STANDARD BASES IN $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

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Abstract. In this paper we study standard bases for submodules of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$ respectively of their localisation with respect to a $\underline{t}$-local monomial ordering. The main step is to prove the existence of a division with remainder generalising and combining the division theorems of Grauert–Hironaka and Mora. Everything else then translates naturally. Setting either $m = 0$ or $n = 0$ we get standard bases for polynomial rings respectively for power series rings as a special case. We then apply this technique to show that the t-initial ideal of an ideal over the Puiseux series field can be read of from a standard basis of its generators. This is an important step in the constructive proof that each point in the tropical variety of such an ideal admits a lifting.

The paper follows the lines of [GrP02] and [DeS07] generalising the results where necessary. Basically, the only original parts for the standard bases are the proofs of Theorem 2.1 and Theorem 3.3, but even here they are easy generalisations of Grauert–Hironaka's respectively Mora's Division Theorem (the latter in the form stated and proved first by Greuel and Pfister, see [GGM$^+$94], [GrP96]; see also [Mor82], [Gra94]). The paper should therefore rather be seen as a unified approach for the existence of standard bases in polynomial and power series rings, and it was written mostly due to the lack of a suitable reference for the existence of standard bases in $K[[t]][x_1,\ldots,x_n]$ which are needed when dealing with tropical varieties. Namely, when we want to show that every point in the tropical variety of an ideal J defined over

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the field of Puiseux series exhibits a lifting to the variety of J, then, assuming that J is generated by elements in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$, we need to know that we can compute the so-called t-initial ideal of J by computing a standard basis of the ideal defined by the generators in $K\big[\big[t^{\frac{1}{N}}\big]\big][x_1,\ldots,x_n]$ (see Theorem 7.10 and [JMM07]).

An important point is that if the input data is polynomial in both $\underline{t}$ and $\underline{x}$ then we can actually compute the standard basis since a standard basis computed in $K[t_1,\ldots,t_m]_{\langle t_1,\ldots,t_m\rangle}[x_1,\ldots,x_n]$ will do (see Corollary 4.7). This was previously known for the case where there are no x_i (see [GrP96]).

In this paper we treat only formal power series, while Grauert (see [Gra72]) and Hironaka (see [Hir64]) considered convergent power series with respect to certain valuations which includes the formal case. It should be rather straightforward how to adjust Theorem 2.1 accordingly. Many authors contributed to the further development (see e.g. [Bec90] for a standard basis criterion in the power series ring) and to generalisations of the theory, e.g. to algebraic power series (see e.g. [Hir77], [AMR77], [ACH]) or to differential operators (see e.g. [GaH05]). This list is by no means complete.

In Section 1 we introduce the basic notions. Section 2 is devoted to the proof of the existence of a determinate division with remainder for polynomials in $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$ which are homogeneous with respect to the x_i. This result is then used in Section 3 to show the existence of weak divisions with remainder for all elements of $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$. In Section 4 we introduce standard bases and prove the basics for these, and we prove Schreyer's Theorem and, thus Buchberger's Criterion in Section 5. In Section 6 we describe some algorithms which rely on the standard basis algorithm, and if the input is polynomial in $\underline{t}$ as well as in $\underline{x}$ then the algorithms terminate. Finally, in Section 7 we apply standard bases to study t-initial ideals of ideals over the Puiseux series field.

1. Basic Notation

Throughout the paper K will be any field, $R = K[[t_1, \ldots, t_m]]$ will denote the ring of formal power series over K and

$$R[x_1, \ldots, x_n] = K[[t_1, \ldots, t_m]][x_1, \ldots, x_n]$$

denotes the ring of polynomials in the indeterminates $x_1, \ldots, x_n$ with coefficients in the power series ring R. We will in general use the shorthand notation $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{t} = (t_1, \ldots, t_m)$, and the usual multi-index notation

$$\underline{t}^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m} \quad \text{and} \quad \underline{x}^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n},$$

for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$.

Definition 1.1
A *monomial ordering* on

$$\text{Mon}(\underline{t}, \underline{x}) = \left\{ \underline{t}^\alpha \cdot \underline{x}^\beta \;\middle|\; \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n \right\}$$

is a total ordering $>$ on $\text{Mon}(\underline{t}, \underline{x})$ which is compatible with the semigroup structure of $\text{Mon}(\underline{t}, \underline{x})$, i.e. such that for all $\alpha, \alpha', \alpha'' \in \mathbb{N}^m$ and $\beta, \beta', \beta'' \in \mathbb{N}^n$

$$\underline{t}^\alpha \cdot \underline{x}^\beta \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \quad \Longrightarrow \quad \underline{t}^{\alpha+\alpha''} \cdot \underline{x}^{\beta+\beta''} \;>\; \underline{t}^{\alpha'+\alpha''} \cdot \underline{x}^{\beta'+\beta''}.$$

We call a monomial ordering $>$ on $\text{Mon}(\underline{t}, \underline{x})$ *$\underline{t}$-local* if its restriction to $\text{Mon}(\underline{t})$ is *local*, i.e.

$$t_i < 1 \quad \text{for all } \; i = 1, \ldots, m.$$

We call a $\underline{t}$-local monomial ordering on $\text{Mon}(\underline{t}, \underline{x})$ a *$\underline{t}$-local weighted degree ordering* if there is a $w = (w_1, \ldots, w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$ such that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $\beta, \beta' \in \mathbb{N}^n$

$$w \cdot (\alpha, \beta) > w \cdot (\alpha', \beta') \quad \Longrightarrow \quad \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'},$$

where $w \cdot (\alpha, \beta) = w_1 \cdot \alpha_1 + \ldots + w_m \cdot \alpha_m + w_{m+1} \cdot \beta_1 + \ldots + w_n \cdot \beta_n$ denotes the standard scalar product. We call w a *weight vector* of $>$.

Example 1.2
The *$\underline{t}$-local lexicographical ordering* $>_{lex}$ on $\text{Mon}(\underline{t}, \underline{x})$ is defined by

$$\underline{t}^\alpha \cdot \underline{x}^\beta \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$$

if and only if

$$\exists\, j \in \{1,\ldots,n\} \;:\; \beta_1 = \beta_1',\ldots,\beta_{j-1} = \beta_{j-1}', \text{ and } \beta_j > \beta_j',$$

or

$$\big(\beta = \beta' \text{ and } \exists\, j \in \{1,\ldots,m\} \;:\; \alpha_1 = \alpha_1',\ldots,\alpha_{j-1} = \alpha_{j-1}', \alpha_j < \alpha_j'\big).$$

Example 1.3
Let $>$ be any $\underline{t}$-local ordering and $w = (w_1,\ldots,w_{m+n}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n$, then

$$\underline{t}^\alpha \cdot \underline{x}^\beta >_w \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}$$

if and only if

$$w \cdot (\alpha,\beta) > w \cdot (\alpha',\beta')$$

or

$$\big(w \cdot (\alpha,\beta) = w \cdot (\alpha',\beta') \text{ and } \underline{t}^\alpha \cdot \underline{x}^\beta > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'}\big)$$

defines a $\underline{t}$-local weighted degree ordering $>_w$ on $\text{Mon}(\underline{t},\underline{x})$ with weight vector w.

Even if we are only interested in standard bases of ideals we have to pass to submodules of free modules in order to have syzygies at hand for the proof of Buchberger's Criterion via Schreyer orderings.

Definition 1.4
We define

$$\text{Mon}^s(\underline{t},\underline{x}) := \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m, i = 1,\ldots,s\big\},$$

where $e_i = (\delta_{ij})_{j=1,\ldots,s}$ is the vector with all entries zero except the i-th one which is one. We call the elements of $\text{Mon}^s(\underline{t},\underline{x})$ *module monomials* or simply *monomials*.
For $p, p' \in \text{Mon}^s(\underline{t},\underline{x}) \cup \{0\}$ we define

$$p \mid p'$$

in words "p divides p'", if and only if

$$\exists\, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n \;:\; \underline{t}^\alpha \cdot \underline{x}^\beta \cdot p = p'.$$

If $p \neq 0$, then we define in this case

$$\frac{p}{p'} := \underline{t}^{\alpha''} \cdot \underline{x}^{\beta''} \in \text{Mon}(\underline{t},\underline{x}).$$

Note that, this is well defined since β and α are uniquely determined if $p \neq 0$, and note also that for $p = \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ and $p' = \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j$ the condition $p \mid p'$ necessarily implies that $i = j$.
Moreover, given two monomials $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i, \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j \in \operatorname{Mon}^s(\underline{t}, \underline{x})$ we define the *lowest common multiple* of the two as

$$\operatorname{lcm}\left(\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i, \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j\right) := \begin{cases} t_1^{\max(\alpha_1,\alpha_1')} \cdots x_n^{\max(\beta_n,\beta_n')}, & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus the least common multiple of two monomials is somehow the *smallest* monomial which is divisible by both monomials, in the sense that it does divide every other monomial with this property.
Given a monomial ordering on $\operatorname{Mon}(\underline{t}, \underline{x})$, a $\underline{t}$*-local monomial ordering* on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ with respect to $>$ is a total ordering $>_m$ on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ which is strongly compatible with the operation of the multiplicative semigroup $\operatorname{Mon}(\underline{t}, \underline{x})$ on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ in the sense that

