SOME OBSTRUCTED EQUISINGULAR FAMILIES OF CURVES ON SURFACES IN $\mathbb{P}^3_{\mathbb{C}}$

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Abstract. Very few examples of obstructed equsingular families of curves on surfaces other than $\mathbb{P}^2_{\mathbb{C}}$ are known. Combining results from [Wes04] and [Hir92] with an idea from [ChC99] we give in the present paper series of examples of families of irreducible curves with simple singularities on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ which are not T-smooth, i.e. do not have the expected dimension, (Section 1) and we compare this with conditions (showing the same asymptotics) which ensure the existence of a T-smooth component (Section 2).

Below we are going to construct two series of equisingular families of curves on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. In both examples the families are obstructed in the sense that they do not have the expected dimension. However, while in the first example at least the existence of such curves was expected, the families in the second example were expected to be empty. It would be interesting to see if the equisingular families contain further components which are well-behaved. However, the families which we construct fail to satisfy the numerical conditions for the existence of such a component given in Section 2 by a factor of two.

1. Examples of obstructed families

Throughout this section Σ will denote a smooth projective surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 2$, and H will be a hyperplane section of Σ. $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ will be a finite set of *simple* singularity types, that is the $\mathcal{S}_i$ are of type A_k (given by $x^2 - y^{k+1} = 0$, $k \geq 1$), D_k (given by $x^2y - y^{k-1} = 0$, $k \geq 4$), or E_k (given by $x^3 - y^4 = 0$, $x^3 - xy^3 = 0$, or $x^3 - y^5 = 0$ for $k = 6, 7, 8$ respectively). In general, for positive integers $r_1, \ldots, r_s$ and d we denote by $V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ the family of irreducible curves in the linear system $|dH|$ with precisely $r = r_1 + \ldots + r_s$ singular points, r_i of which are of the type $\mathcal{S}_i$, $i = 1, \ldots, s$, where $\mathcal{S}_i$ may be any analytic type of an isolated singularity. $V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ is called *T-smooth* or *not obstructed* if it is smooth of the expected dimension

$$\begin{aligned}\operatorname{expdim}\left(V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)\right) &= \dim|dH| - \sum_{i=1}^{s} r_i \cdot \tau(\mathcal{S}_i)\\ &= \frac{nd^2 + (4n - n^2)d}{2} + \frac{n^3 - 6n^2 + 11n - 6}{6} - \sum_{i=1}^{s} r_i \cdot \tau(\mathcal{S}_i),\end{aligned}$$

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where $\tau(\mathcal{S}) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\right\rangle$ is the Tjurina number of the singularity type $\mathcal{S}$ given by the local equation $f = 0$. Note that $\tau(A_k) = \tau(D_k) = \tau(E_k) = k$.

In this note we give examples of such equisingular families of curves which are obstructed in the sense that they have dimension larger than the expected one. We use the idea by which Chiantini and Ciliberto in [ChC99] showed the existence of obstructed families of nodal curves.

Let us fix a plane P in $\mathbb{P}^3_{\mathbb{C}}$, a point p outside P, and a curve C of degree $d > 1$ in P. If we intersect the cone $K_{C,p}$ over C with vertex p with Σ, this gives a curve $C' = K_{C,p} \cap \Sigma$ in $|dH|$ which is determined by the choice of C and p (see Lemma 4). In particular, if C varies in an N-dimensional family in P, then C' varies in an N-dimensional family on Σ, and if C is irreducible, then for a general choice of p the curve C' will be irreducible as well (see Lemma 5). Moreover, if C has a singular point q of (simple) singularity type $\mathcal{S}$ and Σ meets the line joining p and q transversally in n points, then C' will have a singularity of the same type in each of these points.

Example 1

Fix the set $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ and let $m = \max\{\tau(\mathcal{S}) \mid \mathcal{S} \in S\}$. Suppose that $n > 2m + 4$ and $d >> n$, and let $r_1, \ldots, r_s \geq 0$ be such that

$$\frac{d^2 + (4-n)d + 2}{2} \quad \leq \quad \sum_{i=1}^{s} r_i \cdot \tau(\mathcal{S}_i) \quad \leq \quad \frac{d^2 + (4-n)d + 2}{2} + m - 1.$$

Then

$$\sum_{i=1}^{s} r_i \cdot \tau(\mathcal{S}_i) \leq \frac{d^2 + (4-n)d + 2}{2} + m - 1 \leq \frac{d^2}{2} - m \cdot d - 3.$$

