STANDARD BASES IN $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

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ABSTRACT. In this paper we study standard bases for submodules of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$ respectively of their localisation with respect to a $\underline{t}$-local monomial ordering. The main step is to prove the existence of a division with remainder generalising and combining the division theorems of Grauert and Mora. Everything else then translates naturally. Setting either $m=0$ or $n=0$ we get standard bases for polynomial rings respectively for power series rings as a special case. We then apply this technique to show that the t-initial ideal of an ideal over the Puiseux series field can be read of from a standard basis of its generators. This is an important step in the constructive proof that each point in the tropical variety of such an ideal admits a lifting.

The paper follows the lines of [GrP02] and [DeS07] generalising the results where necessary. Basically, the only original parts for the standard bases are the proofs of Theorem 2.1 and Theorem 3.3, but even here they are easy generalisations of Grauert's respectively Mora's Division Theorem (the latter in the form stated and proved first by Greuel and Pfister, see [GrP96]; see also [Grä94]). The paper should therefore rather be seen as a unified approach for the existence of standard bases in polynomial and power series rings, and it was written mostly due to the lack of a suitable reference for the existence of standard bases in $K[[t]][x_1,\ldots,x_n]$ which are needed when dealing with tropical varieties. Namely, when we want to show that every point in the tropical variety of an ideal J defined over the field of Puiseux series exhibits a

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Date: March, 2007.

1991 *Mathematics Subject Classification.* Primary 13P10, 13F25, 16W60; Secondary 12J25, 16W60.

Key words and phrases. Standard basis, monomial ordering, division with remainder.

The author was supported by the IMA, Minneapolis.

lifting to the variety of J, then, assuming that J is generated by elements in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$, we need to know that we can compute the so-called t-initial ideal of J by computing a standard basis of the ideal defined by the generators in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$ (see Theorem 6.10 and [JMM07]).

An important point is that if the input data is polynomial in both $\underline{t}$ and $\underline{x}$ then we can actually compute the standard basis since a standard basis computed in $K[t_1,\ldots,t_m]_{\langle t_1,\ldots,t_m\rangle}[x_1,\ldots,x_n]$ will do (see Corollary 4.7). This was previously known for the case where there are no x_i (see [GrP96]).

In Section 1 we introduce the basic notions. Section 2 is devoted to the proof of the existence of a determinate division with remainder for polynomials in $K[[t_1,\ldots,t_m]][x_1,\ldots,x_m]^s$ which are homogeneous with respect to the x_i. This result is then used in Section 3 to show the existence of weak divisions with remainder for all elements of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_m]^s$. In Section 4 we introduce standard bases and prove the basics for these, and we prove Schreyer's Theorem and, thus Buchberger's Criterion in Section 5. Finally, in Section 6 we apply standard bases to study t-initial ideals of ideals over the Puiseux series field.

1. Basic Notation

Throughout the paper K will be any field, $R = K[[t_1,\ldots,t_m]]$ will denote the ring of formal power series over K and

$$R[x_1,\ldots,x_n] = K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]$$

denotes the ring of polynomials in the indeterminates $x_1,\ldots,x_n$ with coefficients in the power series ring R. We will in general use the short hand notation $\underline{x} = (x_1,\ldots,x_n)$ and $\underline{t} = (t_1,\ldots,t_m)$, and the usual multi index notation

$$\underline{t}^\alpha = t_1^{\alpha_1}\cdots t_m^{\alpha_m} \quad \text{and} \quad \underline{x}^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n},$$

for $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m$ and $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$.

Definition 1.1
A *monomial ordering* on

$$\mathrm{Mon}(\underline{t},\underline{x}) = \left\{\underline{t}^\alpha \cdot \underline{x}^\beta \;\middle|\; \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n\right\}$$

is a total ordering $>$ on $\mathrm{Mon}(\underline{t},\underline{x})$ which is compatible with the semi group structure of $\mathrm{Mon}(\underline{t},\underline{x})$, i.e. such that for all $\alpha, \alpha', \alpha'' \in \mathbb{N}^m$ and $\beta, \beta', \beta'' \in \mathbb{N}^n$

$$\underline{t}^\alpha \cdot \underline{x}^\beta \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \quad\Longrightarrow\quad \underline{t}^{\alpha+\alpha''} \cdot \underline{x}^{\beta+\beta''} \;>\; \underline{t}^{\alpha'+\alpha''} \cdot \underline{x}^{\beta'+\beta''}.$$

We call a monomial ordering $>$ on $\mathrm{Mon}(\underline{t},\underline{x})$ $\underline{t}$*-local* if its restriction to $\mathrm{Mon}(\underline{t})$ is *local*, i.e. $t_i < 1$ for all $i = 1, \ldots, m$. We call a $\underline{t}$-local monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$ a $\underline{t}$*-local weighted degree ordering* if there is a $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$ such that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $\beta, \beta' \in \mathbb{N}^n$

$$w \cdot (\alpha, \beta) > w \cdot (\alpha', \beta') \quad\Longrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'},$$

where $w \cdot (\alpha, \beta) = w_1 \cdot \alpha_1 + \ldots + w_m \cdot \alpha_m + w_{m+1} \cdot \beta_1 + \ldots + w_n \cdot \beta_n$ denotes the standard scalar product. We call w a *weight vector* of $>$.

Example 1.2
The $\underline{t}$*-local lexicographical ordering* $>_{lex}$ on $\mathrm{Mon}(\underline{t},\underline{x})$ is defined by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$$

if and only if

$$\exists\, j \in \{1, \ldots, n\} \;:\; \beta_1 = \beta_1', \ldots, \beta_{j-1} = \beta_{j-1}', \;\text{ and }\; \beta_j > \beta_j',$$

or

$$\big(\beta = \beta' \text{ and } \exists\, j \in \{1, \ldots, m\} \;:\; \alpha_1 = \alpha_1', \ldots, \alpha_{j-1} = \alpha_{j-1}', \alpha_j < \alpha_j'\big).$$

Example 1.3
Let $>$ be any $\underline{t}$-local ordering and $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$, then $\underline{t}^\alpha \cdot \underline{x}^\beta >_w \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$ if and only if $w \cdot (\alpha, \beta) > w \cdot (\alpha', \beta')$ or

$$\big(w \cdot (\alpha, \beta) = w \cdot (\alpha', \beta') \;\text{ and }\; \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}\big)$$

defines a $\underline{t}$-local weighted degree ordering $>_w$ on $\mathrm{Mon}(\underline{t},\underline{x})$ with weight vector w.

Even if we are only interested in standard bases of ideals we have to pass to submodules of free modules in order to have syzygies at hand for the proof of Buchberger's Criterion via Schreyer orderings.

Definition 1.4
We define

$$\mathrm{Mon}^s(\underline{t},\underline{x}) := \left\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m, i = 1,\ldots,s\right\},$$

where $e_i = (\delta_{ij})_{j=1,\ldots,s}$ is the vector with all entries zero except the i-th one which is one. We call the elements of $\mathrm{Mon}^s(\underline{t},\underline{x})$ *module monomials* or simply *monomials*.
For $p, p' \in \mathrm{Mon}^s(\underline{t},\underline{x}) \cup \{0\}$ the notion of divisibility and of the lowest common multiple $\mathrm{lcm}(p,p')$ are defined in the obvious way.
Given a monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$, a $\underline{t}$*-local monomial ordering* on $\mathrm{Mon}^s(\underline{t},\underline{x})$ with respect to $>$ is a total ordering $>_m$ on $\mathrm{Mon}^s(\underline{t},\underline{x})$ which is strongly compatible with the operation of the multiplicative semi group $\mathrm{Mon}(\underline{t},\underline{x})$ on $\mathrm{Mon}^s(\underline{t},\underline{x})$ in the sense that

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i \;>_m\; \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j \quad\Longrightarrow\quad \underline{t}^{\alpha+\alpha''}\cdot\underline{x}^{\beta+\beta''}\cdot e_i \;>_m\; \underline{t}^{\alpha'+\alpha''}\cdot\underline{x}^{\beta'+\beta''}\cdot e_j$$

and

$$\underline{t}^\alpha\cdot\underline{x}^\beta \;>\; \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'} \quad\Longleftrightarrow\quad \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i \;>_m\; \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_i$$

for all $\beta,\beta',\beta'' \in \mathbb{N}^n$, $\alpha,\alpha',\alpha'' \in \mathbb{N}^m, i,j \in \{1,\ldots,s\}$.
Note that due to the second condition the ordering $>_m$ on $\mathrm{Mon}^s(\underline{t},\underline{x})$ determines the ordering $>$ on $\mathrm{Mon}(\underline{t},\underline{x})$ uniquely, and we will therefore usually not distinguish between them, i.e. we will use the same notation $>$ also for $>_m$, and we will not specify the monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$ in advance, but instead refer to it as the *induced monomial ordering on* $\mathrm{Mon}(\underline{t},\underline{x})$.
We call a monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ $\underline{t}$*-local* if the induced monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$ is so.
We call a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ a $\underline{t}$*-local weight ordering* if there is a $w = (w_1,\ldots,w_{m+n+s}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n \times \mathbb{R}^s$ such that for all $\alpha,\alpha' \in \mathbb{N}^m$, $\beta,\beta' \in \mathbb{N}^n$ and $i,j \in \{1,\ldots,s\}$

$$w\cdot(\alpha,\beta,e_i) > w\cdot(\alpha',\beta',e_j) \quad\Longrightarrow\quad \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i > \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j,$$

and we call w a *weight vector* of $>$.

Example 1.5
Let $w \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^{n+s}$ and let $>$ be any $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ such that the induced $\underline{t}$-local monomial ordering on $\operatorname{Mon}(\underline{t}, \underline{x})$ is a $\underline{t}$-local weighted degree ordering with respect to the weight vector $(w_1, \ldots, w_{m+n})$. Then

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \; >_w \; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j$$

if and only if

$$w \cdot (\alpha, \beta, e_i) > w \cdot (\alpha', \beta', e_j)$$

or

$$\big(w \cdot (\alpha, \beta, e_i) = w \cdot (\alpha', \beta', e_j) \;\text{ and }\; \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j\big)$$

defines a $\underline{t}$-local weight monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ with weight vector w. In particular, there exists such a monomial ordering.

Remark 1.6
In the following we will mainly be concerned with monomial orderings on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ and with submodules of free modules over $R[\underline{x}]$, but all these results specialise to $\operatorname{Mon}(\underline{t}, \underline{x})$ and ideals by just setting $s = 1$. □

For a $\underline{t}$-local monomial ordering we can introduce the notions of leading monomial and leading term of elements in $R[\underline{x}]^s$.

Definition 1.7
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$. We call

$$0 \neq f = \sum_{i=1}^{s} \sum_{|\beta|=0}^{d} \sum_{|\alpha|=0}^{\infty} a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in R[\underline{x}]^s,$$

with $a_{\alpha,\beta,i} \in K$, $|\beta| = \beta_1 + \ldots + \beta_n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_m$, the *distributive representation* of f, $\mathcal{M}_f := \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0\big\}$ the set of *monomials of* f and $\mathcal{T}_f := \big\{a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0\big\}$ the set of *terms of* f.
Moreover, $\operatorname{lm}_>(f) := \max\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in \mathcal{M}_f\}$ is called the *leading monomial* of f. Note again, that this maximum exists since the number of β's occurring in f and the number of i's is finite and the ordering is local with respect to $\underline{t}$.