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \;>_m\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j \quad\Longrightarrow\quad \underline{t}^{\alpha+\alpha''} \cdot \underline{x}^{\beta+\beta''} \cdot e_i \;>_m\; \underline{t}^{\alpha'+\alpha''} \cdot \underline{x}^{\beta'+\beta''} \cdot e_j$$

and

$$\underline{t}^\alpha \cdot \underline{x}^\beta \;>\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \quad\Longleftrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \;>_m\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_i$$

for all $\beta, \beta', \beta'' \in \mathbb{N}^n$, $\alpha, \alpha', \alpha'' \in \mathbb{N}^m, i, j \in \{1, \ldots, s\}$.
Note that due to the second condition the ordering $>_m$ on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ determines the ordering $>$ on $\operatorname{Mon}(\underline{t}, \underline{x})$ uniquely, and we will therefore usually not distinguish between them, i.e. we will use the same notation $>$ also for $>_m$, and we will not specify the monomial ordering on $\operatorname{Mon}(\underline{t}, \underline{x})$ in advance, but instead refer to it as the *induced monomial ordering on* $\operatorname{Mon}(\underline{t}, \underline{x})$.
We call a monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ $\underline{t}$*-local* if the induced monomial ordering on $\operatorname{Mon}(\underline{t}, \underline{x})$ is so.
We call a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ a $\underline{t}$*-local weight ordering* if there is a $w = (w_1, \ldots, w_{m+n+s}) \in \mathbb{R}^m_{\leq 0} \times \mathbb{R}^n \times \mathbb{R}^s$ such that for all $\alpha, \alpha' \in \mathbb{N}^m$, $\beta, \beta' \in \mathbb{N}^n$ and $i, j \in \{1, \ldots, s\}$

$$w \cdot (\alpha, \beta, e_i) > w \cdot (\alpha', \beta', e_j) \quad\Longrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i > \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_j,$$

and we call w a *weight vector* of $>$.

Example 1.5
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$.

(a) We can extend $>$ to a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ in two straightforward ways by

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i\;>\;\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j$$

if and only if

$$i<j\quad\text{or}\quad(i=j\;\text{ and }\;\underline{t}^\alpha\cdot\underline{x}^\beta>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}),$$

respectively by

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i\;>\;\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j$$

if and only if

$$\underline{t}^\alpha\cdot\underline{x}^\beta>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\quad\text{or}\quad(\underline{t}^\alpha\cdot\underline{x}^\beta=\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\;\text{ and }\;i<j),$$

the first ordering giving *priority to the components*, the second one giving *priority to the monomials*.

(b) If $w\in\mathbb{R}^m_{\leq 0}\times\mathbb{R}^n$ and $>_w$ is the $\underline{t}$-local weighted degree ordering with respect to $>$ and w from Example 1.3 then the corresponding module monomial ordering giving priority to the monomials is a $\underline{t}$-local weight ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ with weight vector $(w,0,\ldots,0)$.

(c) The module monomial ordering corresponding to $>$ and giving priority to the components is a $\underline{t}$-local weight ordering with respect to the weight vector $(0,\ldots,0,s,s-1,\ldots,1)$.

Example 1.6
Let $w\in\mathbb{R}^m_{\leq 0}\times\mathbb{R}^{n+s}$ and let $>$ be any $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ such that the induced $\underline{t}$-local monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$ is a $\underline{t}$-local weighted degree ordering with respect to the weight vector $(w_1,\ldots,w_{m+n})$. Then

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i\;>_w\;\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j$$

if and only if

$$w\cdot(\alpha,\beta,e_i)>w\cdot(\alpha',\beta',e_j)$$

or

$$\big(w\cdot(\alpha,\beta,e_i)=w\cdot(\alpha',\beta',e_j)\ \text{ and }\ \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j\big)$$

defines a $\underline{t}$-local weight monomial ordering on $\text{Mon}^s(\underline{t},\underline{x})$ with weight vector w. In particular, there exists such a monomial ordering.

Remark 1.7

In the following we will mainly be concerned with monomial orderings on $\text{Mon}^s(\underline{t},\underline{x})$ and with submodules of free modules over $R[\underline{x}]$, but all these results specialise to $\text{Mon}(\underline{t},\underline{x})$ and ideals by just setting $s=1$. □

The following lemma follows easily from the above definitions.

Lemma 1.8

The following conditions for a monomial ordering $>$ *on* $\text{Mon}^s(\underline{t},\underline{x})$ *are equivalent:*

(a) $>$ *is* $\underline{t}$*-local.*
(b) $\underline{t}^\alpha<1$ *for all* $0\neq\alpha\in\mathbb{N}^m$*.*
(c) $\underline{t}^{\alpha+\alpha'}\cdot\underline{x}^\beta<\underline{t}^{\alpha'}\cdot\underline{x}^\beta$ *for all* $\alpha,\alpha'\in\mathbb{N}^m$*,* $\beta\in\mathbb{N}^n$*.*
(d) $\underline{t}^{\alpha+\alpha'}\cdot\underline{x}^\beta\cdot e_i<\underline{t}^{\alpha'}\cdot\underline{x}^\beta\cdot e_i$ *for all* $\alpha,\alpha'\in\mathbb{N}^m$*,* $\beta\in\mathbb{N}^n$*,* $i=1,\ldots,s$*.*

For a $\underline{t}$-local monomial ordering we can introduce the notions of leading monomial and leading term of elements in $R[\underline{x}]$.

Definition 1.9

Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}(\underline{t},\underline{x})$. We call

$$0\neq f=\sum_{|\beta|=0}^{d}\sum_{|\alpha|=0}^{\infty}a_{\alpha,\beta}\cdot\underline{t}^\alpha\cdot\underline{x}^\beta\in R[\underline{x}],$$

with $a_{\alpha,\beta}\in K$, $|\beta|=\beta_1+\ldots+\beta_n$ and $|\alpha|=\alpha_1+\ldots+\alpha_m$, the *distributive representation* of f,

$$\mathcal{M}_f:=\big\{\underline{t}^\alpha\cdot\underline{x}^\beta\mid a_{\alpha,\beta}\neq 0\big\}$$

the set of *monomials of* f and

$$\mathcal{T}_f:=\big\{a_{\alpha,\beta}\cdot\underline{t}^\alpha\cdot\underline{x}^\beta\mid a_{\alpha,\beta}\neq 0\big\}$$

the set of *terms of* f.

Moreover,

$$\text{lm}_>(f):=\max\big\{\underline{t}^\alpha\cdot\underline{x}^\beta\mid \underline{t}^\alpha\cdot\underline{x}^\beta\in\mathcal{M}_f\big\}$$

is called the *leading monomial* of f. Note, that this maximum exists since the number of β's occurring in f is finite and the ordering is local with respect to $\underline{t}$.
If $\mathrm{lm}_>(f) = \underline{t}^\alpha \cdot \underline{x}^\beta$ then we call

$$\mathrm{lc}_>(f) := a_{\alpha,\beta}$$

the *leading coefficient* of f,

$$\mathrm{lt}_>(f) := a_{\alpha,\beta} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta$$

its *leading term*, and

$$\mathrm{tail}_>(f) := f - \mathrm{lt}_>(f)$$

its *tail*.
For the sake of completeness we define

$$\mathrm{lm}_>(0) := 0, \quad \mathrm{lt}_>(0) := 0, \quad \mathrm{lc}_>(0) := 0, \quad \mathrm{tail}_>(f) = 0,$$

and

$$0 < \underline{t}^\alpha \cdot \underline{x}^\beta \quad \forall\, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n.$$

Finally, for a subset $G \subseteq R[\underline{x}]$ we call the ideal

$$L_>(G) = \langle \mathrm{lm}_>(f) \mid f \in G \rangle \lhd K[\underline{t}, \underline{x}]$$

in the polynomial ring $K[\underline{t}, \underline{x}]$ generated by all the leading monomials of elements in G the *leading ideal* of G.

Analogously we define the notions for elements in $R[\underline{x}]^s$.

Definition 1.10
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$. We call

$$0 \neq f = \sum_{i=1}^{s} \sum_{|\beta|=0}^{d} \sum_{|\alpha|=0}^{\infty} a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in R[\underline{x}]^s,$$

with $a_{\alpha,\beta,i} \in K$, $|\beta| = \beta_1 + \ldots + \beta_n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_m$, the *distributive representation* of f,

$$\mathcal{M}_f := \left\{ \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0 \right\}$$

the set of *monomials of* f and

$$\mathcal{T}_f := \left\{ a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid a_{\alpha,\beta,i} \neq 0 \right\}$$

the set of *terms of* f.
Moreover,

$$\operatorname{lm}_>(f) := \max\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \mid \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \in \mathcal{M}_f\}$$

is called the *leading monomial* of f. Note again, that this maximum exists since the number of β's occurring in f and the number of i's is finite and the ordering is local with respect to $\underline{t}$.
If $\operatorname{lm}_>(f) = \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$ then we call

$$\operatorname{lc}_>(f) := a_{\alpha,\beta,i}$$

the *leading coefficient* of f,

$$\operatorname{lt}_>(f) := a_{\alpha,\beta,i} \cdot \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i$$

its *leading term*, and

$$\operatorname{tail}_>(f) := f - \operatorname{lt}_>(f)$$

its *tail*.
For the sake of completeness we again define

$$\operatorname{lm}_>(0) := 0, \quad \operatorname{lt}_>(0) := 0, \quad \operatorname{lc}_>(0) := 0, \quad \operatorname{tail}_>(f) = 0,$$

and

$$0 < \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \quad \forall\, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n, i \in \mathbb{N}.$$

Finally, for a subset $G \subseteq R[\underline{x}]^s$ we call the submodule

$$L_>(G) = \langle \operatorname{lm}_>(f) \mid f \in G \rangle \leq K[\underline{t}, \underline{x}]^s$$

of the free module $K[\underline{t}, \underline{x}]^s$ over the polynomial ring $K[\underline{t}, \underline{x}]$ generated by all the leading monomials of elements in G the *leading submodule* of G.

