REDUCIBLE FAMILIES OF CURVES WITH ORDINARY MULTIPLE POINTS ON SURFACES IN $\mathbb{P}^3_{\mathbb{C}}$

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Abstract. In Keilen (2003), (2005) and (2004) we gave numerical conditions which ensure that an equisingular family is irreducible respectively T-smooth. Combining results from Greuel, Lossen and Shustin (2001) and an idea from Chiantini and Ciliberto (1999) we give in the present paper series of examples of families of irreducible curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ with only ordinary multiple points which are reducible and where at least one component does not have the expected dimension. The examples show that for families of curves with ordinary multiple points the conditions for T-smoothness in Keilen (2004) have the right asymptotics.

Throughout this article Σ will denote a smooth projective surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 2$, and H will be a hyperplane section of Σ. For a positive integer m we denote by M_m the topological singularity type of an ordinary m-fold point, i. e. the singularity has m smooth branches with pairwise different tangents. And for positive integers d and r we denote by $V^{irr}_{|dH|}(rM_m)$ the family of irreducible curves in the linear system $|dH|$ with precisely r singular points all of which are ordinary m-fold points. $V^{irr}_{|dH|}(rM_m)$ is called T-smooth if it is smooth of the expected dimension

$$\operatorname{expdim}\left(V^{irr}_{|dH|}(rM_m)\right) = \dim|dH| - r \cdot \frac{m^2+m-4}{2}.$$

Theorem 1. *For $m \geq 18n$ there is an integer $l_0 = l_0(m,\Sigma)$ such that for all $l \geq l_0$ the family $V^{irr}_{|dH|}(rM_m)$ with $d = 2lm + l$ and $r = 4l^2n$ has at least one T-smooth component and one component of higher dimension.*

Moreover, the T-smooth component dominates $\operatorname{Sym}^r(\Sigma)$ under the map

$$V^{irr}_{|dH|}(rM_m) \longrightarrow \operatorname{Sym}^r(\Sigma) : C \mapsto \operatorname{Sing}(C)$$

sending a curve C to its singular locus, and the fundamental group $\pi_1(\Sigma \setminus C)$ of the complement of any curve $C \in V^{irr}_{|dH|}(rM_m)$ is abelian.

Before we prove the theorem let us compare the result with the conditions for T-smoothness in Keilen (2004) and for irreducibility in Keilen (2005).

Here we have given examples of non-T-smooth families $V_{|dH|}(rM_m)$ where

$$r \cdot m^2 \equiv n \cdot d^2,$$

Date: September, 2003.

1991 *Mathematics Subject Classification.* Primary 14H10, 14H15, 14H20; Secondary 14J26, 14J27, 14J28, 14J70.

Key words and phrases. Equisingular families of curves, fat point schemes.

The results were obtained during a research visit at the universities of Siena and Torino supported by the EAGER-node of North Italy.

if we neglect the terms of lower order in ml. If $n \geq 4$ and the Picard number of Σ is one, then according to Keilen (2004) Corollary 2.3 respectively Corollary 2.4 – neglecting terms of lower order in m and d –

$$r \cdot m^2 < \frac{1}{2n-6} \cdot n \cdot d^2$$

would be a sufficient condition for T-smoothness. Similarly, if $n = 2$, then Σ is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and we may apply Keilen (2004) Theorem 2.5 to find that

$$r \cdot m^2 < \frac{1}{8} \cdot n \cdot d^2$$

implies T-smoothness. Since the families fail to satisfy the conditions only by a constant factor we see that asymptotically in d, m and r the conditions for T-smoothness are proper.

For irreducibility the situation is not quite as good. The conditions in Keilen (2005) Corollary 2.4 for irreducibility if $n \geq 4$ and the Picard number of Σ is one is roughly

$$r \cdot m^2 < \frac{24}{n^2 m^2} \cdot n \cdot d^2,$$

and similarly for $n = 2$ Keilen (2005) Theorem 2.6 it is

$$r \cdot m^2 < \frac{1}{6m^2} \cdot n \cdot d^2.$$

Here the "constant" by which the families fail to satisfy the condition depends on the multiplicity m, so that with respect to m the asymptotics are not proper. However, we should like to point out that it does not depend on the number r of singular points which are imposed.