If $\text{lm}_>(f) = \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ then we call $\text{lc}_>(f) := a_{\alpha,\beta,i}$ the *leading coefficient* of f, $\text{lt}_>(f) := a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ its *leading term*, and $\text{tail}_>(f) := f - \text{lt}_>(f)$ its *tail*.
For the sake of completeness we define $\text{lm}_>(0) := 0$, $\text{lt}_>(0) := 0$, $\text{lc}_>(0) := 0$, $\text{tail}_>(f) = 0$, and $0 < \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \quad \forall\, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n, i \in \mathbb{N}$.
Finally, for a subset $G \subseteq R[\underline{x}]^s$ we call the submodule

$$L_>(G) = \langle \text{lm}_>(f) \mid f \in G \rangle \leq K[\underline{t}, \underline{x}]^s$$

of the free module $K[\underline{t}, \underline{x}]^s$ over the polynomial ring $K[\underline{t}, \underline{x}]$ generated by all the leading monomials of elements in G the *leading submodule* of G.

We know that in general a standard basis of an ideal respectively submodule I will not be a generating set of I itself, but only of the ideal respectively submodule which I generates in the localisation with respect to the monomial ordering. We therefore introduce this notion here as well.

Definition 1.8
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}(\underline{t}, \underline{x})$, then $S_> = \{u \in R[\underline{x}] \mid \text{lt}_>(u) = 1\}$ is the *multiplicative set associated to* $>$, and $R[\underline{x}]_> = S_>^{-1} R[\underline{x}] = \left\{ \frac{f}{u} \;\middle|\; f \in R[\underline{x}], u \in S_> \right\}$ is the *localisation of* $R[\underline{x}]$ *with respect to* $>$.

If $>$ is a $\underline{t}$-local monomial ordering with $x_i > 1$ for all $i = 1, \ldots, n$ (e.g. $>_{lex}$ from Example 1.2), then $S_> \subset R^*$, and therefore $R[\underline{x}]_> = R[\underline{x}]$.
It is straight forward to extend the notions of leading monomial, leading term and leading coefficient to $R[\underline{x}]_>$ and free modules over this ring.

Definition 1.9
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$, $g = \frac{f}{u} \in R[\underline{x}]_>^s$ with $u \in S_>$, and $G \subseteq R[\underline{x}]_>^s$. We then define the *leading monomial*, the *leading coefficient* respectively the *leading term* of g as

$$\text{lm}_>(g) := \text{lm}_>(f), \quad \text{lc}_>(g) := \text{lc}_>(f), \quad \text{resp.} \quad \text{lt}_>(g) := \text{lt}_>(f),$$

and the *leading ideal* (if $s = 1$) respectively *leading submodule* of G

$$L_>(G) = \langle \text{lm}_>(h) \mid h \in G \rangle \leq K[\underline{t}, \underline{x}]^s.$$

These definitions are independent of the chosen representative, since if $g = \frac{f}{u} = \frac{f'}{u'}$ then $u' \cdot f = u \cdot f'$, and hence

$$\mathrm{lt}_>(f) = \mathrm{lt}_>(u')\cdot\mathrm{lt}_>(f) = \mathrm{lt}_>(u'\cdot f) = \mathrm{lt}_>(u\cdot f') = \mathrm{lt}_>(u)\cdot\mathrm{lt}_>(f') = \mathrm{lt}_>(f').$$

Remark 1.10
Note that the leading submodule of a submodule in $R[\underline{x}]^s_>$ is a submodule in a free module over the polynomial ring $K[\underline{t},\underline{x}]$ over the base field, and note that for $J \le R[\underline{x}]^s_>$ we obviously have $L_>(J) = L_>(J \cap R[\underline{x}]^s)$, and similarly for $I \le R[\underline{x}]^s$ we have $L_>(I) = L_>\big(\langle I\rangle_{R[\underline{x}]_>}\big)$, since every element of $\langle I\rangle_{R[\underline{x}]_>}$ is of the form $\frac{f}{u}$ with $f \in I$ and $u \in S_>$.

In order to be able to work either theoretically or even computationally with standard bases it is vital to have a division with remainder and possibly an algorithm to compute it. We will therefore generalise Grauert's and Mora's Division with remainder. For this we first would like to consider the different qualities a division with remainder may satisfy.

Definition 1.11
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, and let $A = R[\underline{x}]$ or $A = R[\underline{x}]_>$, where we consider the latter as a subring of $K[[\underline{t},\underline{x}]]$ in order to have the notion of terms of elements at hand.
Suppose we have $f, g_1, \ldots, g_k, r \in A^s$ and $q_1, \ldots, q_k \in A$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{1}$$

With the notation $r = \sum_{j=1}^s r_j \cdot e_j$, $r_1, \ldots, r_s \in A$, we say that (1) satisfies with respect to $>$ the condition

- (ID1) iff $\mathrm{lm}_>(f) \ge \mathrm{lm}_>(q_i \cdot g_i)$ for all $i = 1, \ldots, k$,
- (ID2) iff $\mathrm{lm}_>(g_i) \not| \mathrm{lm}_>(r)$ for $i = 1, \ldots, k$, unless $r = 0$,
- (DD1) iff for $j < i$ no term of $q_i \cdot \mathrm{lm}_>(g_i)$ is divisible by $\mathrm{lm}_>(g_j)$,
- (DD2) iff no term of r is divisible by $\mathrm{lm}_>(g_i)$ for $i = 1, \ldots, k$.
- (SID2) iff $\mathrm{lm}_>(g_i) \not| \mathrm{lm}_>(r_j \cdot e_j)$ unless $r_j = 0$ for all i and j.

Here, "ID" stands for *indeterminate division with remainder* while "DD" means *determinate division with remainder* and the "S" in (SID2) represents *strong*. Accordingly, we call a representation of f as in (1) a *determinate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if

it satisfies (DD1) and (DD2), while we call it an *indeterminate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if it satisfies (ID1) and (ID2). In any of these cases we call r a *remainder* or a *normal form* of f with respect to $(g_1, \ldots, g_k)$.
If the remainder in a division with remainder of f with respect to $(g_1, \ldots, g_k)$ is zero we call the representation of f a *standard representation.*
Finally, if $A = R[\underline{x}]$ then for $u \in S_>$ we call a division with remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ also a *weak division with remainder* of f with respect to $(g_1, \ldots, g_k)$, a remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak normal form* of f with respect to $(g_1, \ldots, g_k)$, and a standard representation of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak standard representation* of f with respect to $(g_1, \ldots, g_k)$.

It is rather obvious to see that (DD2) $\Longleftarrow$ (SID2)$\Longleftarrow$ (ID2), that (DD1)+(ID2) $\Longleftarrow$ (ID1), and that the coefficients and the remainder of a division satisfying (DD1) and (DD2) is uniquely determined.
We first want to generalise Grauert's Division with Remainder to the case of elements in $R[\underline{x}]$ which are homogeneous with respect to $\underline{x}$. We therefore introduce this notion in the following definition.

Definition 1.12
Let $f = \sum_{i=1}^{s} \sum_{|\beta|=0}^{d} \sum_{\alpha \in \mathbb{N}^m} a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in R[\underline{x}]^s$.

(a) We call $\deg_{\underline{x}}(f) := \max \left\{ |\beta| \;\middle|\; a_{\alpha,\beta,i} \neq 0 \right\}$ the *$\underline{x}$-degree* of f.
(b) $f \in R[\underline{x}]^s$ is called *$\underline{x}$-homogeneous of $\underline{x}$-degree d* if all terms of f have the same $\underline{x}$-degree d. We denote by $R[\underline{x}]^s_d$ the R-submodule of $R[\underline{x}]^s$ of $\underline{x}$-homogeneous elements. Note that by this definition 0 is $\underline{x}$-homogeneous of degree d for all $d \in \mathbb{N}$.
(c) If $>$ is a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ then we call

$$\text{ecart}_>(f) := \deg_{\underline{x}}(f) - \deg_{\underline{x}} \left(\text{lm}_>(f) \right) \geq 0$$

the *ecart* of f. It in some sense measures the failure of the homogeneity of f.

2. Determinate Division with Remainder in $K[[\underline{t}]][\underline{x}]_d^s$

We are now ready to show that for $\underline{x}$-homogeneous elements in $R[\underline{x}]$ there exists a determinate division with remainder. We follow mainly the proof of Grauert's Division Theorem as given in [DeS07].

Theorem 2.1 (HDDwR)
Let $f, g_1, \ldots, g_k \in R[\underline{x}]^s$ be $\underline{x}$-homogeneous, then there exist uniquely determined $q_1, \ldots, q_k \in R[\underline{x}]$ and $r \in R[\underline{x}]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfying (DD1), (DD2) and

(DDH) *$q_1, \ldots, q_k, r$ are $\underline{x}$-homogeneous of $\underline{x}$-degrees* $\deg_{\underline{x}}(q_i) = \deg_{\underline{x}}(f) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ *respectively* $\deg_{\underline{x}}(r) = \deg_{\underline{x}}(f)$.

Proof: The result is obvious if the g_i are terms, and we will reduce the general case to this one. We set $f_0 = f$ and for $\nu > 0$ we define recursively

$$f_\nu = f_{\nu-1} - \sum_{i=1}^{k} q_{i,\nu} \cdot g_i - r_\nu = \sum_{i=1}^{k} q_{i,\nu} \cdot \big(\big(- \mathrm{tail}(g_i)\big),$$

where the $q_{i,\nu} \in R[\underline{x}]$ and $r_\nu \in R[\underline{x}]^s$ are such that

$$f_{\nu-1} = q_{1,\nu} \cdot \mathrm{lt}_>(g_1) + \ldots + q_{k,\nu} \cdot \mathrm{lt}_>(g_k) + r_\nu \tag{2}$$

satisfies (DD1), (DD2) and (DDH). Note that such a representation of $f_{\nu-1}$ exists since the $\mathrm{lt}_>(g_i)$ are terms.
We want to show that f_ν, $q_{i,\nu}$ and r_ν all converge to zero in the $\langle t_1, \ldots, t_m\rangle$-adic topology, that is that for each $N \geq 0$ there exists a $\mu_N \geq 0$ such that for all $\nu \geq \mu_N$

$$f_\nu, r_\nu \in \langle t_1, \ldots, t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{resp.} \quad q_{i,\nu} \in \langle t_1, \ldots, t_m\rangle^N.$$

By Lemma 2.3 there is $\underline{t}$-local weight ordering $>_w$ such that

$$\mathrm{lm}_>(g_i) = \mathrm{lm}_{>_w}(g_i) \quad \text{for all } i = 1, \ldots, k.$$

If we replace in the above construction $>$ by $>_w$, we still get the same sequences $(f_\nu)_{\nu=0}^\infty$, $(q_{i,\nu})_{\nu=1}^\infty$ and $(r_\nu)_{\nu=1}^\infty$, since for the construction of $q_{i,\nu}$ and r_ν only the leading monomials of the g_j are used. In particular, (2) will satisfy (DD1), (DD2) and (DDH) with respect to $>_w$. Due to

(DDH) f_ν is again $\underline{x}$-homogeneous of $\underline{x}$-degree equal to that of $f_{\nu-1}$, and since (DD1) and (DD2) imply (ID1) we have

$$\begin{aligned}&\mathrm{lm}_{>_w}(f_{\nu-1}) \geq \max\{\mathrm{lm}_{>_w}(q_{i,\nu})\cdot\mathrm{lm}_{>_w}(g_i)\mid i=1,\ldots,k\}\\ &> \max\big\{\,\mathrm{lm}_{>_w}(q_{i,\nu})\cdot\mathrm{lm}_{>_w}\big(-\mathrm{tail}(g_i)\big)\,\big|\, i=1,\ldots,k\big\} \geq \mathrm{lm}_{>_w}(f_\nu).\end{aligned}$$

It follows from Lemma 2.4 that f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, i.e. for given N there is a μ_N such that

$$f_\nu \in \langle t_1,\ldots,t_m\rangle^N\cdot R[\underline{x}]^s \quad \text{for all } \nu\geq\mu.$$

But then, by construction for $\nu > \mu_N$

$$r_\nu \in \langle t_1,\ldots,t_m\rangle^N\cdot R[\underline{x}]^s$$

and

$$q_{i,\nu}\in\langle t_1,\ldots,t_m\rangle^{N-d_i},$$

where $d_i = \deg\big(\mathrm{lm}_>(g_i)\big) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ is independent of ν. Thus both, r_ν and $q_{i,\nu}$, converge as well to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology.