Since the monomial ordering is compatible with the semigroup structure on $\operatorname{Mon}(\underline{t}, \underline{x})$ respectively with the operation of the semigroup $\operatorname{Mon}(\underline{t}, \underline{x})$ on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ the statements in the following lemma are obvious.

Lemma 1.11
Let $f \in R[\underline{x}]$ *and* $g \in R[\underline{x}]^s$.

(a) $\operatorname{lm}_>(f \cdot g) = \operatorname{lm}_>(f) \cdot \operatorname{lm}_>(g)$,
(b) $\operatorname{lc}_>(f \cdot g) = \operatorname{lc}_>(f) \cdot \operatorname{lc}_>(g)$,

(c) $\mathrm{lt}_>(f\cdot g)=\mathrm{lt}_>(f)\cdot\mathrm{lt}_>(g)$.

Proof: Since the statements are true when f or g is zero, we may assume that neither of them is so. Note that for any terms $a_{\alpha,\beta}\cdot\underline{t}^\alpha\cdot\underline{x}^\beta$ of f and $b_{\alpha',\beta',i}\cdot\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_i$ of g we have

$$\mathrm{lm}_>(f)\cdot\mathrm{lm}_>(g)\geq\underline{t}^\alpha\cdot\underline{x}^\beta\cdot\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_i$$

with equality if and only if $\mathrm{lm}_>(f)=\underline{t}^\alpha\cdot\underline{x}^\beta$ and $\mathrm{lm}_>(g)=\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_i$. This proves the lemma. □

We know that in general a standard basis of an ideal respectively submodule I will not be a generating set of I itself, but only of the ideal respectively submodule which I generates in the localisation with respect to the monomial ordering. We therefore introduce this notion here as well.

Definition 1.12
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}(\underline{t},\underline{x})$, then

$$S_>=\{u\in R[\underline{x}]\mid\mathrm{lt}_>(u)=1\}$$

is the *multiplicative set associated to* $>$, and

$$R[\underline{x}]_>=S_>^{-1}R[\underline{x}]=\left\{\frac{f}{u}\;\middle|\;f\in R[\underline{x}],u\in S_>\right\}$$

is the *localisation of* $R[\underline{x}]$ *with respect to* $>$.

Remark 1.13
If $>$ is a $\underline{t}$-local monomial ordering with $x_i>1$ for all $i=1,\ldots,n$ (e.g. $>_{lex}$ from Example 1.2), then $S_>\subset R^*$, and therefore $R[\underline{x}]_>=R[\underline{x}]$.

It is straightforward to extend the notions of leading monomial, leading term and leading coefficient to $R[\underline{x}]_>$ and free modules over this ring.

Definition 1.14
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $g=\frac{f}{u}\in R[\underline{x}]_>^s$ with $u\in S_>$, and $G\subseteq R[\underline{x}]_>^s$. We then define the *leading monomial*, the *leading coefficient* respectively the *leading term* of g as

$$\mathrm{lm}_>(g):=\mathrm{lm}_>(f),\quad\mathrm{lc}_>(g):=\mathrm{lc}_>(f),\quad\text{resp.}\quad\mathrm{lt}_>(g):=\mathrm{lt}_>(f),$$

and the *leading ideal* (if $s = 1$) respectively *leading submodule* of G

$$L_>(G) = \langle \mathrm{lm}_>(h) \mid h \in G \rangle \leq K[\underline{t}, \underline{x}]^s.$$

These definitions are independent of the chosen representative, since if $g = \frac{f}{u} = \frac{f'}{u'}$ then $u' \cdot f = u \cdot f'$, and hence

$$\mathrm{lt}_>(f) = \mathrm{lt}_>(u')\cdot\mathrm{lt}_>(f) = \mathrm{lt}_>(u'\cdot f) = \mathrm{lt}_>(u\cdot f') = \mathrm{lt}_>(u)\cdot\mathrm{lt}_>(f') = \mathrm{lt}_>(f').$$

Remark 1.15

Note that the leading submodule of a submodule in $R[\underline{x}]^s_>$ is a submodule in a free module over the polynomial ring $K[\underline{t}, \underline{x}]$ over the base field, and note that for $J \leq R[\underline{x}]^s_>$ we obviously have

$$L_>(J) = L_>(J \cap R[\underline{x}]^s),$$

and similarly for $I \leq R[\underline{x}]^s$ we have

$$L_>(I) = L_>\big(\langle I \rangle_{R[\underline{x}]_>}\big),$$

since every element of $\langle I \rangle_{R[\underline{x}]_>}$ is of the form $\frac{f}{u}$ with $f \in I$ and $u \in S_>$.
□

In order to be able to work either theoretically or even computationally with standard bases it is vital to have a division with remainder and possibly an algorithm to compute it. We will therefore generalise Grauert–Hironaka's and Mora's Division with remainder. For this we first would like to consider the different qualities a division with remainder may satisfy.

Definition 1.16

Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, and let $A = R[\underline{x}]$ or $A = R[\underline{x}]_>$, where we consider the latter as a subring of $K[[\underline{t}, \underline{x}]]$ in order to have the notion of terms of elements at hand.
Suppose we have $f, g_1, \ldots, g_k, r \in A^s$ and $q_1, \ldots, q_k \in A$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{1}$$

With the notation $r = \sum_{j=1}^{s} r_j \cdot e_j$, $r_1, \ldots, r_s \in A$, we say that (1) satisfies with respect to $>$ the condition

(ID1) iff $\mathrm{lm}_>(f) \geq \mathrm{lm}_>(q_i \cdot g_i)$ for all $i = 1, \ldots, k$,
(ID2) iff $\mathrm{lm}_>(g_i) \not| \mathrm{lm}_>(r)$ for $i = 1, \ldots, k$, unless $r = 0$,

(DD1) iff for $j < i$ no term of $q_i \cdot \mathrm{lm}_>(g_i)$ is divisible by $\mathrm{lm}_>(g_j)$,
(DD2) iff no term of r is divisible by $\mathrm{lm}_>(g_i)$ for $i = 1, \ldots, k$.
(SID2) iff $\mathrm{lm}_>(g_i) \nmid \mathrm{lm}_>(r_j \cdot e_j)$ unless $r_j = 0$ for all i and j.

Here, "ID" stands for *indeterminate division with remainder* while "DD" means *determinate division with remainder* and the "S" in (SID2) represents *strong*. Accordingly, we call a representation of f as in (1) a *determinate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if it satisfies (DD1) and (DD2), while we call it an *indeterminate division with remainder* of f with respect to $(g_1, \ldots, g_k)$ if it satisfies (ID1) and (ID2). In any of these cases we call r a *remainder* or a *normal form* of f with respect to $(g_1, \ldots, g_k)$.
If the remainder in a division with remainder of f with respect to $(g_1, \ldots, g_k)$ is zero we call the representation of f a *standard representation.*
Finally, if $A = R[\underline{x}]$ then for $u \in S_>$ we call a division with remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ also a *weak division with remainder* of f with respect to $(g_1, \ldots, g_k)$, a remainder of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak normal form* of f with respect to $(g_1, \ldots, g_k)$, and a standard representation of $u \cdot f$ with respect to $(g_1, \ldots, g_k)$ is called a *weak standard representation* of f with respect to $(g_1, \ldots, g_k)$.

The following lemma should clarify the relations between the above conditions, and it should explain the *determinate* versa the *indeterminate.*

Lemma 1.17
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$*, and suppose we have $f, g_1, \ldots, g_k, r \in R[\underline{x}]^s$ and $q_1, \ldots, q_k \in R[\underline{x}]$ such that*

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{2}$$

Then the following holds true:

(a) *If* (2) *satisfies (DD2) then it satisfies (SID2).*
(b) *If* (2) *satisfies (SID2) then it satisfies (ID2).*
(c) *If* (2) *satisfies (DD1) and (ID2) then it satisfies (ID1).*

(d) *Suppose that* $g_i \neq 0$ *for* $i = 1, \ldots, k$. *If* $f = q'_1 \cdot g_1 + \ldots + q'_k \cdot q_k + r'$ *is a second such representation of* f *and both satisfy (DD1) and (DD2), then* $q_1 = q'_1, \ldots, q_k = q'_k$ *and* $r = r'$. *That is, a determinate division with remainder is uniquely determined, if it exists.*

Proof: (a) This is obvious, if we take into account that 0 has no term and therefore, even if $r = 0$ it is true that no term of r is divisible by any $\mathrm{lm}_>(g_i)$.

(b) This is obvious.

(c) Let $\mathrm{lm}_>(q_j \cdot g_j) = \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ be maximal in $\{\mathrm{lm}_>(q_i \cdot g_i) \mid i = 1, \ldots, k\}$, and suppose that $\mathrm{lm}_>(q_j \cdot g_j) > \mathrm{lm}_>(f)$. Since the monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ does not occur on the left hand side of the equality in (2) it must occur at least in one of the other summands on the right hand side in order to cancel. Suppose it occurs in some $q_i \cdot g_i$, $i \neq j$, then by our choice necessarily

$$\mathrm{lm}_>(q_j) \cdot \mathrm{lm}_>(g_j) = \mathrm{lm}_>(q_j \cdot g_j) = \mathrm{lm}_>(q_i \cdot g_i) = \mathrm{lm}_>(q_i) \cdot \mathrm{lm}_>(g_i),$$

in contradiction to (DD1). Thus $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ must be a term in r, and again by the maximality in our choice necessarily

$$\mathrm{lm}_>(q_j) \cdot \mathrm{lm}_>(g_j) = \mathrm{lm}_>(q_j \cdot g_j) = \mathrm{lm}_>(r),$$

in contradiction to (ID2). This shows that (ID1) is satisfied.

