\subset\Sigma^r$ such that the family[1]

[1] Here $\sim$ means either topological equivalence $\sim_t$ or contact equivalence $\sim_c$. – By a very general subset of Σ^r we mean the complement of at most countably many closed subvarieties.

$V_U^{gen} = V_{|D|,U}^{gen}(\mathcal{S}_1,\dots,\mathcal{S}_r) = \big\{C \in V^{irr} \big| \underline{z} \in U, (C,z_i) \sim S_i, 1 \le i \le r\}$ *is dense in* $V_{|D|}^{irr,fix}(\mathcal{S}_1,\dots,\mathcal{S}_r)$.

Proof: This follows from the remark in Subsection C. □

Lemma III.3
With the notation of Lemma III.2 we assume that

(a) $(D - K_\Sigma)^2 \ge 2 \cdot \sum_{i=1}^k \big(\nu^*(\mathcal{S}_i)+2\big)^2$,
(b) $(D-K_\Sigma).B > \max\big\{\nu^*(\mathcal{S}_i)+1 \;\big|\; i=1,\dots,r\big\}$ *for any irreducible curve* B *with* $B^2 = 0$ *and* $\dim|B|_a > 0$, *and*
(c) $D - K_\Sigma$ *is nef.*

Then there exists a very general subset $U \subset \Sigma^r$ *such that* $V_U^{gen} \subseteq V_{|D|}^{irr,reg}(\mathcal{S}_1,\dots,\mathcal{S}_r)$.

Proof: By [8] Theorem 2.1 we know that there is a very general subset $U \subset \Sigma^r$ such that for $\underline{z} \in U$ and $\underline{\nu} = \big(\nu^*(\mathcal{S}_1)+1,\dots,\nu^*(\mathcal{S}_r)+1\big)$ we have

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma}(D)\big) = 0.$$

However, if $C \in V^{irr}$ and $\underline{z} \in U$ with $(C,z_i) \sim \mathcal{S}_i$, then by the definition of $\nu^*(\mathcal{S}_i)$ we have

$$\mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma} \hookrightarrow \mathcal{J}_{X(C)/\Sigma},$$

and hence the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X(\underline{\nu};\underline{z})/\Sigma}(D)\big)$ implies

$$h^1\big(\Sigma, \mathcal{J}_{X(C)/\Sigma}(D)\big) = 0,$$

i. e. $C \in V^{irr,reg}$. □

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