Hence, by [Wes04] Remark 3.3.5 the family $V = V_d^{irr}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ of irreducible plane curves C of degree d with precisely $r = r_1 + \ldots + r_s$ singular points, r_i of which are of type $\mathcal{S}_i$, is non-empty, and we may estimate its dimension:

$$\begin{aligned}\dim(V) \geq \operatorname{expdim}(V) &= \frac{d(d+3)}{2} - \sum_{i=1}^{s} r_i \cdot \tau(\mathcal{S}_i) \\ &\geq \frac{d(d+3)}{2} - \frac{d^2 + (4-n)d + 2}{2} - m + 1 = \frac{n-1}{2} \cdot d - m.\end{aligned}$$

By the above construction we see that hence the family of curves C' satisfies

$$\dim\left(V_{|dH|}^{irr}(nr_1\mathcal{S}_1, \ldots, nr_s\mathcal{S}_s)\right) \geq \frac{n-1}{2} \cdot d - m.$$

However, the expected dimension of this family is

$$\begin{aligned}&\operatorname{expdim}\left(V_{|dH|}^{irr}(nr_1\mathcal{S}_1, \ldots, nr_s\mathcal{S}_s)\right) \\ &\quad = \frac{nd^2 + (4n - n^2)d}{2} + \frac{n^3 - 6n^2 + 11n - 6}{6} - \sum_{i=1}^{s} n \cdot r_i \cdot \tau(\mathcal{S}_i) \\ &\leq \frac{nd^2 + (4n - n^2)d}{2} + \frac{n^3 - 6n^2 + 11n - 6}{6} - n \cdot \left(\frac{d^2 + (4-n)d + 2}{2}\right)\end{aligned}$$

$$= \frac{n^3 - 6n^2 + 5n - 6}{6}.$$

For $d >> n$, more precisely for

$$d > \frac{n^3 - 6n^2 + 5n - 6 + 6m}{3n - 3},$$

the expected dimension will be smaller than the actual dimension, which proves that the family is obstructed.
In particular, if $S = \{\mathcal{S}\}$, $\mathcal{S} \in \{A_k, D_k, E_k\}$, and

$$r = \left\lceil \frac{d^2 + (4-n)d + 2}{2k} \right\rceil,$$

then $V_{|dH|}(nr\mathcal{S})$ is obstructed, once $d >> n > 3k + 4$.

Note that in the previous example

$$\operatorname{expdim}\big(V_{|dH|}(nr_1\mathcal{S}_1, \ldots, nr_s\mathcal{S}_s)\big) \geq \frac{n^3 - 6n^2 + 5n - 6}{6} - n \cdot (m-1) > 0,$$

that is, the existence of curves in $|dH|$ with the given singularities was expected. This not so in the following example.

Example 2
Let k be an *even*, positive integer, $m \geq 1$, $d = 2(k+1)^m$, and

$$r = \frac{3 \cdot (k+1) \cdot \big((k+1)^{2m} - 1\big)}{(k+1)^2 - 1}.$$

Hirano proved in [Hir92] the existence of an irreducible plane curve of degree d with precisely r singular points all of type A_k. Thus the above construction shows that

$$V^{irr}_{|dH|}(nrA_k)$$

is non-empty. However, the expected dimension is

$$\begin{aligned} \operatorname{expdim}\big(V^{irr}_{|dH|}(nrA_k)\big) &= \frac{nd^2 + (4n - n^2)d}{2} + \frac{n^3 - 6n^2 + 11n - 6}{6} - knr \\ &= \left(2 - \frac{3 \cdot (k^2 + k)}{k^2 + 2k}\right) \cdot (k+1)^{2m} + o\big((k+1)^m\big), \end{aligned}$$

which is negative for m sufficiently large, since

$$\frac{3 \cdot (k^2 + k)}{k^2 + 2k} > 2.$$

This shows that $V^{irr}_{|dH|}(nrA_k)$ is obstructed for sufficiently large k.