But then

$$q_i := \sum_{\nu=1}^\infty q_{i,\nu}\in R[\underline{x}] \quad\text{and}\quad r:=\sum_{\nu=1}^\infty r_\nu\in R[\underline{x}]^s$$

are $\underline{x}$-homogeneous of $\underline{x}$-degrees $\deg_{\underline{x}}(q_i) = \deg_{\underline{x}}(f) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ respectively $\deg_{\underline{x}}(r) = \deg_{\underline{x}}(f)$ unless they are zero, and

$$f = q_1\cdot g_1+\ldots+q_k\cdot g_k + r$$

satisfies (DD1), (DD2) and (DDH).

The uniqueness of the representation is obvious. □

The following lemmata contain technical results used throughout the proof of the previous theorem.

Lemma 2.2

If $>$ is a monomial ordering on $\mathrm{Mon}^s(\underline{z})$ with $\underline{z}=(\underline{t},\underline{x})$, and $M\subset \mathrm{Mon}^s(\underline{z})$ is finite, then there exists $w\in\mathbb{Z}^{m+n+s}$ with

$$w_i<0,\quad \textit{if } z_i<1,\qquad \textit{and}\qquad w_i>0,\quad \textit{if } z_i>1,$$

such that for $\underline{z}^\gamma \cdot e_i, \underline{z}^{\gamma'} \cdot e_j \in M$ *we have*

$$\underline{z}^\gamma \cdot e_i \;>\; \underline{z}^{\gamma'} \cdot e_j \quad\Longleftrightarrow\quad w \cdot (\gamma, e_i) \;>\; w \cdot (\gamma', e_j).$$

In particular, if $>$ *is* $\underline{t}$*-local then every* $\underline{t}$*-local weight ordering on* $\mathrm{Mon}^s(\underline{t}, \underline{x})$ *with weight vector* w *coincides on* M *with* $>$.

Proof: The proof goes analogous to [GrP02, Lemma 1.2.11], using [Bay82, (1.7)] (for this note that in the latter the requirement that $>$ is a well-ordering is superfluous). □

Lemma 2.3
Let $>$ *be a* $\underline{t}$*-local ordering on* $\mathrm{Mon}^s(\underline{t}, \underline{x})$ *and let* $g_1, \ldots, g_k \in R[\underline{x}]^s$ *be* $\underline{x}$*-homogeneous (not necessarily of the same degree), then there is a* $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ *such that any* $\underline{t}$*-local weight ordering with weight vector* w*, say* $>_w$*, induces the same leading monomials as* $>$ *on* $g_1, \ldots, g_k$*, i.e.*

$$\mathrm{lm}_>(g_i) = \mathrm{lm}_{>_w}(g_i) \quad \textit{for all } i = 1, \ldots, k.$$

Proof: Consider the monomial ideals $I_i = \langle \mathcal{M}_{\mathrm{tail}(g_i)} \rangle$ in $K[\underline{t}, \underline{x}]$ generated by all monomials of $\mathrm{tail}(g_i)$, $i = 1, \ldots, k$. By Dickson's Lemma (see e.g. [GrP02, Lemma 1.2.6]) I_i is generated by a finite subset, say $B_i \subset \mathcal{M}_{\mathrm{tail}(g_i)}$, of the monomials of $\mathrm{tail}(g_i)$. If we now set

$$M = B_1 \cup \ldots \cup B_k \cup \{\mathrm{lm}_>(g_1), \ldots, \mathrm{lm}_>(g_k)\},$$

then by Lemma 2.2 there is $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ such that any $\underline{t}$-local weight ordering, say $>_w$, with weight vector w coincides on M with $>$. Let now $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ be any monomial occurring in $\mathrm{tail}(g_i)$. Then there is a monomial $\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \in B_i$ such that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \;\Big|\; \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

which in particular implies that $e_\nu = e_\mu$. Since g_i is $\underline{x}$-homogeneous it follows first that $|\beta| = |\beta'|$ and thus that $\beta = \beta'$. Moreover, since $>_w$ is $\underline{t}$-local it follows that $\underline{t}^{\alpha'} \geq_w \underline{t}^\alpha$ and thus that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu.$$

But since $>$ and $>_w$ coincide on $\{\mathrm{lm}_>(g_i)\} \cup B_i \subset M$ we necessarily have that

$$\mathrm{lm}_>(g_i) \;>_w\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

and hence $\mathrm{lm}_{>_w}(g_i) = \mathrm{lm}_>(g_i)$. □

Lemma 2.4
Let $>$ be a $\underline{t}$-local weight ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ with weight vector $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$, and let $(f_\nu)_{\nu\in\mathbb{N}}$ be a sequence of $\underline{x}$-homogeneous elements of fixed $\underline{x}$-degree d in $R[\underline{x}]^s$ such that

$$\mathrm{lm}_>(f_\nu) > \mathrm{lm}_>(f_{\nu+1}) \quad \text{for all } \nu \in \mathbb{N}.$$

Then f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, i.e.

$$\forall\, N \geq 0\ \exists\, \mu_N \geq 0\ :\ \forall\, \nu \geq \mu_N \ \text{ we have }\ f_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s.$$

In particular, the element $\sum_{\nu=0}^\infty f_\nu \in R[\underline{x}]^s_d$ exists.

Proof: Since $w_1,\ldots,w_m < 0$ the set of monomials

$$M_k = \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \;\big|\; w\cdot(\alpha,\beta,e_i) > -k, |\beta| = d\big\}.$$

is finite for a any fixed $k \in \mathbb{N}$.
Let $N \geq 0$ be fixed, set $\tau = \max\{|w_1|,\ldots,|w_{m+n+s}|\}$ and $k := (N + nd + 1)\cdot\tau$, then for any monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ of $\underline{x}$-degree d

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \notin M_k \quad\Longrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s, \qquad (3)$$

since

$$\sum_{i=1}^m \alpha_i \cdot w_i \leq -k - \sum_{i=1}^n \beta_i \cdot w_{m+i} - w_{m+n+j} \leq -k + (nd+1)\cdot\tau$$

and thus

$$|\alpha| = \sum_{i=1}^m \alpha_i \geq \sum_{i=1}^m \alpha_i \cdot \frac{-w_i}{\tau} \geq \frac{k}{\tau} - nd - 1 = N.$$

Moreover, since M_k is finite and the $\mathrm{lm}_>(f_\nu)$ are pairwise different there are only finitely many ν such that $\mathrm{lm}_>(f_\nu) \in M_k$. Let μ be maximal among those ν, then by (3)

$$\mathrm{lm}_>(f_\nu) \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{for all} \quad \nu > \mu.$$

But since $>$ is a $\underline{t}$-local weight ordering we have that $\mathrm{lm}_>(f_\nu) \notin M_k$ implies that no monomial of f_ν is in M_k, and thus $f_\nu \in \langle t_1,\ldots,t_m\rangle^N \cdot R[\underline{x}]^s$ for all $\nu > \mu$ by (3). This shows that f_ν converges to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology.

Since f_ν converges to zero in the $\langle t_1,\dots,t_m\rangle$-adic topology, for every monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ there is only a finite number of ν's such that $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ is a monomial occurring in f_ν. Thus the sum $\sum_{\nu=0}^\infty f_\nu$ exists and is obviously $\underline{x}$-homogeneous of degree d. □

From the proof of Theorem 2.1 we can deduce an algorithm for computing the determinate division with remainder up to arbitrary order, or if we don't require termination then it will "compute" the determinate division with remainder completely. Since for our purposes termination is not important, we will simply formulate the non-terminating algorithm.

Algorithm 2.5 (HDDwR)

INPUT: (f,G) with $G=\{g_1,\dots,g_k\}$ and $f,g_1,\dots,g_k\in R[\underline{x}]^s$ $\underline{x}$-homogeneous, $>$ a $\underline{t}$-local monomial ordering

OUTPUT: $(q_1,\dots,q_k,r)\in R[\underline{x}]^k\times R[\underline{x}]^s$ such that

$$f=q_1\cdot g_1+\ldots+q_k\cdot g_k+r$$

is a homogeneous determinate division with remainder of f satisfying (DD1), (DD2) and (DDH).

INSTRUCTIONS:

- $f_0:=f$
- $r:=0$
- FOR $i=1,\dots,k$ DO $q_i:=0$
- $\nu:=0$
- WHILE $f_\nu\neq 0$ DO
  - $q_{0,\nu}:=0$
  - FOR $i=1,\dots,k$ DO
    * $h_{i,\nu}:=\sum_{p\in\mathcal{T}_{f_\nu}\,:\,\mathrm{lm}_>(g_i)\,|\,p}p$
    * $q_{i,\nu}:=\frac{h_{i,\nu}}{\mathrm{lt}_>(g_i)}$
    * $q_i:=q_i+q_{i,\nu}$
  - $r_\nu:=f_\nu-q_{1,\nu}\cdot\mathrm{lt}_>(g_1)-\ldots-q_{k,\nu}\cdot\mathrm{lt}_>(g_k)$
  - $r:=r+r_\nu$
  - $f_{\nu+1}:=f_\nu-q_{1,\nu}\cdot g_1-\ldots-q_{k,\nu}\cdot g_k-r_\nu$
  - $\nu:=\nu+1$

Remark 2.6
If $m = 0$, i.e. if the input data $f, g_1, \ldots, g_k \in K[\underline{x}]^s$, then Algorithm 2.5 terminates since for a given degree there are only finitely many monomials of this degree and therefore there cannot exist an infinite sequence of homogeneous polynomials $(f_\nu)_{\nu\in\mathbb{N}}$ of the same degree with

$$\operatorname{lm}_>(f_1) > \operatorname{lm}_>(f_2) > \operatorname{lm}_>(f_3) > \ldots.$$

3. Division with Remainder in $K[[\underline{t}]][\underline{x}]^s$

We will use the existence of homogeneous determinate divisions with remainder to show that in $R[\underline{x}]^s$ weak normal forms exist. In order to be able to apply this existence result we have to homogenise, and we need to extend our monomial ordering to the homogenised monomials.