(d) Obviously the representation

$$0 = f - f = (q_1 - q'_1) \cdot g_1 + \ldots + (q_k - q'_k) \cdot g_k + (r - r')$$

still satisfies the conditions (DD1) and (DD2). But then by (a) and (b) it satisfies (ID2) and by (c) it satisfies (ID1). This implies

$$0 = \mathrm{lm}_>(0) \geq \mathrm{lm}_>(q_i - q'_i) \cdot \mathrm{lm}_>(g_i) \geq 0,$$

and since by our assumption $g_i \neq 0$ this implies $\mathrm{lm}_>(q_i - q'_i) = 0$ and hence $q_i = q'_i$. And therefore, finally, also $r = r'$.

□

We first want to generalise Grauert–Hironaka's Division with Remainder to the case of elements in $R[\underline{x}]$ which are homogeneous with respect to $\underline{x}$. We therefore introduce this notion in the following definition.

Definition 1.18
Let $f = \sum_{i=1}^{s} \sum_{|\beta|=0}^{d} \sum_{\alpha\in\mathbb{N}^m} a_{\alpha,\beta,i} \cdot \underline{t}^{\alpha} \cdot \underline{x}^{\beta} \cdot e_i \in R[\underline{x}]^s$.

(a) We call

$$\deg_{\underline{x}}(f) := \max\left\{|\beta| \;\middle|\; a_{\alpha,\beta,i} \neq 0\right\}$$

the *$\underline{x}$-degree* of f.

(b) $f \in R[\underline{x}]^s$ is called *$\underline{x}$-homogeneous of $\underline{x}$-degree d* if for any $0 \neq \lambda \in K$ we have

$$f(\underline{t}, \lambda \cdot \underline{x}) = \lambda^d \cdot f,$$

or, equivalently, if all terms of f have the same $\underline{x}$-degree d. We denote by $R[\underline{x}]_d^s$ the R-submodule of $R[\underline{x}]^s$ of $\underline{x}$-homogeneous elements of $\underline{x}$-degree d. Note that by this definition 0 is $\underline{x}$-homogeneous of degree d for all $d \in \mathbb{N}$.

(c) If $>$ is a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ then we call

$$\mathrm{ecart}_{>}(f) := \deg_{\underline{x}}(f) - \deg_{\underline{x}}\left(\mathrm{lm}_{>}(f)\right) \geq 0$$

the *ecart* of f. It in some sense measures the failure of the homogeneity of f.

2. Determinate Division with Remainder in $K[[\underline{t}]][\underline{x}]_d^s$

We are now ready to show that for $\underline{x}$-homogeneous elements in $R[\underline{x}]$ there exists a determinate division with remainder. We follow mainly the proof of Grauert–Hironaka's Division Theorem as given in [DeS07].

Theorem 2.1 (HDDwR)
Let $f, g_1, \ldots, g_k \in R[\underline{x}]^s$ be $\underline{x}$-homogeneous, then there exist uniquely determined $q_1, \ldots, q_k \in R[\underline{x}]$ and $r \in R[\underline{x}]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfying

(DD1) *For $j < i$ no term of $q_i \cdot \mathrm{lm}_{>}(g_i)$ is divisible by $\mathrm{lm}_{>}(g_j)$,*

(DD2) *no term of r is divisible by any $\mathrm{lm}_{>}(g_i)$, and*

(DDH) $q_1,\ldots,q_k,r$ *are* $\underline{x}$*-homogeneous of* $\underline{x}$*-degrees* $\deg_{\underline{x}}(q_i)=\deg_{\underline{x}}(f)-\deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ *respectively* $\deg_{\underline{x}}(r)=\deg_{\underline{x}}(f)$.

We call such a representation of f a homogeneous determinate division with remainder*.*

Proof: We first consider the case that $g_1,\ldots,g_k\in\mathrm{Mon}^s(\underline{t},\underline{x})$ are monomials. Then define recursively for $i=1,\ldots,k$

$$q_i:=\frac{h_i}{g_i}\in R[\underline{x}],$$

where h_i is the sum of all terms of $f-\sum_{j=1}^{i-1}q_j\cdot g_j$ which are divisible by g_i, $i=1,\ldots,k$. Thus with $r:=f-\sum_{i=1}^{k}q_i\cdot g_i$

$$f=q_1\cdot g_1+\ldots+q_k\cdot g_k+r$$

obviously satisfies the above conditions (DD1), (DD2) and (DDH).
This result generalises immediately to the case where the g_i are terms, i.e. constant multiples of monomials.
Let us now consider the general case. We set $f_0=f$ and for $\nu>0$ we define recursively

$$f_\nu=f_{\nu-1}-\sum_{i=1}^{k}q_{i,\nu}\cdot g_i-r_\nu=\sum_{i=1}^{k}q_{i,\nu}\cdot\big(\big(-\mathrm{tail}(g_i)\big),$$

where the $q_{i,\nu}\in R[\underline{x}]$ and $r_\nu\in R[\underline{x}]^s$ are such that

$$f_{\nu-1}=q_{1,\nu}\cdot\mathrm{lt}_>(g_1)+\ldots+q_{k,\nu}\cdot\mathrm{lt}_>(g_k)+r_\nu \tag{3}$$

satisfies (DD1), (DD2) and (DDH). Note that such a representation of $f_{\nu-1}$ exists since the $\mathrm{lt}_>(g_i)$ are terms.
We want to show that f_ν, $q_{i,\nu}$ and r_ν all converge to zero in the $\langle t_1,\ldots,t_m\rangle$-adic topology, that is, for each $N\geq 0$ there exists a $\mu_N\geq 0$ such that for all $\nu\geq\mu_N$

$$f_\nu,r_\nu\in\langle t_1,\ldots,t_m\rangle^N\cdot R[\underline{x}]^s\quad\text{resp.}\quad q_{i,\nu}\in\langle t_1,\ldots,t_m\rangle^N.$$

By Lemma 2.5 there is $\underline{t}$-local weight ordering $>_w$ such that

$$\mathrm{lm}_>(g_i)=\mathrm{lm}_{>_w}(g_i)\quad\text{for all } i=1,\ldots,k.$$

If we replace in the above construction $>$ by $>_w$, we still get the same sequences $(f_\nu)_{\nu=0}^\infty$, $(q_{i,\nu})_{\nu=1}^\infty$ and $(r_\nu)_{\nu=1}^\infty$, since for the construction of

$q_{i,\nu}$ and r_ν only the leading monomials of the g_j are used. In particular, (3) will satisfy (DD1), (DD2) and (DDH) with respect to $>_w$. Due to (DDH) f_ν is again $\underline{x}$-homogeneous of $\underline{x}$-degree equal to that of $f_{\nu-1}$, and since (DD1) and (DD2) imply (ID1) by Lemma 1.17 we have

$$\begin{aligned}&\mathrm{lm}_{>_w}(f_{\nu-1}) \geq \max\{\mathrm{lm}_{>_w}(q_{i,\nu}) \cdot \mathrm{lm}_{>_w}(g_i) \mid i = 1, \ldots, k\}\\ &> \max\big\{\mathrm{lm}_{>_w}(q_{i,\nu}) \cdot \mathrm{lm}_{>_w}\big(-\mathrm{tail}(g_i)\big) \bigm| i = 1, \ldots, k\big\} \geq \mathrm{lm}_{>_w}(f_\nu).\end{aligned}$$

It follows from Lemma 2.6 that f_ν converges to zero in the $\langle t_1, \ldots, t_m\rangle$-adic topology, i.e. for given N there is a μ_N such that

$$f_\nu \in \langle t_1, \ldots, t_m\rangle^N \cdot R[\underline{x}]^s \quad \text{for all } \nu \geq \mu_N.$$

But then, by construction for $\nu > \mu_N$

$$r_\nu \in \langle t_1, \ldots, t_m\rangle^N \cdot R[\underline{x}]^s$$

and

$$q_{i,\nu} \in \langle t_1, \ldots, t_m\rangle^{N-d_i},$$

where $d_i = \deg\big(\mathrm{lm}_>(g_i)\big) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ is independent of ν. Thus both, r_ν and $q_{i,\nu}$, converge as well to zero in the $\langle t_1, \ldots, t_m\rangle$-adic topology.

But then

$$q_i := \sum_{\nu=1}^{\infty} q_{i,\nu} \in R[\underline{x}] \quad \text{and} \quad r := \sum_{\nu=1}^{\infty} r_\nu \in R[\underline{x}]^s$$

are $\underline{x}$-homogeneous of $\underline{x}$-degrees $\deg_{\underline{x}}(q_i) = \deg_{\underline{x}}(f) - \deg_{\underline{x}}\big(\mathrm{lm}_>(g_i)\big)$ respectively $\deg_{\underline{x}}(r) = \deg_{\underline{x}}(f)$ unless they are zero, and

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

satisfies (DD1), (DD2) and (DDH).

The uniqueness of the representation follows from Lemma 1.17. □

The following lemmata contain technical results used throughout the proof of the previous theorem.

Lemma 2.2

Let Γ be a directed graph with vertex set V and edge set $E \subset V \times V$. If for every vertex the number of outward pointing edges coincides with the number of inward pointing edges, then F is a disjoint union of cycles.