2. Some remarks on conditions for T-smoothness

Unless otherwise specified in this section Σ will be an arbitrary smooth projective surface, H a very ample divisor on Σ, and $\mathcal{S}_1, \ldots, \mathcal{S}_s$ arbitrary (not necessarily different) topological or analytical singularity types. As in Section 1 we denote for $d \geq 0$ by $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_s)$ the equisingular family of irreducible curves in $|dH|$ with precisely s singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_s$, and again the expected dimension is

$$\operatorname{expdim}\big(V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_s)\big) = \dim|dH| - \sum_{i=1}^{s} \tau(\mathcal{S}_i).$$

$V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_s)$ is called T-smooth if it is smooth of the expected dimension.
By [Kei01] Theorem 1.2 and 2.3 (which is a slight improvement of [KeT02] Theorem 3.3 and Theorem 4.3) there is a curve $C \in V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_s)$ if

- $d \cdot H^2 - g(H) \geq m_i + m_j$, and
- $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big) = 0$ for $\underline{z} \in \Sigma^r$ very general,

where $\underline{m} = (m_1, \ldots, m_s)$ with $m_i = e^*(\mathcal{S}_i)$, a certain invariant which only depends on $\mathcal{S}_i$. Moreover, $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_s)$ is T-smooth at this curve C (see e.g. [Shu99] Theorem 1). Finally, by [AlH00] Theorem 1.1 there is a number $d(m)$ depending only on $m = \max\{m_1, \ldots, m_s\}$, such that for all $d \geq d(m)$ and for $\underline{z} \in \Sigma^r$ very general the map

$$H^0\big(\Sigma, \mathcal{O}_\Sigma((d-1)H)\big) \longrightarrow H^0\big(\Sigma, \mathcal{O}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big)$$

has maximal rank. In particular, if

$$\dim|(d-1)H| \geq \deg\big(X(\underline{m};\underline{z})\big) = \sum_{i=1}^{s} \frac{m_i \cdot (m_i+1)}{2},$$

then $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}((d-1)H)\big) = 0$. This proves the following Proposition.

Proposition 3
Let $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ be a finite set of pairwise different topological or analytical singularity types. Then there exists a number $d(S)$ such that for all $d \geq d(S)$ and $r_1, \ldots, r_s \geq 0$ satisfying

$$\sum_{i=1}^{s} r_i \cdot \frac{e^*(\mathcal{S}_i) \cdot \big(e^*(\mathcal{S}_i)+1\big)}{2} < \dim|(d-1)H| \tag{1}$$

the equisingular family $V^{irr}_{|dH|}(r_1\mathcal{S}_1, \ldots, r_s\mathcal{S}_s)$ has a non-empty T-smooth component.

In [Shu03] upper bounds for $e^*(\mathcal{S})$ are given. For a non-simple analytical singularity type we have

$$e^*(\mathcal{S}) = e^a(\mathcal{S}) \leq 3\sqrt{\mu(\mathcal{S})} - 2$$

where $\mu(\mathcal{S})$ is the Milnor number of $\mathcal{S}$, and for any topological singularity type

$$e^*(\mathcal{S}) = e^s(\mathcal{S}) \leq \frac{9}{\sqrt{6}} \cdot \sqrt{\delta(\mathcal{S})} - 1,$$

where $\delta(\mathcal{S})$ is the delta invariant of $\mathcal{S}$.
For *simple* singularity types there are the better bounds

$\mathcal{S}$	$e^*(\mathcal{S})$	$\mathcal{S}$	$e^*(\mathcal{S})$
A_1	2	D_4	3
A_2	3	D_5	4
$A_k, k = 3, \ldots, 7$	4	$D_k, k \leq 6, \ldots, 10$	5
$A_k, k = 8, \ldots, 10$	5	$D_k, k \leq 11, \ldots, 13$	6
$A_k, k \geq 1$	$\leq 2 \cdot \lfloor\sqrt{k+5}\rfloor$	$D_k, k \geq 1$	$\leq 2 \cdot \lfloor\sqrt{k+7}\rfloor + 1$
E_6	4	E_7	4
E_8	5		

In particular, if $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_s\}$ is a finite set of *simple* singularities, then there is a $d(S)$ such that for all $d \geq d(S)$ and all $r_1, \ldots, r_s \geq 0$ satisfying

$$2 \cdot \sum_{i=1}^{s} r_i \cdot \Big(\tau(\mathcal{S}_i) + o\big(\sqrt{\tau(\mathcal{S}_i)}\big)\Big) \leq \dim |dH| \qquad (2)$$

the family $V^{irr}_{|dH|}(nr_1\mathcal{S}_1, \ldots, nr_s\mathcal{S}_s)$ has a non-empty T-smooth component.
The families in Example 1 fail to satisfy this condition roughly by the factor 2. We thus cannot conclude that these families are reducible as we could in a similar situation in [Kei05a].
However, if we compare Condition 1 respectively 2 to the conditions in [GLS00] or [Kei05b] which ensure that the equisingular family is T-smooth at *every* point, the latter basically involve the square of the Tjurina number and are therefore much more restrictive. This, of course, was to be expected.