Definition 3.1
Let $\underline{x}_h = (x_0, \underline{x}) = (x_0, \ldots, x_n)$.

(a) For $0 \neq f \in R[\underline{x}]^s$. We define the *homogenisation* f^h of f to be

$$f^h := x_0^{\deg_{\underline{x}}(f)} \cdot f\left(\underline{t}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in R[\underline{x}_h]^s_{\deg_{\underline{x}}(f)}$$

and $0^h := 0$. If $T \subset R[\underline{x}]^s$ then we set $T^h := \{f^h \mid f \in T\}$.

(b) We call the $R[\underline{x}]$-linear map $d : R[\underline{x}_h]^s \longrightarrow R[\underline{x}]^s : g \mapsto g^d := g_{|x_0=1}$ the *dehomogenisation* with respect to x_0.

(c) Given a $\underline{t}$-local monomial ordering $>$ on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ we define a $\underline{t}$-local monomial ordering $>_h$ on $\operatorname{Mon}^s(\underline{t}, \underline{x}_h)$ by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot x_0^a \cdot e_i \ >_h\ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot x_0^{a'} \cdot e_j$$

if and only if

$$|\beta| + a > |\beta'| + a'$$

or

$$\big(|\beta| + a = |\beta'| + a' \ \text{ and } \ \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j\big),$$

and we call it the *homogenisation of* $>$.

In the following remark we want to gather some straight forward properties of homogenisation and dehomogenisation.

Remark 3.2
Let $f, g \in R[\underline{x}]^s$ and $F \in R[\underline{x}_h]^s_k$. Then:

(a) $f = (f^h)^d$.
(b) $F = (F^d)^h \cdot x_0^{\deg_{\underline{x}_h}(F) - \deg_{\underline{x}}(F^d)}$.
(c) $\mathrm{lm}_{>_h}(f^h) = x_0^{\mathrm{ecart}(f)} \cdot \mathrm{lm}_>(f)$.
(d) $\mathrm{lm}_{>_h}(g^h) | \mathrm{lm}_{>_h}(f^h) \Longleftrightarrow \mathrm{lm}_>(g) | \mathrm{lm}_>(f) \ \wedge \ \mathrm{ecart}(g) \leq \mathrm{ecart}(f)$.
(e) $\mathrm{lm}_{>_h}(F) = x_0^{\mathrm{ecart}(F^d) + \deg_{\underline{x}_h}(F) - \deg_{\underline{x}}(f)} \cdot \mathrm{lm}_>(F^d)$.

Theorem 3.3 (Division with Remainder)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s$. Then any $f \in R[\underline{x}]^s$ has a weak division with remainder with respect to $g_1, \ldots, g_k$.

Proof: The proof follows from the correctness and termination of Algorithm 3.4, which assumes the existence of the homogeneous determinate division with remainder from Theorem 2.1 respectively Algorithm 2.5. □

The following algorithm relies on the HDDwR-Algorithm, and it only terminates under the assumption that we are able to produce homogeneous determinate divisions with remainder, which implies that it is not an algorithm that can be applied in practise.

Algorithm 3.4 (DwR - Mora's Division with Remainder)
INPUT: (f, G) with $G = \{g_1, \ldots, g_k\}$ and $f, g_1, \ldots, g_k \in R[\underline{x}]^s$, $>$ a $\underline{t}$-local monomial ordering
OUTPUT: $(u, q_1, \ldots, q_k, r) \in S_> \times R[\underline{x}]^k \times R[\underline{x}]^s$ such that

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is a weak division with remainder of f.
INSTRUCTIONS:

- $T := (g_1, \ldots, g_k)$
- $D := \{g_i \in T \mid \mathrm{lm}_>(g_i) \ \text{divides} \ \mathrm{lm}_>(f)\}$
- IF $f \neq 0$ AND $D \neq \emptyset$ DO
  - IF $e := \min\{\mathrm{ecart}_>(g_i) \mid g_i \in D\} - \mathrm{ecart}_>(f) > 0$ THEN
    - $(Q_1', \ldots, Q_k', R') := \mathrm{HDDwR}\left(x_0^e \cdot f^h, (\mathrm{lt}_{>_h}(g_1^h), \ldots, \mathrm{lt}_{>_h}(g_k^h))\right)$
    - $f' := \left(x_0^e \cdot f^h - \sum_{i=1}^k Q_i' \cdot g_i^h\right)^d$

    - $(u'', q''_1, \ldots, q''_{k+1}, r) := \mathrm{DwR}\big(f', (g_1, \ldots, g_k, f)\big)$
    - $q_i := q''_i + u'' \cdot {Q'_i}^d, \quad i = 1, \ldots, k$
    - $u := u'' - q''_{k+1}$
  - ELSE
    - $(Q'_1, \ldots, Q'_k, R') := \mathrm{HDDwR}\big(f^h, (g_1^h, \ldots, g_k^h)\big)$
    - $(u, q''_1, \ldots, q''_{k+1}, r) := \mathrm{DwR}\big((R')^d, T\big)$
    - $q_i := q''_i + u \cdot {Q'_i}^d, \quad i = 1, \ldots, k$
- ELSE $(u, q_1, \ldots, q_k, r) = (1, 0, \ldots, 0, f)$

Proof: Let us first prove the *termination*. For this we denote the numbers, ring elements and sets, which occur in the ν-th recursion step by a subscript ν, e.g. e_ν, f_ν or T_ν. Since

$$T_1^h \subseteq T_2^h \subseteq T_3^h \subseteq \ldots$$

also their leading submodules in $K[\underline{t}, \underline{x}_h]^s$ form an ascending chain

$$L_{>_h}(T_1^h) \subseteq L_{>_h}(T_2^h) \subseteq L_{>_h}(T_3^h) \subseteq \ldots,$$

and since the polynomial ring is noetherian there must be an N such that

$$L_{>_h}(T_\nu^h) = L_{>_h}(T_N^h) \quad \forall\, \nu \geq N.$$

If $g_{i,N} \in T_N$ such that $\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N)$ with $\mathrm{ecart}_>(g_{i,N}) \leq \mathrm{ecart}_>(f_N)$, then

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \;\big|\; \mathrm{lm}_{>_h}(f_N^h).$$

We thus have either $\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$ for some $g_i \in D^N \subseteq T^{N+1}$ or $f_N \in T_{N+1}$, and hence

$$\mathrm{lm}_{>_h}(f_N^h) \in L_{>_h}(T_{N+1}^h) = L_{>_h}(T_N^h).$$

This ensures the existence of a $g_{i,N} \in T_N$ such that

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$$

which in turn implies that

$$\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N),$$

$e_N \leq \mathrm{ecart}_>(g_{i,N}) - \mathrm{ecart}_>(f_N) \leq 0$ and $T_N = T_{N+1}$. By induction we conclude

$$T_\nu = T_N \quad \forall\, \nu \geq N,$$

and

$$e_\nu \leq 0 \quad \forall\, \nu \geq N. \tag{4}$$

Since in the N-th recursion step we are in the first "ELSE" case we have $(R'_N)^d = f_{N+1}$, and by the properties of HDDwR we know that for all $g \in T_N$

$$x_0^{\mathrm{ecart}_>(g)} \cdot \mathrm{lm}_>(g) = \mathrm{lm}_{>_h}(g^h) \;\not|\; \mathrm{lm}_{>_h}(R'_N)$$

and that

$$\mathrm{lm}_{>_h}(R'_N) = x_0^a \cdot \mathrm{lm}_{>_h}(f^h_{N+1}) = x_0^{a+\mathrm{ecart}_>(f_{N+1})} \cdot \mathrm{lm}_>(f_{N+1})$$

for some $a \geq 0$. It follows that, whenever $\mathrm{lm}_>(g) \mid \mathrm{lm}_>(f_{N+1})$, then necessarily

$$\mathrm{ecart}_>(g) > a + \mathrm{ecart}_>(f_{N+1}) \geq \mathrm{ecart}_>(f_{N+1}). \tag{5}$$

Suppose now that $f_{N+1} \neq 0$ and $D_{N+1} \neq \emptyset$. Then we may choose $g_{i,N+1} \in D_{N+1} \subseteq T_{N+1} = T_N$ such that

$$\mathrm{lm}_>(g_{i,N+1}) \;\big|\; \mathrm{lm}_>(f_{N+1})$$

and

$$e_{N+1} = \mathrm{ecart}_>(g_{i,N+1}) - \mathrm{ecart}_>(f_{N+1}).$$

According to (4) e_{N+1} is non-positive, while according to (5) it must be strictly positive. Thus we have derived a contradiction which shows that either $f_{N+1} = 0$ or $D_{N+1} = \emptyset$, and in any case the algorithm stops.
Next we have to prove the *correctness*. We do this by induction on the number of recursions, say N, of the algorithm.
If $N = 1$ then either $f = 0$ or $D = \emptyset$, and in both cases

$$1 \cdot f = 0 \cdot g_1 + \ldots + 0 \cdot g_k + f$$

is a weak division with remainder of f satisfying (ID1) and (ID2). We may thus assume that $N > 1$ and $e = \min\{\mathrm{ecart}_>(g) \mid g \in D\} - \mathrm{ecart}_>(f)$.
If $e \leq 0$ then by Theorem 2.1

$$f^h = Q'_1 \cdot g^h_1 + \ldots + Q'_k \cdot g^h_k + R'$$

satisfies (DD1), (DD2) and (DDH). (DD1) implies that for each $i = 1,\ldots,k$ we have

$$x_0^{\operatorname{ecart}_>(f)}\cdot\operatorname{lm}_>(f)=\operatorname{lm}_{>_h}(f^h)\geq \operatorname{lm}_{>_h}(Q_i')\cdot\operatorname{lm}_{>_h}(g_i^h)=x_0^{a_i+\operatorname{ecart}_>(g_i)}\cdot\operatorname{lm}_>\left({Q_i'}^d\right)\cdot\operatorname{lm}_>(g_i)$$

for some $a_i\geq 0$, and since f^h and $Q_i'\cdot g_i^h$ are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree by (DDH) the definition of the homogenised ordering implies that necessarily

$$\operatorname{lm}_>(f)\geq\operatorname{lm}_>\left({Q_i'}^d\right)\cdot\operatorname{lm}_>(g_i)\quad\forall\, i=1,\ldots,k.$$

Note that

$$(R')^d=\left(f^h-\sum_{i=1}^k Q_i'\cdot g_i^h\right)^d=f-\sum_{i=1}^k {Q_i'}^d\cdot g_i,$$

and thus

$$\operatorname{lm}_>\left((R')^d\right)=\operatorname{lm}_>\left(f-\sum_{i=1}^k {Q_i'}^d\cdot g_i\right)\leq\operatorname{lm}_>(f).$$

Moreover, by induction

$$u\cdot(R')^d=q_1''\cdot g_1+\ldots q_k''\cdot g_k+r$$

satisfies (ID1) and (ID2). But (ID1) implies that

$$\operatorname{lm}_>(f)\ \geq\ \operatorname{lm}_>\left((R')^d\right)\ \geq\ \operatorname{lm}_>(q_i''\cdot g_i),$$

so that

$$u\cdot f=\sum_{i=1}^k\left(q_i''+u\cdot {Q_i'}^d\right)\cdot g_i+r$$

satisfies (ID1) and (ID2).