The families in Theorem 1 thus exhibit the same properties as the families of plane curves provided in Greuel, Lossen and Shustin (2001), which we use to construct the non-T-smooth component. The idea is to intersect a family of cones in $\mathbb{P}^3_{\mathbb{C}}$ over the plane curves provided by Greuel, Lossen and Shustin (2001) with Σ and to calculate the dimension of the resulting family. Under the conditions on m and l requested this family turns out to be of higher dimension than the expected one. The same idea was used by Chiantini and Ciliberto (1999) in order to give examples of nodal families of curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ which are not of the expected dimension. We then combine an asymptotic h^1-vanishing result by Alexander and Hirschowitz (2000) with an existence statement from Keilen and Tyomkin (2002) to show that there is also a T-smooth component, where actually the curves have their singularities in very general position.

Proof of Theorem 1. Fix a general plane P in $\mathbb{P}^3_{\mathbb{C}}$ and a general point p. By Greuel, Lossen and Shustin (2001) there is an integer $l_1 = l_1(m)$ such that for any $l \geq \max\{l_1, m\}$ the family of irreducible curves in P of degree $2lm + l$ with $4l^2$ ordinary m-fold points as only singularities has a component W of dimension

$$\begin{aligned} \dim(W) &\geq (m+1) \cdot \frac{(l+1) \cdot (l+2)}{2} + (2l+1) \cdot (2l+2) - 4 \\ &= \frac{l^2 m + 9l^2 + 3lm + 15l + 2m - 4}{2}. \end{aligned}$$

Let $\mathcal{W}$ be the family of cones with vertex p over curves C in W, then $\dim(\mathcal{W}) = \dim(W)$, since a cone is uniquely determined by the curve C and the vertex p.

Moreover, any cone in $\mathcal{W}$ has precisely $4l^2$ lines of multiplicity m, so that when we intersect it with Σ we get in general a curve in Σ with $4l^2n$ ordinary m-fold points. In particular, *if any of these curves is irreducible*, $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ must have a component W' of dimension

$$\dim(W') \geq \dim(W) \geq \frac{l^2m + 9l^2 + 3lm + 15l + 2m - 4}{2}.$$

However, since the dimension of the linear system $|dH|$ is

$$\dim|dH| = \binom{d+3}{3} - \binom{d+3-n}{3} - 1,$$

and since

$$\tau^{es}(M_m) = \frac{m\cdot(m+1)}{2} - 2$$

is the expected number of conditions imposed by an ordinary m-fold point, the expected dimension of $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ is

$$\begin{aligned}\operatorname{expdim}\Big(V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)\Big) &= \dim|(2lm+l)H| - 4l^2n\tau^{es}(M_m)\\ &= \frac{17l^2n + \big(4n-n^2\big)\cdot l\cdot(2m+1)}{2} + \frac{n^3-6n^2+11n}{6}.\end{aligned}$$

Due to the conditions on m and l this number is strictly smaller than the dimension of W'. This proves the existence of a non-T-smooth component, as soon as we show that *some cone in* $\mathcal{W}$ *intersects* Σ *in an irreducible curve.*

Let us assume the contrary is the case, that is, all curves cut out on Σ by cones in $\mathcal{W}$ are reducible. By $\pi:\Sigma\to P$ we denote the projection from the point p, and $\mathrm{pr}: W\to \mathrm{Sym}^{4l^2}(P): C\mapsto \mathrm{Sing}(C)$ denotes the map associating to a curve in W its singular locus. For $\underline{z}\in\mathrm{Im}(\mathrm{pr})$ we then let $W_{\underline{z}} = \mathrm{pr}^{-1}(\underline{z})$ be the family of curves in W which have $\underline{z}$ as singular locus. This is an open dense subset of the linear system $Y = |\mathcal{J}_{X(m,\underline{z}),P}(2lm+l)|$ of curves in the plane P of degree $2lm+l$ which have multiplicity at least m in each z_i. If we now consider the cones over elements in Y and intersect them with Σ we get a linear system $\mathcal{D} = \{\pi^{-1}(C)\mid C\in Y\}$ on Σ of the same dimension, since a cone over a curve in Y is uniquely determined by the curve and the vertex p. Since $m\geq 18n$ we thus have

$$\begin{aligned}\dim(\mathcal{D}) &= \dim(W_{\underline{z}})\\ &\geq \dim(W) - \dim\big(\mathrm{Im}(\mathrm{pr})\big) \geq \dim(W) - \dim\Big(\mathrm{Sym}^{4l^2}(P)\Big)\\ &\geq \frac{l^2m + 9l^2 + 3lm + 15l + 2m - 4}{2} - 8l^2\\ &= \frac{(m-7)\cdot l^2 + 3lm + 15l + 2m - 4}{2} > n. \qquad (1)\end{aligned}$$