Proof: We do the proof by induction on the number $\#E$ of edges. If there is no edge, then the statement holds by default, and we may thus assume that $\#E > 0$. Choose any vertex, say $v_0 \in V$, which has an outward pointing edge, say $(v_0, v_1) \in E$. By the assumption v_1 also has an outward pointing edge, say (v_1, v_2), and we can inductively proceed to construct a sequence of vertices $(v_\nu)_{\nu\in\mathbb{N}}$ with $(v_\nu, v_{\nu+1}) \in E$. Since the number of vertices is finite, there is minimal $\mu \geq 0$ such that

$$v_\mu = v_\nu \quad \text{for some} \quad 0 \leq \nu < \mu.$$

But then

$$C = \big((v_\nu, v_{\nu+1}), \ldots, (v_{\mu-1}, v_\mu)\big)$$

is a cycle, and if we remove C from E then the remaining graph still satisfies that for each vertex the number of inward pointing and outward pointing edges coincides. Thus by induction it is a disjoint union of cycles and then so is Γ. □

Lemma 2.3
We use the notation $\underline{z} = (\underline{t}, \underline{x})$ *and* $\mathbb{E}^s = \{e_1, \ldots, e_s\}$.
Let $>$ *be a monomial ordering on* $\operatorname{Mon}^s(\underline{z})$ *and consider the set*

$$\Delta_> := \big\{(\gamma, e_i) - (\gamma', e_j) \in \mathbb{Z}^{m+n+s} \mid \underline{z}^\gamma \cdot e_i > \underline{z}^{\gamma'} \cdot e_j\big\}.$$

Then:

$$(0, \ldots, 0) \notin \left\{ \sum_{i=1}^k n_i \cdot \delta_i \;\middle|\; \delta_i \in \Delta_>, n_i \in \mathbb{Z}_{>0}, k > 0 \right\}.$$

Proof: For the convenience of the reader we reproduce here the proof given in Dave Bayer's thesis [Bay82, (1.7)]. Suppose there exist, not necessarily pairwise different,

$$\delta_i = (\gamma_{i,2}, e_{j_{i,2}}) - (\gamma_{i,1}, e_{j_{i,1}}) \in \Delta_>, \quad i = 1, \ldots, k,$$

with $\gamma_{i,1}, \gamma_{i,2} \in \mathbb{N}^{m+n}$, $j_{i,1}, j_{i,2} \in \{1, \ldots, s\}$ and

$$\underline{z}^{\gamma_{i,2}} \cdot e_{j_{i,2}} \;>\; \underline{z}^{\gamma_{i,1}} \cdot e_{j_{i,1}},$$

such that

$$\left(\sum_{i=1}^k \gamma_{i,2} - \gamma_{i,1}, \sum_{i=1}^k e_{j_{i,2}} - e_{j_{i,1}} \right) = \sum_{i=1}^k \delta_i = (0, \ldots, 0).$$

It is our first aim to show that we may assume that $e_{j_{i,1}} = e_{j_{i,2}}$ for all $i = 1, \ldots, k$.
For this we define a directed graph Γ whose vertex set is $\mathbb{E}^s$ and such that for each $i = 1, \ldots, k$ there is an edge from $e_{j_{i,1}}$ to $e_{j_{i,2}}$. Since by assumption $\sum_{i=1}^k e_{j_{i,2}} - e_{j_{i,1}} = (0, \ldots, 0)$, for each vertex e_ν the number of edges pointing towards e_ν is equal to the number of edges pointing away from e_i. Thus by Lemma 2.2 Γ is a disjoint union of cycles, each given by a subset of $\{\delta_1, \ldots, \delta_k\}$. Let $\{\delta_{i_1}, \ldots, \delta_{i_\mu}\}$ represent such a cycle with

$$j_{i_1,2} = j_{i_2,1}, \ldots, j_{i_{\mu-1},2} = j_{i_\mu,1}, j_{i_\mu,2} = j_{i_1,1}. \tag{4}$$

Set $\gamma = \gamma_{i_1,1} + \ldots + \gamma_{i_\mu,1} \in \mathbb{N}^{m+n}$, $\varepsilon_0 := (\gamma, e_{j_{i_1,1}})$ and recursively

$$\varepsilon_\nu := \varepsilon_{\nu-1} + \delta_{i_\nu} = \left(\sum_{\kappa=1}^{\nu} (\gamma_{\kappa,2} - \gamma_{\kappa,1}) + \gamma, e_{j_{i_\nu,2}} \right) \in \mathbb{N}^{m+n} \times \mathbb{E}^s$$

for $\nu = 1, \ldots, \mu$. By assumption

$$\underline{z}^{\gamma_{i_\nu,2}} \cdot e_{j_{i_\nu,2}} \;>\; \underline{z}^{\gamma_{i_\nu,1}} \cdot e_{j_{i_\nu,1}}$$

and multiplying both sides with $\underline{z}^{\sum_{\kappa=1}^{\nu-1}(\gamma_{i_\kappa,2} - \gamma_{i_\kappa,1}) + \gamma - \gamma_{i_\nu,1}} \in \mathrm{Mon}(\underline{z})$ we get

$$\underline{z}^{\sum_{\kappa=1}^{\nu}(\gamma_{i_\kappa,2} - \gamma_{i_\kappa,1}) + \gamma} \cdot e_{j_{i_\nu,2}} \;>\; \underline{z}^{\sum_{\kappa=1}^{\nu-1}(\gamma_{i_\kappa,2} - \gamma_{i_\kappa,1}) + \gamma} \cdot e_{j_{i_\nu,1}}$$

for $\nu = 1, \ldots, \mu$. Transitivity of the monomial ordering and (4) imply then

$$\underline{z}^{\sum_{\kappa=1}^{\mu} \gamma_{i_\kappa,2}} \cdot e_{j_{i_\mu,2}} = \underline{z}^{\sum_{\kappa=1}^{\mu}(\gamma_{i_\kappa,2} - \gamma_{i_\kappa,1}) + \gamma} \cdot e_{j_{i_\mu,2}} \;>\; \underline{z}^{\gamma} \cdot e_{j_{i_1,1}} = \underline{z}^{\gamma} \cdot e_{j_{i_\mu,2}}.$$

But then

$$\sum_{\nu=1}^{\mu} \delta_{i_\nu} = \varepsilon_\mu - \varepsilon_0 \in \Delta_>,$$

and we may replace $\{\delta_{i_1}, \ldots, \delta_{i_\mu}\}$ by its sum which is an element in $\Delta_>$ such that the last s components are all zero. Doing this with each of the cycles whose disjoint union is Γ we may assume that from the beginning $e_{j_{i,1}} = e_{j_{i,2}}$ for all $i = 1, \ldots, k$.
If $e_{j_{i,1}} = e_{j_{i,2}}$ then $\underline{z}^{\gamma_{i,2}} \cdot e_{j_{i,2}} > \underline{z}^{\gamma_{i,1}} \cdot e_{j_{i,1}}$ implies

$$\underline{z}^{\gamma_{i,2}} \;>\; \underline{z}^{\gamma_{i,1}}$$

with respect to the monomial ordering which $>$ induces on $\text{Mon}(\underline{z})$. The compatibility of $>$ with respect to the semigroup structure of $\text{Mon}(\underline{z})$ then leads to

$$\underline{z}^{\sum_{i=1}^k \gamma_{i,2}} > \underline{z}^{\sum_{i=1}^k \gamma_{i,1}},$$

while $\sum_{i=1}^k \delta_i = 0$ implies

$$\sum_{i=1}^k \gamma_{i,2} = \sum_{i=1}^k \gamma_{i,1},$$

which gives the desired contradiction. □

Lemma 2.4
If $>$ is a monomial ordering on $\text{Mon}^s(\underline{z})$ with $\underline{z} = (\underline{t}, \underline{x})$, and $M \subset \text{Mon}^s(\underline{z})$ is finite, then there exists $w \in \mathbb{Z}^{m+n+s}$ with

$$w_i < 0, \quad \text{if } z_i < 1, \quad \text{and} \quad w_i > 0, \quad \text{if } z_i > 1,$$

such that for $\underline{z}^\gamma \cdot e_i, \underline{z}^{\gamma'} \cdot e_j \in M$ we have

$$\underline{z}^\gamma \cdot e_i > \underline{z}^{\gamma'} \cdot e_j \iff w \cdot (\gamma, e_i) > w \cdot (\gamma', e_j).$$

In particular, if $>$ is $\underline{t}$-local then every $\underline{t}$-local weight ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ with weight vector w coincides on M with $>$.

Proof: We set $M' = M \cup \{e_1, z_1 \cdot e_1, \ldots, z_{m+n} \cdot e_1\}$, and we consider the finite subset

$$\Delta_M := \left\{(\gamma, e_i) - (\gamma', e_j) \mid \underline{z}^\gamma \cdot e_i, \underline{z}^{\gamma'} \cdot e_j \in M', \underline{z}^\gamma \cdot e_i > \underline{z}^{\gamma'} \cdot e_j\right\} \subset \Delta_>$$

of the set $\Delta_>$ from Lemma 2.3.
Lemma 2.3 implies that the convex hull of Δ_M over $\mathbb{Q}$,

$$\text{conv}_{\mathbb{Q}}(\Delta_M) := \left\{ \sum_{\delta \in \Delta_M} \lambda_\delta \cdot \delta \;\middle|\; \lambda_\delta \in \mathbb{Q}_{\geq 0}, \sum_{\delta \in \Delta_M} \lambda_\delta = 1 \right\},$$

does not contain the zero vector, since multiplying such a vanishing convex combination with the greatest common denominator of the coefficients would lead to a positive integer combination as excluded in Lemma 2.3.