It remains to consider the case $e>0$. Then by Theorem 2.1

$$x_0^e\cdot f^h=Q_1'\cdot\operatorname{lt}_{>_h}(g_1^h)+\ldots+Q_k'\cdot\operatorname{lt}_{>_h}(g_k^h)+R' \tag{6}$$

satisfies (DD1), (DD2) and (DDH). (DD1) and (DD2) imply (ID1) for this representation, which means that for some $a_i\geq 0$

$$x_0^{e+\operatorname{ecart}_>(f)}\cdot\operatorname{lm}_>(f)=\operatorname{lm}_{>_h}(x_0^e\cdot f^h)\geq \operatorname{lm}_{>_h}(Q_i')\cdot\operatorname{lm}_{>_h}\left(\operatorname{lt}_{>_h}(g_i^h)\right)=x_0^{a_i+\operatorname{ecart}_>(g_i)}\cdot\operatorname{lm}_>({Q_i'}^d)\cdot\operatorname{lm}_>(g_i),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree with by (DDH) we again necessarily have

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_>\left({Q'_i}^d\right) \cdot \mathrm{lm}_>(g_i).$$

Moreover, by induction

$$u'' \cdot \left(f - \sum_{i=1}^{k} {Q'_i}^d \cdot g_i \right) = \sum_{i=1}^{k} q''_i \cdot g_i + q''_{k+1} \cdot f + r \tag{7}$$

satisfies (ID1) and (ID2).
Since $\mathrm{lt}_>(u'') = 1$ we have

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_>\left(q''_i + u'' \cdot {Q'_i}^d\right) \cdot \mathrm{lm}_>(g_i),$$

for $i = 1, \ldots, k$ and therefore

$$(u'' - q''_{k+1}) \cdot f = \sum_{i=1}^{k} \left(q''_i + u'' \cdot {Q'_i}^d\right) \cdot g_i + r$$

satisfies (ID1) and (ID2) as well. It remains to show that $u = u'' - q''_{k+1} \in S_>$, or equivalently that

$$\mathrm{lt}_>(u'' - q''_{k+1}) = 1.$$

By assumption there is a $g_i \in D$ such that $\mathrm{lm}_>(g_i) \mid \mathrm{lm}_>(f)$ and $\mathrm{ecart}_>(g_i) - \mathrm{ecart}_>(f) = e$. Therefore, $\mathrm{lm}_{>_h}(g_i^h) \mid x_0^e \cdot \mathrm{lm}_{>_h}(f^h)$ and thus in the representation (6) the leading term of $x_0^e \cdot f^h$ has been cancelled by some $Q'_j \cdot \mathrm{lt}_{>_h}(g_j^h)$, which implies that

$$\mathrm{lm}_{>_h}(f^h) > \mathrm{lm}_{>_h}\left(f^h - \sum_{i=1}^{k} Q'_i \cdot g_i^h \right),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree, unless the right hand side is zero, we must have

$$\mathrm{lm}_>(f) > \mathrm{lm}_>\left(f - \sum_{i=1}^{k} {Q'_i}^d \cdot g_i \right) \geq \mathrm{lm}_>(q''_{k+1} \cdot f),$$

where the latter inequality follows from (ID1) for (7). Thus however $\mathrm{lm}_>(q''_{k+1}) < 1$, and since $\mathrm{lm}_>(u'') = 1$ we conclude that

$$\mathrm{lt}_>(u'' - q''_{k+1}) = \mathrm{lt}_>(u'') = 1.$$

This finishes the proof. □

Remark 3.5
As we have pointed out our algorithms are not useful for computational purposes since Algorithm 2.5 does not in general terminate after a finite number of steps. If, however, the input data are in fact polynomials in $\underline{t}$ and $\underline{x}$, then we can replace the t_i by x_{n+i} and apply Algorithm 3.4 to $K[x_1,\ldots,x_{n+m}]^s$, so that it terminates due to Remark 2.6 the computed weak division with remainder

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is then *polynomial* in the sense that $u, q_1, \ldots, q_k \in K[\underline{t}, \underline{x}]$ and $r \in K[\underline{t}, \underline{x}]^s$. In fact, Algorithm 3.4 is then only a variant of the usual Mora algorithm.

In the proof of Schreyer's Theorem we will need the existence of weak divisions with remainder satisfying (SID2), the proof is the same as [GrP02, Remark 2.3.4].

Corollary 3.6
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s_>$. Then any $f \in R[\underline{x}]^s_>$ has a division with remainder with respect to $g_1, \ldots, g_k$ satisfying (SID2).

4. Standard Bases in $K[[\underline{t}]][\underline{x}]^s$

Definition 4.1
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $I \leq R[\underline{x}]^s$ and $J \leq R[\underline{x}]^s_>$ be submodules. A *standard basis of* I is a finite subset $G \subset I$ such that $L_>(I) = L_>(G)$. A *standard basis of* J is a finite subset $G \subset J$ such that $L_>(J) = L_>(G)$. A finite subset $G \subseteq R[\underline{x}]^s_>$ is called a *standard basis* with respect to $>$ if G is a standard basis of $\langle G \rangle \leq R[\underline{x}]^s_>$.

The existence of standard bases is immediate from Hilbert's Basis Theorem.

Proposition 4.2
If $>$ is a $\underline{t}$-local monomial ordering then every submodule of $R[\underline{x}]^s$ and of $R[\underline{x}]^s_>$ has a standard basis.

Standard bases are so useful since they are generating sets for submodules of $R[\underline{x}]^s_>$ and since submodule membership can be tested by division with remainder.

Proposition 4.3
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $I, J \leq R[\underline{x}]^s_>$ submodules, $G = (g_1, \ldots, g_k) \subset J$ a standard basis of J and $f \in R[\underline{x}]^s_>$ with division with remainder $f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$. Then:

(a) *$f \in J$ if and only if $r = 0$.*
(b) *$J = \langle G \rangle$.*
(c) *If $I \subseteq J$ and $L_>(I) = L_>(J)$, then $I = J$.*

Proof: Word by word as in [GrP02, Lemma 1.6.7]. □

In order to work, even theoretically, with standard bases it is vital to have a good criterion to decide whether a generating set is standard basis or not. In order to formulate Buchberger's Criterion it is helpful to have the notion of an *s-polynomial*.

Definition 4.4
Let $>$ be a $\underline{t}$-local monomial ordering on $R[\underline{x}]^s$ and $f, g \in R[\underline{x}]^s$. We define the *s-polynomial* of f and g as

$$\mathrm{spoly}(f,g) := \frac{\mathrm{lcm}\big(\mathrm{lm}_>(f), \mathrm{lm}_>(g)\big)}{\mathrm{lt}_>(f)} \cdot f - \frac{\mathrm{lcm}\big(\mathrm{lm}_>(f), \mathrm{lm}_>(g)\big)}{\mathrm{lt}_>(g)} \cdot g.$$

Theorem 4.5 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $J \leq R[\underline{x}]^s_>$ a submodule and $g_1, \ldots, g_k \in J$. The following statements are equivalent:

(a) *$G = (g_1, \ldots, g_k)$ is a standard basis of J.*
(b) *Every normal form with respect to G of any element in J is zero.*
(c) *Every element in J has a standard representation with respect to G.*
(d) *$J = \langle G \rangle$ and $\mathrm{spoly}(g_i, g_j)$ has a standard representation for all $i < j$.*

Proof: In Proposition 4.3 we have shown that (a) implies (b), and the implication (b) to (c) is trivially true. And, finally, if $f \in J$ has a standard representation with respect to G, then $\mathrm{lm}_>(f) \in L_>(G)$, so that (c) implies (a). Since $\text{spoly}(g_i, g_j) \in J$ condition (d) follows from (c), and the hard part is to show that (d) implies actually (c). This is postponed to Theorem 5.3. □

Since for $G \subset R[\underline{x}]^s$ we have $L_>\big(\langle G\rangle_{R[\underline{x}]}\big) = L_>\big(\langle G\rangle_{R[\underline{x}]_>}\big)$ we get the following corollary.

Corollary 4.6 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in I \leq R[\underline{x}]^s$. Then the following statements are equivalent:

(a) *$G = (g_1, \ldots, g_k)$ is a standard basis of I.*
(b) *Every weak normal form with respect to G of any element in I is zero.*
(c) *Every element in I has a weak standard representation with respect to G.*
(d) *$\langle I\rangle_{R[\underline{x}]_>} = \langle G\rangle_{R[\underline{x}]_>}$ and $\text{spoly}(g_i, g_j)$ has a weak standard representation for all $i < j$.*

When working with polynomials in $\underline{x}$ as well as in $\underline{t}$ we can actually compute divisions with remainder and standard bases (see Remark 3.5), and they are also standard bases of the corresponding submodules considered over $R[\underline{x}]$ by the following corollary.

Corollary 4.7
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ and let $G \subset K[\underline{t}, \underline{x}]^s$ be finite. Then G is a standard basis of $\langle G\rangle_{K[\underline{t},\underline{x}]}$ if and only if G is a standard basis of $\langle G\rangle_{R[\underline{x}]}$.

Proof: Let $G = (g_1, \ldots, g_k)$. By Theorem 3.3 and Remark 3.5 each $\text{spoly}(g_i, g_j)$ has a weak division with remainder with respect to G such that the coefficients and remainders involved are polynomials in $\underline{x}$ as well as in $\underline{t}$. But by Corollary 4.6 G is a standard basis of either of $\langle G\rangle_{K[\underline{t},\underline{x}]}$ and $\langle G\rangle_{R[\underline{x}]}$ if and only if all these remainders are actually zero. □

And thus it makes sense to formulate the classical standard basis algorithm also for the case $R[\underline{x}]$.

Algorithm 4.8 (STD – Standard Basis Algorithm)
INPUT: $(f_1, \ldots, f_k) \in \big(R[\underline{x}]^s\big)^k$ and $>$ a $\underline{t}$-local monomial ordering.

OUTPUT: $(f_1,\ldots,f_l) \in \left(R[\underline{x}]^s\right)^l$ a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$.

INSTRUCTIONS:

- $G = (f_1,\ldots,f_k)$
- $P = \big((f_i,f_j) \mid 1 \leq i < j \leq k\big)$
- WHILE $P \neq \emptyset$ DO
  - Choose some pair $(f,g) \in P$
  - $P = P \setminus \{(f,g)\}$
  - $(u,\underline{q},r) = \mathrm{DwR}\left(\mathrm{spoly}(f,g), G\right)$
  - IF $r \neq 0$ THEN
    - $P = P \cup \{(f,r) \mid f \in G\}$
    - $G = G \cup \{r\}$

Remark 4.9
If the input of STD are polynomials in $K[\underline{t},\underline{x}]$ then the algorithm works in practise due to Remark 3.5, and it computes a standard basis G of $\langle f_1,\ldots,f_k\rangle_{K[\underline{t},\underline{x}]}$ which due to Corollary 4.7 is also a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$, since G still contains the generators $f_1,\ldots,f_k$.
Having division with remainder, standard bases and Buchberger's Criterion at hand one can, from a theoretical point of view, basically derive all the standard algorithms from computer algebra also for free modules over $R[\underline{x}]$ respectively $R[\underline{x}]_>$. Moreover, if the input is polynomial in $\underline{t}$ and $\underline{x}$, then the corresponding operations computed over $K[\underline{t},\underline{x}]_>$ will also lead to generating sets for the corresponding operations over $R[\underline{x}]_>$.