By our assumption the elements of $\mathcal{D}$ are all reducible, and therefor by Bertini's Theorem (see van der Waerden (1973) §47, Satz 3, Satz 4) there is an irreducible one-dimensional algebraic family of curves in Σ such that irreducible components of elements in $\mathcal{D}$ all belong to this family. In particular, the dimension of $\mathcal{D}$ is bounded by the number of components of a general element of $\mathcal{D}$. However, a general element of $\mathcal{D}$ maps via π onto an irreducible curve in P and none of the components is contracted by π, thus the maximal number of components is $\deg(\pi) =$

$\deg(\Sigma) = n$. Hence, $\dim(\mathcal{D}) \leq n$, in contradiction to Equation (1). We conclude that $\mathcal{D}$ contains irreducible elements, and thus the general element of $\mathcal{D}$ and hence of W' is irreducible.

Next we have to show that $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ also has a T-smooth component, after possibly enlarging l_1.

For $\underline{z} = (z_1, \ldots, z_r) \in \Sigma^r$ we denote by $X(m;\underline{z})$ the zero-dimensional scheme with ideal sheaf $\mathcal{J}_{X(m;\underline{z})}$ given by the stalks

$$\mathcal{J}_{X(m;\underline{z}),z} = \begin{cases} \mathfrak{m}^m_{\Sigma,z}, & \text{if } z \in \{z_1,\ldots,z_r\},\\ \mathcal{O}_{\Sigma,z}, & \text{else,}\end{cases}$$

where $\mathcal{O}_{\Sigma,z}$ denotes the local ring of Σ at z and $\mathfrak{m}_{\Sigma,z}$ is its maximal ideal.

By Alexander and Hirschowitz (2000) Theorem 1.1 there is an integer $l_2 = l_2(m,\Sigma)$ such that for $l \geq l_2$ and $\underline{z} \in \Sigma^r$ in very general position the canonical map

$$H^0\big(\Sigma, \mathcal{O}_\Sigma((2lm+l-1)H)\big) \longrightarrow H^0\big(\Sigma, \mathcal{O}_{X(m;\underline{z})}((2lm+l-1)H)\big)$$

has maximal rank. In particular, since $h^1\big(\Sigma, \mathcal{O}_\Sigma((2lm+l-1)H)\big) = 0$ we have

$$h^1\big(\Sigma, \mathcal{J}_{X(m;\underline{z})}(2lm+l-1)H)\big) = 0,$$

once $\deg\big(X(m;\underline{z})\big) \leq h^0\big(\Sigma, \mathcal{O}_\Sigma((2lm+l-1)H)\big)$, which is equivalent to

$$\frac{4l^2n\cdot m\cdot(m+1)}{2} \leq \binom{2lm+l+2}{3} - \binom{2lm+l+2-n}{3},$$

or alternatively

$$\frac{nl\cdot\big(l-(n-2)\cdot(2m+1)\big)}{2} + \frac{n^3-3n^2+2n}{6} \geq 0.$$

The latter inequality is fulfilled as soon as $l \geq (n-2)\cdot(2m+1)$. Moreover, under this hypothesis we have

$$(2lm+l)\cdot H^2 - 2g(H) = (2lm+l)\cdot n - (n-1)\cdot(n-2) \geq 2m,$$

where $g(H)$ denotes the geometric genus of H, and

$$(2lm+l)^2\cdot H^2 > 4l^2nm^2. \tag{2}$$

Thus Keilen and Tyomkin (2002) Theorem 3.3 (see also Keilen (2001) Theorem 1.2) implies that $V^{irr}_{|(2lm+l)H|}\big(4l^2nM_m\big)$ has a non-empty T-smooth component, more precisely it contains a curve in a T-smooth component with singularities in $z_1,\ldots,z_r$. In particular, since there is only a finite number of components and $\underline{z}$ is in very general position, some T-smooth component must dominate $\mathrm{Sym}^r(\Sigma)$. Actually, due to Lossen (1998) Proposition 2.1 (e) and since $h^1(\Sigma,\mathcal{O}_\Sigma) = 0$ every T-smooth component dominates $\mathrm{Sym}^r(\Sigma)$.

Thus the statement follows with

$$l_0(m,\Sigma) := \max\big\{l_1(m), l_2(m,\Sigma), (\deg(\Sigma)-2)\cdot(2m+1), m\big\}.$$

It just remains to show that the fundamental group of the complement of a curve $C \in V^{irr}_{|dH|}(rM_m)$ is abelian. Note first of all that by the Lefschetz Hyperplane Section Theorem Σ is simply connected. But then $\pi_1(\Sigma\setminus C)$ is abelian by Nori (1983) Proposition 6.5 because of (2). $\square$

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