However, a convex set like $\text{conv}_{\mathbb{Q}}(\Delta_M)$ which does not contain zero lies in the positive half space defined by a linear form

$$l_w : \mathbb{Q}^{m+n+s} \longrightarrow \mathbb{Q} : v \mapsto w \cdot v = \sum_{i=1}^{m+n+s} w_i \cdot v_i \tag{5}$$

given by $w \in \mathbb{Q}^{m+n+s}$ (see e.g. [Val76, Thm 2.10]), i.e.

$$\delta \in \Delta_M \quad \Longleftrightarrow \quad w \cdot \delta > 0.$$

Multiplying with the common denominator of its entries we may assume that $w \in \mathbb{Z}^{m+n+s}$.
Moreover, if for $i = 1, \ldots, m+n$ we set $\delta_i := z_i \cdot e_1 - e_1$, then by (5)

$$z_i < 1 \quad \Longleftrightarrow \quad -\delta_i \in \Delta_M \quad \Longleftrightarrow \quad w_i = \delta_i.w_i < 0$$

and

$$z_i > 1 \quad \Longleftrightarrow \quad \delta_i \in \Delta_M \quad \Longleftrightarrow \quad w_i = \delta_i.w_i > 0.$$

□

Lemma 2.5
Let $>$ be a $\underline{t}$-local ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ and let $g_1, \ldots, g_k \in R[\underline{x}]^s$ be $\underline{x}$-homogeneous (not necessarily of the same degree), then there is a $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ such that any $\underline{t}$-local weight ordering with weight vector w, say $>_w$, induces the same leading monomials as $>$ on $g_1, \ldots, g_k$, i.e.

$$\text{lm}_>(g_i) = \text{lm}_{>_w}(g_i) \quad \textit{for all } i = 1, \ldots, k.$$

Proof: Consider the monomial ideals $I_i = \langle \mathcal{M}_{\text{tail}(g_i)} \rangle$ in $K[\underline{t}, \underline{x}]$ generated by all monomials of $\text{tail}(g_i)$, $i = 1, \ldots, k$. By Dickson's Lemma (see e.g. [GrP02, Lemma 1.2.6]) I_i is generated by a finite subset, say $B_i \subset \mathcal{M}_{\text{tail}(g_i)}$, of the monomials of $\text{tail}(g_i)$. If we now set

$$M = B_1 \cup \ldots \cup B_k \cup \{\text{lm}_>(g_1), \ldots, \text{lm}_>(g_k)\},$$

then by Lemma 2.4 there is $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$ such that any $\underline{t}$-local weight ordering, say $>_w$, with weight vector w coincides on M with $>$. Let now $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu$ be any monomial occurring in $\text{tail}(g_i)$. Then there is a monomial $\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \in B_i$ such that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \mid \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

which in particular implies that $e_\nu = e_\mu$. Since g_i is $\underline{x}$-homogeneous it follows first that $|\beta| = |\beta'|$ and thus that $\beta = \beta'$. Moreover, since $>_w$ is $\underline{t}$-local it follows that $\underline{t}^{\alpha'} \geq_w \underline{t}^\alpha$ and thus that

$$\underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu.$$

But since $>$ and $>_w$ coincide on $\{\mathrm{lm}_>(g_i)\} \cup B_i \subset M$ we necessarily have that

$$\mathrm{lm}_>(g_i) \;>_w\; \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot e_\mu \geq_w \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_\nu,$$

and hence $\mathrm{lm}_{>_w}(g_i) = \mathrm{lm}_>(g_i)$. □

Lemma 2.6

Let $>$ be a $\underline{t}$-local weight ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ with weight vector $w \in \mathbb{Z}^m_{<0} \times \mathbb{Z}^{n+s}$, and let $(f_\nu)_{\nu\in\mathbb{N}}$ be a sequence of $\underline{x}$-homogeneous elements of fixed $\underline{x}$-degree d in $R[\underline{x}]^s$ such that

$$\mathrm{lm}_>(f_\nu) > \mathrm{lm}_>(f_{\nu+1}) \quad \textit{for all } \nu \in \mathbb{N}.$$

Then f_ν converges to zero in the $\langle t_1, \ldots, t_m\rangle$-adic topology, i.e.

$$\forall\, N \geq 0\; \exists\, \mu_N \geq 0 \;:\; \forall\, \nu \geq \mu_N \;\; \textit{we have} \;\; f_\nu \in \langle t_1, \ldots, t_m\rangle^N \cdot R[\underline{x}]^s.$$

In particular, the element $\sum_{\nu=0}^\infty f_\nu \in R[\underline{x}]^s_d$ exists.

Proof: Since $w_1, \ldots, w_m < 0$ the set of monomials

$$M_k = \big\{\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_i \;\big|\; w\cdot(\alpha,\beta,e_i) > -k, |\beta| = d\big\}.$$

is finite for a any fixed $k \in \mathbb{N}$.
Let $N \geq 0$ be fixed, set $\tau = \max\{|w_1|, \ldots, |w_{m+n+s}|\}$ and $k := (N + nd + 1)\cdot\tau$, then for any monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ of $\underline{x}$-degree d

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \notin M_k \quad\Longrightarrow\quad \underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j \in \langle t_1, \ldots, t_m\rangle^N \cdot R[\underline{x}]^s, \qquad (6)$$

since

$$\sum_{i=1}^m \alpha_i \cdot w_i \leq -k - \sum_{i=1}^n \beta_i \cdot w_{m+i} - w_{m+n+j} \leq -k + (nd+1)\cdot\tau$$

and thus

$$|\alpha| = \sum_{i=1}^m \alpha_i \geq \sum_{i=1}^m \alpha_i \cdot \frac{-w_i}{\tau} \geq \frac{k}{\tau} - nd - 1 = N.$$

Moreover, since M_k is finite and the $\text{lm}_>(f_\nu)$ are pairwise different there are only finitely many ν such that $\text{lm}_>(f_\nu) \in M_k$. Let μ be maximal among those ν, then by (6)

$$\text{lm}_>(f_\nu) \in \langle t_1, \ldots, t_m \rangle^N \cdot R[\underline{x}]^s \quad \text{for all} \quad \nu > \mu.$$

But since $>$ is a $\underline{t}$-local weight ordering we have that $\text{lm}_>(f_\nu) \notin M_k$ implies that no monomial of f_ν is in M_k, and thus $f_\nu \in \langle t_1, \ldots, t_m \rangle^N \cdot R[\underline{x}]^s$ for all $\nu > \mu$ by (6). This shows that f_ν converges to zero in the $\langle t_1, \ldots, t_m \rangle$-adic topology.

Since f_ν converges to zero in the $\langle t_1, \ldots, t_m \rangle$-adic topology, for every monomial $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ there is only a finite number of ν's such that $\underline{t}^\alpha \cdot \underline{x}^\beta \cdot e_j$ is a monomial occurring in f_ν. Thus the sum $\sum_{\nu=0}^\infty f_\nu$ exists and is obviously $\underline{x}$-homogeneous of degree d. □

From the proof of Theorem 2.1 we can deduce an algorithm for computing the determinate division with remainder up to arbitrary order, or if we do not require termination then it will "compute" the determinate division with remainder completely. Since for our purposes termination is not important, we will simply formulate the non-terminating algorithm.

Algorithm 2.7 (HDDwR)

INPUT: (f, G) with $G = \{g_1, \ldots, g_k\}$ and $f, g_1, \ldots, g_k \in R[\underline{x}]^s$ $\underline{x}$-homogeneous, $>$ a $\underline{t}$-local monomial ordering

OUTPUT: $(q_1, \ldots, q_k, r) \in R[\underline{x}]^k \times R[\underline{x}]^s$ such that

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is a homogeneous determinate division with remainder of f satisfying (DD1), (DD2) and (DDH).

INSTRUCTIONS:

- $f_0 := f$
- $r := 0$
- FOR $i = 1, \ldots, k$ DO $q_i := 0$
- $\nu := 0$
- WHILE $f_\nu \neq 0$ DO
  - $q_{0,\nu} := 0$

- FOR $i = 1, \ldots, k$ DO
  - $h_{i,\nu} := \sum_{p \in \mathcal{T}_{f_\nu} \,:\, \mathrm{lm}_>(g_i) \mid p} p$
  - $q_{i,\nu} := \frac{h_{i,\nu}}{\mathrm{lt}_>(g_i)}$
  - $q_i := q_i + q_{i,\nu}$
- $r_\nu := f_\nu - q_{1,\nu} \cdot \mathrm{lt}_>(g_1) - \ldots - q_{k,\nu} \cdot \mathrm{lt}_>(g_k)$
- $r := r + r_\nu$
- $f_{\nu+1} := f_\nu - q_{1,\nu} \cdot g_1 - \ldots - q_{k,\nu} \cdot g_k - r_\nu$
- $\nu := \nu + 1$

Remark 2.8
If $m = 0$, i.e. if the input data $f, g_1, \ldots, g_k \in K[\underline{x}]^s$, then Algorithm 2.7 terminates since for a given degree there are only finitely many monomials of this degree and therefore there cannot exist an infinite sequence of homogeneous polynomials $(f_\nu)_{\nu\in\mathbb{N}}$ of the same degree with

$$\mathrm{lm}_>(f_1) > \mathrm{lm}_>(f_2) > \mathrm{lm}_>(f_3) > \ldots.$$

3. Division with Remainder in $K[[\underline{t}]][\underline{x}]^s$

We will use the existence of homogeneous determinate divisions with remainder to show that in $R[\underline{x}]^s$ weak normal forms exist. In order to be able to apply this existence result we have to homogenise, and we need to extend our monomial ordering to the homogenised monomials.

Definition 3.1
Let $\underline{x}_h = (x_0, \underline{x}) = (x_0, \ldots, x_n)$.