3. Some remarks on cones

In this section we collect some basic properties on cones used for the construcion in Section 1, in particular the dimension counts.
For points $p_1, \ldots, p_r \in \mathbb{P}^3_{\mathbb{C}}$ we will denote by $\overline{p_1 \ldots p_r}$ the linear span in $\mathbb{P}^3_{\mathbb{C}}$ of $p_1, \ldots, p_r$, i.e. the smallest linear subspace containing $p_1, \ldots, p_r$.
Let $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane, $C \subset P$ a curve, and $p \in \mathbb{P}^3_{\mathbb{C}} \setminus P$ a point. Then we denote by

$$K_{C,p} = \bigcup_{q \in C} \overline{qp}$$

the cone over C with vertex p. Note that

$$K_{C,p} = \bigcup_{q \in K_{C,p}} \overline{qp}$$

and that

$$K_{C,p} \cap P = C.$$

We first show that C and p fix the cone uniquely except when C is a line.

Lemma 4
Let $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane, and $C \subseteq P$ be an irreducible curve which is not a line. Then for $p, p' \in \mathbb{P}^3_{\mathbb{C}}$ with $p \neq p'$ we have that $K_{C,p} \neq K_{C,p'}$.

Proof: Suppose there are points $p \neq p'$ such that $K_{C,p} = K_{C,p'}$. Choose a point $x \in C \setminus \overline{pp'}$ and let $E = \overline{xpp'}$. Then for any point $y \in \overline{xp} \subset K_{C,p} = K_{C,p'}$ we have

$$\overline{yp'} \subset K_{C,p'},$$

and thus $E = \bigcup_{y \in \overline{xp}} \overline{yp'} \subset K_{C,p'}$. This, however, implies that the line

$$l = E \cap P \subseteq K_{C,p'} \cap P = C$$

is contained in C, and since C is irreducible we would have $C = l$ in contradiction to our assumption that C is not a line. Hence, $K_{C,p} \neq K_{C,p'}$ for $p \neq p'$. $\square$

Finally we show that for a general p the cone $K_{C,p}$ intersects Σ in an irreducible curve.

Lemma 5
Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth projective surface, $P \subset \mathbb{P}^3_{\mathbb{C}}$ be a plane such that $P \neq \Sigma$, and $C \subseteq P$ an irreducible curve which is not a line and not contained in Σ. Then for $p \in \mathbb{P}^3_{\mathbb{C}} \setminus P$ general $K_{C,p} \cap \Sigma$ is irreducible.

Proof: Consider the linear system $\mathcal{L}$ in $\mathbb{P}^3_{\mathbb{C}}$ which is given as the closure of

$$\left\{ K_{C,p} \mid p \in \mathbb{P}^3_{\mathbb{C}} \setminus P \right\},$$

and set for $q \in \mathbb{P}^3_{\mathbb{C}} \setminus P$

$$\mathcal{L}_q = \{D \in \mathcal{L} \mid q \in D\}.$$

First we show that for $q' \in C$ and $q \notin P$

$$\bigcap_{p \in \overline{qq'}} K_{C,p} = C \cup \overline{qq'}. \tag{3}$$

Choose pairwise different point $p_1, \ldots, p_n \in \overline{qq'} \setminus \{q, q'\}$. Suppose that there is a $z \in \bigcap_{i=1}^n K_{C,p_i} \setminus (C \cup \overline{qq'})$. Since $z \in K_{C,p_i}$ there is a unique intersection point

$$x_i = \overline{zp_i} \cap C,$$

and these points $x_1, \ldots, x_n$ are pairwise different, since $z \notin \overline{qq'} = \overline{p_i p_j}$ for $i \neq j$. However,

$$x_i \in \overline{zp_i} \subset \overline{zp_ip_j} = \overline{zqq'}$$

and $x_i \in C \subset P$, so that

$$q', x_1, \ldots, x_n \in P \cap \overline{zqq'}$$

and $q', x_1, \ldots, x_n$ are pairwise different collinear points on C. Since C is irreducible but not a line, this implies $\deg(C) \geq n + 1$. In particular, if $n \geq \deg(C)$, then

$$\bigcap_{i=1}^{n} K_{C,p_i} = C \cup \overline{qq'},$$

which implies (3).