5. SCHREYER'S THEOREM FOR $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

In this section we want to prove Schreyer's Theorem for $R[\underline{x}]^s$ which proves Buchberger's Criterion and shows at the same time that a standard basis of a submodule gives rise to a standard basis of the syzygy module defined by it with respect to a special ordering.

Definition 5.1 (Schreyer Ordering)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k \in R[\underline{x}]^s_>$. We define a *Schreyer ordering* with respect to $>$ and $(g_1,\ldots,g_k)$,

say $>_S$, on $\operatorname{Mon}^k(\underline{t},\underline{x})$ by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \varepsilon_i \;>_S\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \varepsilon_j$$

if and only if

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \operatorname{lm}_>(g_i) \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \operatorname{lm}_>(g_j)$$

or

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \operatorname{lm}_>(g_i) \;=\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \operatorname{lm}_>(g_j) \;\text{ and }\; i < j,$$

where $\varepsilon_i = (\delta_{ij})_{j=1,\ldots,k}$ is the canonical basis with i-th entry one and the rest zero.
Moreover, we define the *syzygy module of* $(g_1, \ldots, g_k)$ to be

$$\operatorname{syz}(g_1, \ldots, g_k) := \{(q_1, \ldots, q_k) \in R[\underline{x}]^k_> \mid q_1 \cdot g_1 + \ldots + q_k \cdot g_k = 0\},$$

and we call the elements of $\operatorname{syz}(g_1, \ldots, g_k)$ *syzygies* of $g_1, \ldots, g_k$.

Remark 5.2
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s_>$. Let us fix for each $i < j$ a division with remainder of $\operatorname{spoly}(g_i, g_j)$, say

$$\operatorname{spoly}(g_i, g_j) = \sum_{\nu=1}^{k} q_{i,j,\nu} \cdot g_\nu + r_{ij}, \tag{8}$$

and define

$$m_{ji} := \frac{\operatorname{lcm}\big(\operatorname{lm}_>(g_i), \operatorname{lm}_>(g_j)\big)}{\operatorname{lm}_>(g_i)},$$

so that

$$\operatorname{spoly}(g_i, g_j) = \frac{m_{ji}}{\operatorname{lc}_>(g_i)} \cdot g_i - \frac{m_{ij}}{\operatorname{lc}_>(g_j)} \cdot g_j.$$

Then

$$s_{ij} := \frac{m_{ji}}{\operatorname{lc}_>(g_i)} \cdot \varepsilon_i - \frac{m_{ij}}{\operatorname{lc}_>(g_j)} \cdot \varepsilon_j - \sum_{\nu=1}^{k} q_{i,j,\nu} \cdot \varepsilon_\nu \in R[\underline{x}]^k_>$$

has the property

$$s_{ij} \in \operatorname{syz}(g_1, \ldots, g_k) \quad\Longleftrightarrow\quad r_{ij} = 0.$$

Theorem 5.3 (Schreyer)
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t},\underline{x})$*,* $g_1,\ldots,g_k \in R[\underline{x}]^s_>$ *and suppose that* $\operatorname{spoly}(g_i,g_j)$ *has a weak standard representation with respect to* $G=(g_1,\ldots,g_k)$ *for each* $i<j$.
Then G is a standard basis, and with the notation in Remark 5.2 $\{s_{ij}\mid i<j\}$ *is a standard basis of* $\operatorname{syz}(g_1,\ldots,g_k)$ *with respect to* $>_S$.

Proof: The same as in [GrP02, Theorem 2.5.9]. □

6. Application to t-Initial Ideals

In this section we want to show that for an ideal J over the field of Puiseux series which is generated by elements in $K[[t^{\frac{1}{N}}]][\underline{x}]$ respectively in $K[t^{\frac{1}{N}},\underline{x}]$ the t-initial ideal (a notion we will introduce further down) with respect to $w\in\mathbb{Q}_{<0}\times\mathbb{Q}^n$ can be computed from a standard basis of the generators.

Definition 6.1
We consider for $0\neq N\in\mathbb{N}$ the discrete valuation ring

$$R_N\big[\big[t^{\frac{1}{N}}\big]\big]=\left\{\sum_{\alpha=0}^{\infty}a_\alpha\cdot t^{\frac{\alpha}{N}}\;\Big|\;a_\alpha\in K\right\}$$

of power series in the unknown $t^{\frac{1}{N}}$ with *discrete valuation*

$$\operatorname{val}\left(\sum_{\alpha=0}^{\infty}a_\alpha\cdot t^{\frac{\alpha}{N}}\right)=\operatorname{ord}_t\left(\sum_{\alpha=0}^{\infty}a_\alpha\cdot t^{\frac{\alpha}{N}}\right)=\min\left\{\frac{\alpha}{N}\;\Big|\;a_\alpha\neq 0\right\}\in\frac{1}{N}\cdot\mathbb{Z},$$

and we denote by $L_N=\operatorname{Quot}(R_N)$ its quotient field. If $N\mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L=K\{\{t\}\}=\varinjlim L_N=\bigcup_{N\geq 0}L_N$$

the *field of (formal) Puiseux series* over K.

Remark 6.2
If $0\neq N\in\mathbb{N}$ then $S_N=\{1,t^{\frac{1}{N}},t^{\frac{2}{N}},t^{\frac{2}{N}},\ldots\}$ is a multiplicative subset of R_N, and obviously $L_N=S_N^{-1}R_N=\{t^{\frac{-\alpha}{N}}\cdot f\mid f\in R_N,\alpha\in\mathbb{N}\}$, since $R_N^*=\{\sum_{\alpha=0}^{\infty}a_\alpha\cdot t^{\frac{\alpha}{N}}\mid a_0\neq 0\}$. The valuations of R_N extend to L_N, and thus L, by $\operatorname{val}\left(\frac{f}{g}\right)=\operatorname{val}(f)-\operatorname{val}(g)$ for $f,g\in R_N$ with $g\neq 0$.

Definition 6.3
For $0 \neq N \in \mathbb{N}$ if we consider $t^{\frac{1}{N}}$ as a variable, we get the set of monomials $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) = \left\{t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta} \;\middle|\; \alpha \in \mathbb{N}, \beta \in \mathbb{N}^n\right\}$ in $t^{\frac{1}{N}}$ and $\underline{x}$. If $N \mid M$ then obviously $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) \subset \operatorname{Mon}\left(t^{\frac{1}{M}}, \underline{x}\right)$.

Remark and Definition 6.4
Let $0 \neq N \in \mathbb{N}$, $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.
We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta}$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta}.$$

As a K-vector space the formal power series ring $K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right]$ is just

$$K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right] = \prod_{q \in \mathbb{R}} V_{q,w,N},$$

and we can thus write any power series $f \in K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right]$ in a unique way as

$$f = \sum_{q \in \mathbb{R}} f_{q,w} \quad \text{with} \quad f_{q,w} \in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f \in K\left[\left[t^{\frac{1}{N'}}, \underline{x}\right]\right]$ for some other $0 \neq N' \in \mathbb{N}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.
Moreover, if $0 \neq f \in R_N[\underline{x}] \subset K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right]$, then there is a maximal $\hat{q} \in \mathbb{R}$ such that $f_{\hat{q},w} \neq 0$ and $f_{q,w} \in W_{q,w,N}$ for all $q \in \mathbb{R}$, since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w}) = q \in \mathbb{R}$,

$$\operatorname{in}_w(f) = f_{\hat{q},w} \in K\left[t^{\frac{1}{N}}, \underline{x}\right]$$

the *w-initial form* of f or the *initial form of f w.r.t. w*, and

$$\operatorname{ord}_w(f) = \hat{q} = \max\{\deg_w(f_{q,w}) \mid f_{q,w} \neq 0\}$$

the w-*order* of f. For $I \subseteq R_N[\underline{x}]$ we call

$$\mathrm{in}_w(I) = \big\langle \mathrm{in}_w(f) \bigm| f \in I \big\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big]$$

the w-*initial ideal* of I. Note that its definition depends on N! Moreover, we call

$$\text{t-in}_w(f) = \mathrm{in}_w(f)(1, \underline{x}) = \mathrm{in}_w(f)_{|t=1} \in K[\underline{x}]$$

the *t-initial form of f w.r.t. w*, and if $f = t^{\frac{-\alpha}{N}} \cdot g \in L[\underline{x}]$ with $g \in R_N[\underline{x}]$ we set $\text{t-in}_w(f) := \text{t-in}_w(g)$. This definition does not depend on the particular representation of f. If $I \subseteq L[\underline{x}]$ is an ideal, then

$$\text{t-in}_w(I) = \langle \text{t-in}_w(f) \mid f \in I \rangle \lhd K[\underline{x}]$$

is the *t-initial ideal* of I, which does not depend on any N.
Note also that the product of two w-quasihomogeneous elements $f_{q,w} \cdot f_{q',w} \in V_{q+q',w,N}$, and in particular, $\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$ for $f, g \in R_N[\underline{x}]$, and for $f, g \in L[\underline{x}]$ $\text{t-in}_w(f \cdot g) = \text{t-in}_w(f) \cdot \text{t-in}_w(g)$. An immediate consequence of this is the following lemma.

Lemma 6.5
If $0 \neq f = \sum_{i=1}^{k} g_i \cdot h_i$ *with* $f, g_i, h_i \in R_N[\underline{x}]$ *and* $\mathrm{ord}_w(f) \geq \mathrm{ord}_w(g_i \cdot h_i)$ *for all* $i = 1, \ldots, k$*, then*

$$\mathrm{in}_w(f) \in \big\langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k) \big\rangle \lhd K\big[t^{\frac{1}{N}}, \underline{x}\big].$$

Proof: Due to the direct product decomposition we have that

$$\mathrm{in}_w(f) = f_{\hat{q},w} = \sum_{i=1}^{k} (g_i \cdot h_i)_{\hat{q},w}$$

where $\hat{q} = \mathrm{ord}_w(f)$. By assumption $\mathrm{ord}_w(g_i) + \mathrm{ord}_w(h_i) = \mathrm{ord}_w(g_i \cdot h_i) \leq \mathrm{ord}_w(f) = \hat{q}$ with equality if and only if $(g_i \cdot h_i)_{\hat{q},w} \neq 0$. In that case necessarily $(g_i \cdot h_i)_{\hat{q},w} = \mathrm{in}_w(g_i) \cdot \mathrm{in}_w(h_i)$, which finishes the proof. □

In order to be able to apply standard bases techniques we need to fix a t-local monomial ordering which refines a given weight vector w.

Definition 6.6
Fix any *global* monomial ordering, say $>$, on $\mathrm{Mon}(\underline{x})$ and let $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$.