(a) For $0 \neq f \in R[\underline{x}]^s$. We define the *homogenisation* f^h of f to be

$$f^h := x_0^{\deg_{\underline{x}}(f)} \cdot f\left(\underline{t}, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in R[\underline{x}_h]^s_{\deg_{\underline{x}}(f)}$$

and $0^h := 0$. If $T \subset R[\underline{x}]^s$ then we set

$$T^h := \left\{ f^h \;\middle|\; f \in T \right\}.$$

(b) We call the $R[\underline{x}]$-linear map

$$d : R[\underline{x}_h]^s \longrightarrow R[\underline{x}]^s : g \mapsto g^d := g_{|x_0=1} = g(\underline{t}, 1, \underline{x})$$

the *dehomogenisation* with respect to x_0.

(c) Given a $\underline{t}$-local monomial ordering $>$ on $\text{Mon}^s(\underline{t},\underline{x})$ we define a $\underline{t}$-local monomial ordering $>_h$ on $\text{Mon}^s(\underline{t},\underline{x}_h)$ by

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot x_0^a\cdot e_i\ >_h\ \underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot x_0^{a'}\cdot e_j$$

if and only if

$$|\beta|+a>|\beta'|+a'$$

or

$$\big(|\beta|+a=|\beta'|+a'\ \text{ and }\ \underline{t}^\alpha\cdot\underline{x}^\beta\cdot e_i>\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot e_j\big),$$

and we call it the *homogenisation of* $>$.

In the following remark we want to gather some straightforward properties of homogenisation and dehomogenisation.

Remark 3.2
Let $f,g\in R[\underline{x}]^s$ and $F\in R[\underline{x}_h]^s_k$. Then:

(a) $f=(f^h)^d$.
(b) $F=(F^d)^h\cdot x_0^{\deg_{\underline{x}_h}(F)-\deg_{\underline{x}}(F^d)}$.
(c) $\text{lm}_{>_h}(f^h)=x_0^{\text{ecart}(f)}\cdot\text{lm}_>(f)$.
(d) $\text{lm}_{>_h}(g^h)\,|\,\text{lm}_{>_h}(f^h)\iff\text{lm}_>(g)\,|\,\text{lm}_>(f)\ \wedge\ \text{ecart}(g)\le\text{ecart}(f)$.
(e) $\text{lm}_{>_h}(F)=x_0^{\text{ecart}(F^d)+\deg_{\underline{x}_h}(F)-\deg_{\underline{x}}(F^d)}\cdot\text{lm}_>(F^d)$.

Theorem 3.3 (Division with Remainder)
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k\in R[\underline{x}]^s$. Then any $f\in R[\underline{x}]^s$ has a weak division with remainder with respect to $g_1,\ldots,g_k$, i.e. there exist $q_1,\ldots,q_k\in R[\underline{x}]$, $r\in R[\underline{x}]^s$ and $u\in S_>$ such that

$$u\cdot f=q_1\cdot g_1+\ldots+q_k\cdot g_k+r$$

satisfies

(ID1) $\text{lm}_>(f)\ge\text{lm}_>(q_i\cdot g_i)$ *for* $i=1,\ldots,k$, *and*
(ID2) $\text{lm}_>(r)\not\in\langle\text{lm}_>(g_1),\ldots,\text{lm}_>(g_k)\rangle$ *unless* $r=0$.

Proof: The proof follows from the correctness and termination of Algorithm 3.4, which assumes the existence of the homogeneous determinate division with remainder from Theorem 2.1 respectively Algorithm 2.7. □

The following algorithm relies on the HDDwR-Algorithm, and it only terminates under the assumption that we are able to produce homogeneous determinate divisions with remainder, which implies that it is not an algorithm that can be applied in practice.

Algorithm 3.4 (DwR - Mora's Division with Remainder)
INPUT: (f, G) with $G = \{g_1, \ldots, g_k\}$ and $f, g_1, \ldots, g_k \in R[\underline{x}]^s$, $>$ a $\underline{t}$-local monomial ordering
OUTPUT: $(u, q_1, \ldots, q_k, r) \in S_> \times R[\underline{x}]^k \times R[\underline{x}]^s$ such that

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is a weak division with remainder of f.
INSTRUCTIONS:

- $T := (g_1, \ldots, g_k)$
- $D := \{g_i \in T \mid \text{lm}_>(g_i) \text{ divides } \text{lm}_>(f)\}$
- IF $f \neq 0$ AND $D \neq \emptyset$ DO
  - IF $e := \min\{\text{ecart}_>(g_i) \mid g_i \in D\} - \text{ecart}_>(f) > 0$ THEN
    - $(Q'_1, \ldots, Q'_k, R') := \text{HDDwR}\left(x_0^e \cdot f^h, (\text{lt}_{>_h}(g_1^h), \ldots, \text{lt}_{>_h}(g_k^h))\right)$
    - $f' := \left(x_0^e \cdot f^h - \sum_{i=1}^k Q'_i \cdot g_i^h\right)^d$
    - $(u'', q''_1, \ldots, q''_{k+1}, r) := \text{DwR}\left(f', (g_1, \ldots, g_k, f)\right)$
    - $q_i := q''_i + u'' \cdot Q'^d_i, \quad i = 1, \ldots, k$
    - $u := u'' - q''_{k+1}$
  - ELSE
    - $(Q'_1, \ldots, Q'_k, R') := \text{HDDwR}\left(f^h, (g_1^h, \ldots, g_k^h)\right)$
    - $(u, q''_1, \ldots, q''_k, r) := \text{DwR}\left((R')^d, T\right)$
    - $q_i := q''_i + u \cdot Q'^d_i, \quad i = 1, \ldots, k$
- ELSE $(u, q_1, \ldots, q_k, r) = (1, 0, \ldots, 0, f)$

Proof: Let us first prove the *termination*. For this we denote the numbers, ring elements and sets, which occur in the ν-th recursion step by a subscript ν, e.g. e_ν, f_ν or T_ν. Since

$$T_1^h \subseteq T_2^h \subseteq T_3^h \subseteq \ldots$$

also their leading submodules in $K[\underline{t}, \underline{x}_h]^s$ form an ascending chain

$$L_{>_h}(T_1^h) \subseteq L_{>_h}(T_2^h) \subseteq L_{>_h}(T_3^h) \subseteq \ldots,$$

and since the polynomial ring is noetherian there must be an N such that

$$L_{>_h}(T_\nu^h) = L_{>_h}(T_N^h) \quad \forall\, \nu \geq N.$$

If $g_{i,N} \in T_N$ such that $\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N)$ with $\mathrm{ecart}_>(g_{i,N}) \leq \mathrm{ecart}_>(f_N)$, then

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \;\Big|\; \mathrm{lm}_{>_h}(f_N^h).$$

We thus have either $\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$ for some $g_i \in D^N \subseteq T^{N+1}$ or $f_N \in T_{N+1}$, and hence

$$\mathrm{lm}_{>_h}(f_N^h) \in L_{>_h}(T_{N+1}^h) = L_{>_h}(T_N^h).$$

This ensures the existence of a $g_{i,N} \in T_N$ such that

$$\mathrm{lm}_{>_h}(g_{i,N}^h) \mid \mathrm{lm}_{>_h}(f_N^h)$$

which in turn implies that

$$\mathrm{lm}_>(g_{i,N}) \mid \mathrm{lm}_>(f_N),$$

$e_N \leq \mathrm{ecart}_>(g_{i,N}) - \mathrm{ecart}_>(f_N) \leq 0$ and $T_N = T_{N+1}$. By induction we conclude

$$T_\nu = T_N \quad \forall\, \nu \geq N,$$

and

$$e_\nu \leq 0 \quad \forall\, \nu \geq N. \tag{7}$$

Since in the N-th recursion step we are in the first "ELSE" case we have $(R'_N)^d = f_{N+1}$, and by the properties of HDDwR we know that for all $g \in T_N$

$$x_0^{\mathrm{ecart}_>(g)} \cdot \mathrm{lm}_>(g) = \mathrm{lm}_{>_h}(g^h) \;\not|\; \mathrm{lm}_{>_h}(R'_N)$$

and that

$$\mathrm{lm}_{>_h}(R'_N) = x_0^a \cdot \mathrm{lm}_{>_h}(f_{N+1}^h) = x_0^{a+\mathrm{ecart}_>(f_{N+1})} \cdot \mathrm{lm}_>(f_{N+1})$$

for some $a \geq 0$. It follows that, whenever $\mathrm{lm}_>(g) \mid \mathrm{lm}_>(f_{N+1})$, then necessarily

$$\mathrm{ecart}_>(g) > a + \mathrm{ecart}_>(f_{N+1}) \geq \mathrm{ecart}_>(f_{N+1}). \tag{8}$$

Suppose now that $f_{N+1} \neq 0$ and $D_{N+1} \neq \emptyset$. Then we may choose $g_{i,N+1} \in D_{N+1} \subseteq T_{N+1} = T_N$ such that

$$\mathrm{lm}_>(g_{i,N+1}) \bigm| \mathrm{lm}_>(f_{N+1})$$

and

$$e_{N+1} = \mathrm{ecart}_>(g_{i,N+1}) - \mathrm{ecart}_>(f_{N+1}).$$

According to (7) e_{N+1} is non-positive, while according to (8) it must be strictly positive. Thus we have derived a contradiction which shows that either $f_{N+1} = 0$ or $D_{N+1} = \emptyset$, and in any case the algorithm stops.
Next we have to prove the *correctness*. We do this by induction on the number of recursions, say N, of the algorithm.
If $N = 1$ then either $f = 0$ or $D = \emptyset$, and in both cases

$$1 \cdot f = 0 \cdot g_1 + \ldots + 0 \cdot g_k + f$$

is a weak division with remainder of f satisfying (ID1) and (ID2). We may thus assume that $N > 1$ and $e = \min\{\mathrm{ecart}_>(g) \mid g \in D\} - \mathrm{ecart}_>(f)$.
If $e \leq 0$ then by Theorem 2.1

$$f^h = Q_1' \cdot g_1^h + \ldots + Q_k' \cdot g_k^h + R'$$

satisfies (DD1), (DD2) and (DDH). (DD1) implies that for each $i = 1, \ldots, k$ we have

$$x_0^{\mathrm{ecart}_>(f)} \cdot \mathrm{lm}_>(f) = \mathrm{lm}_{>_h}(f^h) \geq \mathrm{lm}_{>_h}(Q_i') \cdot \mathrm{lm}_{>_h}(g_i^h) = x_0^{a_i + \mathrm{ecart}_>(g_i)} \cdot \mathrm{lm}_> \left({Q_i'}^d\right) \cdot \mathrm{lm}_>(g_i)$$

for some $a_i \geq 0$, and since f^h and $Q_i' \cdot g_i^h$ are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree by (DDH) the definition of the homogenised ordering implies that necessarily