Note that by (3) for $q \in \mathbb{P}^3_{\mathbb{C}} \setminus P$

$$\bigcap_{D\in\mathcal{L}_q} D \subseteq \bigcap_{K_{C,p}\in\mathcal{L}_q} K_{C,p} = \bigcap_{q'\in C} \bigcap_{p\in\overline{qq'}} K_{C,p} = \bigcap_{q'\in C} \left(C \cup \overline{qq'}\right) = C \cup \{q\},$$

and thus

$$\bigcap_{D\in\mathcal{L}} D \subseteq \bigcap_{q\in\mathbb{P}^3_{\mathbb{C}}\setminus P} \bigcap_{D\in\mathcal{L}_q} D = C. \tag{4}$$

Consider now the linear systems

$$\mathcal{L}_\Sigma = \{D\cap\Sigma \mid D\in\mathcal{L}\} \quad\text{and}\quad \mathcal{L}_{q,\Sigma} = \{D\cap\Sigma \mid D\in\mathcal{L}_q\} = \{D\in\mathcal{L}_\Sigma \mid q\in D\}.$$

Suppose that $\mathcal{L}_\Sigma$ does not contain any irreducible curve. By (4) and since $C \not\subset \Sigma$ the linear system $\mathcal{L}_\Sigma$ has no fixed component. Thus by Bertini's Theorem $\mathcal{L}_\Sigma$ must be composed with a pencil $\mathcal{B}$, and since for a general point $q\in\Sigma$ the pencil $\mathcal{B}$ contains only one element, say $\widetilde{C}$, through q, the linear system $\mathcal{L}_{q,\Sigma}$ has a fixed component $\widetilde{C}$. But then

$$\widetilde{C} \subseteq \bigcap_{D\in\mathcal{L}_q} D\cap\Sigma = C\cap\Sigma.$$

However, $C\cap\Sigma$ is zero-dimensional, while $\widetilde{C}$ has dimension one.
This proves that $\mathcal{L}_\Sigma$ contains an irreducible element, and thus its general element is irreducible. In particular, for $p\in\mathbb{P}^3_{\mathbb{C}}\setminus P$ general $K_{C,p}\cap\Sigma$ is irreducible. □

References

[AlH00] James Alexander and André Hirschowitz, *An asymptotic vanishing theorem for generic unions of multiple points*, Inventiones Math. **140** (2000), no. 2, 303–325.

[ChC99] Luca Chiantini and Ciro Ciliberto, *On the severi varieties of surfaces in* $\mathbb{P}^3$, J. Algebraic Geom. **8** (1999), no. 1, 67–83.

[GLS00] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Castelnuovo function, zero-dimensional schemes, and singular plane curves*, J. Algebraic Geom. **9** (2000), no. 4, 663–710.

[Hir92] Atsuko Hirano, *Constructions of plane curves with cusps*, Saitama Math. J. **10** (1992), 21–24.

[Kei01] Thomas Keilen, *Families of curves with prescribed singularities*, Ph.D. thesis, Universität Kaiserslautern, 2001, http://www.mathematik.uni-kl.de/~keilen/download/Thesis/thesis.ps.gz.

[Kei05a] Thomas Keilen, *Reducible families of curves with ordinary multiple points on surfaces in* $\mathbb{P}^3$, To appear in: Comm. in Alg. (2005), http://www.mathematik.uni-kl.de/~keilen/download/Keilen004/Keilen004.ps.gz.

[Kei05b] Thomas Keilen, *Smoothness of equisingular families of curves*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2467–2481, http://www.mathematik.uni-kl.de/~keilen/download/Keilen003/Keilen003.ps.gz.

[KeT02] Thomas Keilen and Ilya Tyomkin, *Existence of curves with prescribed singularities*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1837–1860, http://www.mathematik.uni-kl.de/~keilen/download/KeilenTyomkin001/KeilenTyomkin001.ps.gz.

[Shu99] Eugenii Shustin, *Lower deformations of isolated hypersurface singularities*, Algebra i Analiz **10** (1999), no. 5, 221–249.

[Shu03] Eugenii Shustin, *Analytic order of singular and critical points*, Trans. Amer. Math. Soc. **356** (2003), no. 3, 953–985.

[Wes04] Eric Westenberger, *Families of hypersurfaces with many prescribed singularities*, Ph.D. thesis, TU Kaiserslautern, 2004, http:// www. mathematik. uni-kl. de/ ˜westenb/ files/ Westenberger_Dissertation.ps.gz.

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