We define a t-local monomial ordering, say $>_w$, on $\operatorname{Mon}\left(t^{\frac{1}{N}},\underline{x}\right)$ by

$$t^{\frac{\alpha}{N}}\cdot\underline{x}^{\beta}\;>_w\;t^{\frac{\alpha'}{N}}\cdot\underline{x}^{\beta'}$$

if and only if

$$w\cdot\left(\frac{\alpha}{N},\beta\right)>w\cdot\left(\frac{\alpha'}{N},\beta'\right)$$

or

$$w\cdot\left(\frac{\alpha}{N},\beta\right)=w\cdot\left(\frac{\alpha'}{N},\beta'\right)\quad\text{and}\quad\underline{x}^{\beta}\;>\;\underline{x}^{\beta'}.$$

Note that this ordering is indeed t-local since $w_0<0$, and that it depends on w and on $>$, but assuming that $>$ is fixed we will refrain from writing $>_{w,>}$ instead of $>_w$.

Remark 6.7
If $N\mid M$ then $\operatorname{Mon}\left(t^{\frac{1}{N}},\underline{x}\right)\subset\operatorname{Mon}\left(t^{\frac{1}{M}},\underline{x}\right)$, as already mentioned. For $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$ we may thus consider the ordering $>_w$ on both $\operatorname{Mon}\left(t^{\frac{1}{N}},\underline{x}\right)$ and on $\operatorname{Mon}\left(t^{\frac{1}{M}},\underline{x}\right)$, and let us call them for a moment $>_{w,N}$ respectively $>_{w,M}$. It is important to note, that the restriction of $>_{w,M}$ to $\operatorname{Mon}\left(t^{\frac{1}{N}},\underline{x}\right)$ coincides with $>_{w,N}$. We therefore omit the additional subscript in our notation.

We now fix some global monomial ordering $>$ on $\operatorname{Mon}(\underline{x})$, and given a vector $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$ we will throughout this section always denote by $>_w$ the monomial ordering from Definition 6.6.

Proposition 6.8
If $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$ and $f\in R_N[\underline{x}]$ with $\operatorname{lt}_{>_w}(f)=1$, then $\operatorname{in}_w(f)=1$.

Proof: Suppose this is not the case then there exists a monomial of f, say $1\neq t^{\alpha}\cdot\underline{x}^{\beta}\in\mathcal{M}_f$, such that $w\cdot(\alpha,\beta)\geq w\cdot(0,\ldots,0)=0$, and since $\operatorname{lm}_{>_w}(f)=1$ we must necessarily have equality. But since $>$ is global $\underline{x}^{\beta}>1$, which implies that also $t^{\alpha}\cdot\underline{x}^{\beta}>_w 1$, in contradiction to $\operatorname{lm}_{>_w}(f)=1$. □

Proposition 6.9
Let $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$, $I\trianglelefteq R_N[\underline{x}]$ be an ideal, and let $G=\{g_1,\ldots,g_k\}$ be a standard basis of I with respect to $>_w$ then

$$\operatorname{in}_w(I)=\big\langle\operatorname{in}_w(g_1),\ldots,\operatorname{in}_w(g_k)\big\rangle\trianglelefteq K\left[t^{\frac{1}{N}},\underline{x}\right],$$

and in particular,

$$\text{t-in}_w(I) = \left\langle \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k) \right\rangle \trianglelefteq K[\underline{x}].$$

Proof: If G is standard basis of I then by Corollary 4.6 every element $f \in I$ has a weak standard representation of the form $u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k$, where $\text{lt}_{>_w}(u) = 1$ and $\text{lm}_{>_w}(u \cdot f) \geq \text{lm}_{>_w}(q_i \cdot g_i)$. The latter in particular implies that

$$\text{ord}_w(u \cdot f) = \deg_w\left(\text{lm}_{>_w}(u \cdot f)\right) \geq \deg_w\left(\text{lm}_{>_w}(q_i \cdot g_i)\right) = \text{ord}_w(q_i \cdot g_i).$$

We conclude therefore by Lemma 6.5 and Proposition 6.8 that

$$\text{in}_w(f) = \text{in}_w(u \cdot f) \in \left\langle \text{in}_w(g_1), \ldots, \text{in}_w(g_k) \right\rangle.$$

For the part on the t-initial ideals just note that if $f \in I$ then by the above $\text{in}_w(f) = \sum_{i=1}^k h_i \cdot \text{in}_w(g_i)$ for some $h_i \in K\left[t^{\frac{1}{N}}, \underline{x}\right]$, and thus

$$\text{t-in}_w(f) = \sum_{i=1}^k h_i(1, \underline{x}) \cdot \text{t-in}_w(g_i) \in \langle \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k) \rangle_{K[\underline{x}]}.$$

□

Theorem 6.10
Let $J \trianglelefteq L[\underline{x}]$ and $I \trianglelefteq R_N[\underline{x}]$ be ideals with $J = \langle I \rangle_{L[\underline{x}]}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of I with respect to $>_w$. Then

$$\text{t-in}_w(J) = \text{t-in}_w(I) = \left\langle \text{t-in}_w(G) \right\rangle \lhd K[\underline{x}].$$

Proof: Since $R_N[\underline{x}]$ is noetherian, we may add a finite number of elements of I to G so as to assume that $G = (g_1, \ldots, g_k)$ generates I. Since by Proposition 6.9 we already know that the t-initial forms of any standard basis of I with respect to $>_w$ generate $\text{t-in}_w(I)$ this does not change the right hand side. But then by assumption $J = \langle G \rangle_{L[\underline{x}]}$, and given an element $f \in J$ we can write it as

$$f = \sum_{i=1}^k t^{\frac{-\alpha}{N \cdot M}} \cdot a_i \cdot g_i$$

for some $M >> 0$, $a_i \in R_{N \cdot M}$ and $\alpha \in \mathbb{N}$. It follows that

$$t^{\frac{\alpha}{N \cdot M}} \cdot f = \sum_{i=1}^k a_i \cdot g_i \in \langle G \rangle_{R_{N \cdot M}[\underline{x}]}.$$

Since G is a standard basis over $R_N[\underline{x}]$ with respect to $>_w$ on $\operatorname{Mon}\big(t^{\frac{1}{N}},\underline{x}\big)$ by Buchberger's Criterion 4.6 $\operatorname{spoly}(g_i,g_j)$, $i<j$, has a weak standard representation $u_{ij}\cdot\operatorname{spoly}(g_i,g_j)=\sum_{\nu=1}^k q_{ij\nu}\cdot g_\nu$ with $u_{ij},q_{ij\nu}\in R_N[\underline{x}]\subseteq R_{N\cdot M}[\underline{x}]$ and $\operatorname{lt}_{>_w}(u_{ij})=1$. Taking Remark 6.7 into account these are also weak standard representations with respect to the corresponding monomial ordering $>_w$ on $\operatorname{Mon}(t^{\frac{1}{N\cdot M}},\underline{x})$, and again by Buchberger's Criterion 4.6 there exists a weak standard representation $u\cdot t^{\frac{\alpha}{N\cdot M}}\cdot f=\sum_{i=1}^k q_i\cdot g_i$. By Propositions 6.5 and 6.8 this implies that

$$t^{\frac{\alpha}{N\cdot M}}\cdot \operatorname{in}_w(f)=\operatorname{in}_w\big(u\cdot t^{\frac{\alpha}{N\cdot M}}\cdot f\big)\in\big\langle \operatorname{in}_w(G)\big\rangle.$$

Setting $t=1$ we get $\text{t-in}_w(f)=\big(t^{\frac{k}{N\cdot M}}\cdot\operatorname{in}_w(f)\big)_{|t=1}\in\big\langle \text{t-in}_w(G)\big\rangle$. □

Corollary 6.11

Let $J=\langle I'\rangle_{L[\underline{x}]}$ with $I'\trianglelefteq K\big[t^{\frac{1}{N}},\underline{x}\big]$, $w\in\mathbb{R}_{<0}\times\mathbb{R}^n$ and G is a standard basis of I' with respect to $>_w$ on $\operatorname{Mon}\big(t^{\frac{1}{N}},\underline{x}\big)$, then

$$\text{t-in}_w(J)=\text{t-in}_w(I')=\big\langle \text{t-in}_w(G)\big\rangle\trianglelefteq K[\underline{x}].$$

Proof: Enlarge G to a finite generating set G' of I', then G' is still a standard basis of I'. By Corollary 4.7 G' is then also a standard basis of

$$I:=\langle G'\rangle_{R_N[\underline{x}]}=\langle f_1,\ldots,f_k\rangle_{R_N[\underline{x}]},$$

and Theorem 6.10 applied to I thus shows that

$$\text{t-in}(J)=\big\langle \text{t-in}_w(G')\big\rangle.$$

However, if $f\in G'\subset I'$ is one of the additional elements then it has a weak standard representation

$$u\cdot f=\sum_{g\in G}q_g\cdot g$$

with respect to G and $>_w$, since G is a standard basis of I'. Applying Propositions 6.5 and 6.8 then shows that $\operatorname{in}_w(f)\in\langle\operatorname{in}_w(G)\rangle$, which finishes the proof. □

Remark 6.12

Note that if $I\trianglelefteq R_N[\underline{x}]$ and $J=\langle I\rangle_{L[\underline{x}]}$, then

$$J\cap R_N[\underline{x}]=I:\big\langle t^{\frac{1}{N}}\big\rangle^\infty,$$

but the saturation is in general necessary.

Since $L_N \subset L$ is a field extension Corollary 6.13 implies $J \cap L_N[\underline{x}] = \langle I \rangle_{L_N[\underline{x}]}$, and it suffices to see that

$$\langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}} \right\rangle^\infty.$$

If $I \cap S_N \neq \emptyset$ then both sides of the equation coincide with $R_N[\underline{x}]$, so that we may assume that $I \cap S_N$ is empty. Recall that $L_N = S_N^{-1} R_N$, so that if $f \in R_N[\underline{x}]$ with $t^{\frac{\alpha}{N}} \cdot f \in I$ for some α, then

$$f = \frac{t^{\frac{\alpha}{N}} \cdot f}{t^{\frac{\alpha}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}].$$

Conversely, if $f = \frac{g}{t^{\frac{k}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}]$ with $g \in I$, then $g = t^{\frac{\alpha}{N}} \cdot f \in I$ and thus f is in the right hand side.

Corollary 6.13
Let $F \subset F'$ be a field extension and $I \trianglelefteq F[\underline{x}]$. Then $I = \langle I \rangle_{F'[\underline{x}]} \cap F[\underline{x}]$.

Proof: The result is obvious if I is generated by monomials. For the general case fix any global monomial ordering $>$ on $\mathrm{Mon}(\underline{x})$ and set $I^e = \langle I \rangle_{F'[\underline{x}]}$. Since $I \subseteq I^e \cap F[\underline{x}] \subseteq I^e$ we also have

$$L_>(I) \subseteq L_>\big(I^e \cap F[\underline{x}]\big) \subseteq L_>(I^e) \cap F[\underline{x}]. \tag{9}$$

If we choose a standard basis $G = (g_1, \ldots, g_k)$ of I, then by Buchberger's Criterion G is also a Gröbner basis of I^e and thus

$$L_>(I) = \langle \mathrm{lm}_>(g_i) \mid i = 1, \ldots, k \rangle_{F[\underline{x}]}$$

and

$$L_>(I^e) = \langle \mathrm{lm}_>(g_i) \mid i = 1, \ldots, k \rangle_{F'[\underline{x}]} = \big\langle L_>(I) \big\rangle_{F'[\underline{x}]}.$$

Since the latter is a monomial ideal, we have

$$L_>(I^e) \cap F[\underline{x}] = L_>(I).$$

In view of (9) this shows that

$$L_>(I) = L_>\big(I^e \cap F[\underline{x}]\big),$$

and since $I \subseteq I^e \cap F[\underline{x}]$ this finishes the proof by Proposition 4.3. □

We can actually show more, namely, that for each $I \trianglelefteq R_N[\underline{x}]$ and each $M > 0$ (see Corollary 6.15)

$$\langle I \rangle_{R_{M \cdot N}[\underline{x}]} \cap R_N[\underline{x}] = I,$$

and if I is saturated with respect to $t^{\frac{1}{N}}$ then (see Corollary 6.18)

$$\operatorname{in}_w\big(\langle I\rangle_{R_{M\cdot N}[\underline{x}]}\big)=\big\langle \operatorname{in}_w(G)\big\rangle,$$

if G is a standard basis of I with respect to $>_w$.
For this we need the following simple observation.

Lemma 6.14
$R_{N\cdot M}[\underline{x}]$ is a free $R_N[\underline{x}]$-module with basis $\big\{1,t^{\frac{1}{N\cdot M}},\ldots,t^{\frac{M-1}{N\cdot M}}\big\}$.

Corollary 6.15
If $I\trianglelefteq R_N[\underline{x}]$ then $\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\cap R_N[\underline{x}]=I$.

Proof: If $f=g\cdot h\in\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\cap R_N[\underline{x}]$ with $g\in I$ and $h\in R_{N\cdot M}[\underline{x}]$ then by Lemma 6.14 there are uniquely determined $h_i\in R_N$ such that $h=\sum_{i=0}^{M-1}h_i\cdot t^{\frac{i}{N\cdot M}}$, and hence $f=\sum_{i=0}^{M-1}(g\cdot h_i)\cdot t^{\frac{i}{N\cdot M}}$ with $g\cdot h_i\in R_N[\underline{x}]$. By assumption $f\in R_N[\underline{x}]=R_{N\cdot M}[\underline{x}]\cap\langle 1\rangle_{R_N[\underline{x}]}$ and by Lemma 6.14 we thus have $g\cdot h_i=0$ for all $i=1,\ldots,M-1$. But then $f=g\cdot h_0\in I$. □

Lemma 6.16
Let $I\trianglelefteq R_N[\underline{x}]$ be an ideal such that $I=I:\big\langle t^{\frac{1}{N}}\big\rangle^\infty$, then for any $M\geq 1$

$$\langle I\rangle_{R_{N\cdot M}[\underline{x}]}=\langle I\rangle_{R_{N\cdot M}[\underline{x}]}:\big\langle t^{\frac{1}{N\cdot M}}\big\rangle^\infty.$$

Proof: Let $f,h\in R_{N\cdot M}[\underline{x}]$, $\alpha\in\mathbb{N}$, $g\in I$ such that

$$t^{\frac{\alpha}{N\cdot M}}\cdot f=g\cdot h. \tag{10}$$

We have to show that $f\in\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. For this purpose do division with remainder in order to get $\alpha=a\cdot M+b\quad$ with $\quad 0\leq b<M$. By Lemma 6.14 there are $h_i,f_i\in R_N[\underline{x}]$ such that $f=\sum_{i=0}^{M-1}f_i\cdot t^{\frac{i}{N\cdot M}}$ and $h=\sum_{i=0}^{M-1}h_i\cdot t^{\frac{i}{N\cdot M}}$. (10) then translates into

$$\sum_{i=0}^{M-1-b}t^{\frac{b+i}{N\cdot M}}\cdot t^{\frac{a}{N}}\cdot f_i+\sum_{i=M-b}^{M-1}t^{\frac{b+i-M}{N\cdot M}}\cdot t^{\frac{a+1}{N}}\cdot f_i=\sum_{i=0}^{M-1}g\cdot h_i\cdot t^{\frac{i}{N\cdot M}},$$

and since $\big\{1,t^{\frac{1}{N\cdot M}},\ldots,t^{\frac{M-1}{N\cdot M}}\big\}$ is $R_N[\underline{x}]$-linearly independent we can compare coefficients to find $t^{\frac{a}{N}}\cdot f_i=g\cdot h_{b+i}\in I$ for $i=0,\ldots,M-b-1$, and $t^{\frac{a+1}{N}}\cdot f_i=g\cdot h_{b+i-M}\in I$ for $i=M-b,\ldots,M-1$. In any case, since I is saturated with respect to $t^{\frac{1}{N}}$ by assumption we conclude that $f_i\in I$ for all $i=0,\ldots,M-1$, and therefore $f\in\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. □

Corollary 6.17
Let $J \trianglelefteq L[\underline{x}]$ be an ideal such that $J = \langle J \cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of $J \cap R_N[\underline{x}]$ with respect to $>_w$.
Then for all $M \geq 1$

$$\mathrm{in}_w\left(J \cap R_{N\cdot M}[\underline{x}]\right) = \left\langle\, \mathrm{in}_w(G)\right\rangle \lhd K\left[t^{\frac{1}{N\cdot M}}, \underline{x}\right]$$

and

$$\text{t-in}_w\left(J \cap R_{N\cdot M}[\underline{x}]\right) = \left\langle\, \text{t-in}_w(G)\right\rangle = \text{t-in}_w\left(J \cap R_N[\underline{x}]\right) \lhd K[\underline{x}].$$

Proof: Enlarge G to a generating set G' of $I = J \cap R_N[\underline{x}]$ over $R_N[\underline{x}]$ by adding a finite number of elements of I. Then

$$\left\langle L_{>_w}(G')\right\rangle \subseteq \left\langle L_{>_w}(I)\right\rangle = \left\langle L_{>_w}(G)\right\rangle \subseteq \left\langle L_{>_w}(G')\right\rangle$$

shows that G' is still a standard basis of I with respect to $>_w$. So we can assume that $G = G'$.
By Proposition 6.9 it suffices to show that G is also a standard basis of $J \cap R_{N\cdot M}[\underline{x}]$. Since by assumption $J = \langle I\rangle_{L[\underline{x}]} = \langle G\rangle_{L[\underline{x}]}$, Corollary 6.13 implies that

$$J \cap L_{N\cdot M}[\underline{x}] = \langle G\rangle_{L_{N\cdot M}[\underline{x}]} = S_{N\cdot M}^{-1}\langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Moreover, by Remark 6.12 the ideal $I = \langle G\rangle_{R_N[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N}}$ and by Lemma 6.16 therefore also $\langle G\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, which implies that

$$J \cap R_{N\cdot M}[\underline{x}] = S_{N\cdot M}^{-1}\langle G\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_{N\cdot M}[\underline{x}] = \langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Since $G = (g_1, \ldots, g_k)$ is a standard basis of I every $\mathrm{spoly}(g_i, g_j)$, $i < j$, has a weak standard representation with respect to G and $>_w$ over $R_N[\underline{x}]$ by Buchberger's Criterion 4.6, and these are of course also weak standard representations over $R_{N\cdot M}[\underline{x}]$, so that again by Buchberger's Criterion G is a standard basis of $\langle G\rangle_{R_{N\cdot M}[\underline{x}]} = J \cap R_{N\cdot M}[\underline{x}]$. □

Corollary 6.18
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \left\langle t^{\frac{1}{N}}\right\rangle^{\infty}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of I with respect to $>_w$.
Then for all $M \geq 1$

$$\mathrm{in}_w\left(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\right) = \left\langle\, \mathrm{in}_w(G)\right\rangle \lhd K\left[t^{\frac{1}{N\cdot M}}, \underline{x}\right]$$

and

$$\text{t-in}_w\left(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\right)=\left\langle \text{t-in}_w(G)\right\rangle=\text{t-in}_w(I)\lhd K[\underline{x}]$$

Proof: If we consider $J=\langle I\rangle_{L[\underline{x}]}$ then by Remark 6.12 $J\cap R_N[\underline{x}]=I$, and moreover, by Lemma 6.16 also $\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, so that applying Remark 6.12 once again we also find $J\cap R_{N\cdot M}[\underline{x}]=\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. The result therefore follows from Corollary 6.17. □

Corollary 6.19
Let $J\unlhd L[\underline{x}]$ *be an ideal such that* $J=\langle J\cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, *let* $w=(-1,0,\ldots,0)$ *and let* $M\geq 1$. *Then*

$$1\in\text{in}_\omega\left(J\cap R_N[\underline{x}]\right)\quad\Longleftrightarrow\quad 1\in\text{in}_\omega\left(J\cap R_{N\cdot M}[\underline{x}]\right).$$

Proof: Suppose that $f\in J\cap R_{N\cdot M}[\underline{x}]$ with $\text{in}_\omega(f)=1$, and let $G=(g_1,\ldots,g_k)$ be standard basis of $J\cap R_N[\underline{x}]$ with respect to $>_w$. By Corollary 6.17

$$1=\text{in}_\omega(f)\in\left\langle \text{in}_\omega(g_1),\ldots,\text{in}_\omega(g_k)\right\rangle\lhd K\left[t^{\frac{1}{N\cdot M}},\underline{x}\right],$$

and since this ideal and 1 are w-quasihomogeneous, there exist w-quasihomogeneous elements $h_1,\ldots,h_k\in K\left[t^{\frac{1}{N\cdot M}},\underline{x}\right]$ such that

$$1=\sum_{i=1}^k h_i\cdot\text{in}_\omega(g_i),$$

where each summand on the right hand side (possibly zero) is w-quasihomogeneous of w-degree zero. Since $w=(-1,0,\ldots,0)$ this forces $h_i\in K[\underline{x}]$ for all $i=1,\ldots,k$ and thus $1\in\text{in}_\omega(J\cap R_N[\underline{x}])$. The converse is clear anyhow. □

We want to conclude the section by a remark on the saturation.

Proposition 6.20
If $f_1,\ldots,f_k\in K[t,\underline{x}]$ *and* $I=\langle f_1,\ldots,f_k\rangle\unlhd K[t]_{\langle t\rangle}[\underline{x}]$ *then*

$$\langle I\rangle_{R_1[\underline{x}]}:\langle t\rangle^\infty=\left\langle I:\langle t\rangle^\infty\right\rangle_{R_1[\underline{x}]}.$$

Proof: Let $>_1$ be any global monomial ordering on $\text{Mon}(\underline{x})$ and define a t-local monomial ordering on $\text{Mon}\left(t,\underline{x}\right)$ by

$$t^\alpha\cdot\underline{x}^\beta\;>\;t^{\alpha'}\cdot\underline{x}^{\beta'}$$

if and only if

$$\underline{x}^\alpha >_1 \underline{x}^{\alpha'} \text{ or } \big(\underline{x}^\alpha = \underline{x}^{\alpha'} \text{ and } \alpha < \alpha'\big).$$

Then

$$\{f \in R_1[\underline{x}] \mid \mathrm{lt}_>(f) = 1\} = \{1 + t \cdot p \mid p \in K[t]\},$$

and thus

$$R_1[\underline{x}]_> = R_1[\underline{x}] \quad \text{and} \quad K[t,\underline{x}]_> = K[t]_{\langle t\rangle}[\underline{x}].$$

Using Remark 4.9 we can compute at the same time a standard basis of $\langle I\rangle_{R_1[\underline{x}]} : \langle t\rangle^\infty$ and of $\langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]} : \langle t\rangle^\infty$ with respect to $>$. Since a standard basis is a generating set in the localised ring the result follows. □

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