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_> \left({Q_i'}^d\right) \cdot \mathrm{lm}_>(g_i) \quad \forall\, i = 1, \ldots, k.$$

Note that

$$(R')^d = \left(f^h - \sum_{i=1}^k Q_i' \cdot g_i^h\right)^d = f - \sum_{i=1}^k {Q_i'}^d \cdot g_i,$$

and thus

$$\operatorname{lm}_>\left((R')^d\right) = \operatorname{lm}_>\left(f - \sum_{i=1}^k {Q'_i}^d \cdot g_i\right) \leq \operatorname{lm}_>(f).$$

Moreover, by induction

$$u \cdot (R')^d = q''_1 \cdot g_1 + \ldots q''_k \cdot g_k + r$$

satisfies (ID1) and (ID2). But (ID1) implies that

$$\operatorname{lm}_>(f) \;\geq\; \operatorname{lm}_>\left((R')^d\right) \;\geq\; \operatorname{lm}_>(q''_i \cdot g_i),$$

so that

$$u \cdot f = \sum_{i=1}^k \left(q''_i + u \cdot {Q'_i}^d\right) \cdot g_i + r$$

satisfies (ID1) and (ID2).
It remains to consider the case $e > 0$. Then by Theorem 2.1

$$x_0^e \cdot f^h = Q'_1 \cdot \operatorname{lt}_{>_h}(g_1^h) + \ldots + Q'_k \cdot \operatorname{lt}_{>_h}(g_k^h) + R' \tag{9}$$

satisfies (DD1), (DD2) and (DDH). (DD1) and (DD2) imply (ID1) for this representation, which means that for some $a_i \geq 0$

$$x_0^{e+\operatorname{ecart}_>(f)} \cdot \operatorname{lm}_>(f) = \operatorname{lm}_{>_h}(x_0^e \cdot f^h) \geq$$
$$\operatorname{lm}_{>_h}(Q'_i) \cdot \operatorname{lm}_{>_h}\left(\operatorname{lt}_{>_h}(g_i^h)\right) = x_0^{a_i+\operatorname{ecart}_>(g_i)} \cdot \operatorname{lm}_>({Q'_i}^d) \cdot \operatorname{lm}_>(g_i),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree with by (DDH) we again necessarily have

$$\operatorname{lm}_>(f) \geq \operatorname{lm}_>\left({Q'_i}^d\right) \cdot \operatorname{lm}_>(g_i).$$

Moreover, by induction

$$u'' \cdot \left(f - \sum_{i=1}^k {Q'_i}^d \cdot g_i\right) = \sum_{i=1}^k q''_i \cdot g_i + q''_{k+1} \cdot f + r \tag{10}$$

satisfies (ID1) and (ID2).
Since $\operatorname{lt}_>(u'') = 1$ we have

$$\operatorname{lm}_>(f) \geq \operatorname{lm}_>\left(q''_i + u'' \cdot {Q'_i}^d\right) \cdot \operatorname{lm}_>(g_i),$$

for $i = 1, \ldots, k$ and therefore

$$(u'' - q''_{k+1}) \cdot f = \sum_{i=1}^{k} \left(q''_i + u'' \cdot Q'^d_i\right) \cdot g_i + r$$

satisfies (ID1) and (ID2) as well. It remains to show that $u = u'' - q''_{k+1} \in S_>$, or equivalently that

$$\mathrm{lt}_>(u'' - q''_{k+1}) = 1.$$

By assumption there is a $g_i \in D$ such that $\mathrm{lm}_>(g_i) \mid \mathrm{lm}_>(f)$ and $\mathrm{ecart}_>(g_i) - \mathrm{ecart}_>(f) = e$. Therefore, $\mathrm{lm}_{>_h}(g_i^h) \mid x_0^e \cdot \mathrm{lm}_{>_h}(f^h)$ and thus in the representation (9) the leading term of $x_0^e \cdot f^h$ has been cancelled by some $Q'_j \cdot \mathrm{lt}_{>_h}(g_j^h)$, which implies that

$$\mathrm{lm}_{>_h}(f^h) > \mathrm{lm}_{>_h}\left(f^h - \sum_{i=1}^{k} Q'_i \cdot g_i^h\right),$$

and since both sides are $\underline{x}_h$-homogeneous of the same $\underline{x}_h$-degree, unless the right-hand side is zero, we must have

$$\mathrm{lm}_>(f) > \mathrm{lm}_>\left(f - \sum_{i=1}^{k} Q'^d_i \cdot g_i\right) \geq \mathrm{lm}_>(q''_{k+1} \cdot f),$$

where the latter inequality follows from (ID1) for (10). Thus however $\mathrm{lm}_>(q''_{k+1}) < 1$, and since $\mathrm{lm}_>(u'') = 1$ we conclude that

$$\mathrm{lt}_>(u'' - q''_{k+1}) = \mathrm{lt}_>(u'') = 1.$$

This finishes the proof. □

Remark 3.5
As we have pointed out our algorithms are not useful for computational purposes since Algorithm 2.7 does not in general terminate after a finite number of steps. If, however, the input data are in fact polynomials in $\underline{t}$ and $\underline{x}$, then we can replace the t_i by x_{n+i} and apply Algorithm 3.4 to $K[x_1, \ldots, x_{n+m}]^s$, so that it terminates due to Remark 2.8. The computed weak division with remainder

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

is then *polynomial* in the sense that $u, q_1, \ldots, q_k \in K[\underline{t}, \underline{x}]$ and $r \in K[\underline{t}, \underline{x}]^s$. In fact, Algorithm 3.4 is then only a variant of the usual Mora algorithm.

In the proof of Schreyer's Theorem we will need the existence of weak divisions with remainder satisfying (SID2).

Corollary 3.6
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s$. Then any $f \in R[\underline{x}]^s$ has a weak division with remainder with respect to $g_1, \ldots, g_k$ satisfying (SID2).

Proof: We do the proof by induction on s where for $s = 1$ the condition (SID2) coincides with (ID2) and thus the result follows from Theorem 3.3. We may therefore assume that $s > 1$.
By Theorem 3.3 there exists a week division with remainder

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r, \tag{11}$$

and obviously, there is a $j \in \{1, \ldots, s\}$ such that $\mathrm{lm}_>(r) = \mathrm{lm}_>(r_j \cdot e_j)$, unless $r = 0$ – in which case we are done. In order to keep the notation short we assume that $j = s$ and we may assume that the g_i are ordered in such a way that for some $1 \leq l \leq k$

$$\mathrm{lm}_>(g_i) \in R[\underline{x}] \cdot e_s \quad \Longleftrightarrow \quad i > l,$$

i.e. only for the last $k - l$ of the g_i the leading monomial depends on e_s.
Consider now the projection

$$\pi : R[\underline{x}]^s \longrightarrow R[\underline{x}]^{s-1} : (p_1, \ldots, p_s) \mapsto (p_1, \ldots, p_{s-1}),$$

the inclusion

$$\iota : R[\underline{x}]^{s-1} \longrightarrow R[\underline{x}]^s : (p_1, \ldots, p_{s-1}) \mapsto (p_1, \ldots, p_{s-1}, 0),$$

and the restriction, say $>_*$, of $>$ to $\mathrm{Mon}^{s-1}(\underline{t}, \underline{x})$ defined by

$$p \ >_* \ p' \quad :\Longleftrightarrow \quad \iota(p) > \iota(p')$$

for $p, p' \in \mathrm{Mon}^{s-1}(\underline{t}, \underline{x})$ – which is again a $\underline{t}$-local monomial ordering. Note, also that for $p \in R[\underline{x}]^{s-1}$ we obviously have

$$\mathrm{lm}_> \big(\iota(p)\big) = \iota\big(\mathrm{lm}_{>_*}(p)\big). \tag{12}$$

Moreover, due to the ordering of the g_i we have for $i = 1, \ldots, l$

$$\mathrm{lm}_>(g_i) = \mathrm{lm}_>\Big(\iota\big(\pi(g_i)\big)\Big).$$

By induction hypothesis there exists a weak division with remainder of $\pi(r) = (r_1, \ldots, r_{s-1})$ with respect to $>_*$, say

$$u' \cdot \pi(r) = q_1' \cdot \pi(g_1) + \ldots + q_l' \cdot \pi(g_l) + r' \tag{13}$$

with $u' \in S_{>_*} = S_>$, $q_1', \ldots, q_l' \in R[\underline{x}]$ and $r' = (r_1', \ldots, r_{s-1}') \in R[\underline{x}]^{s-1}$, satisfying (ID1) and (SID2).
We want to show that

$$u \cdot u' \cdot f = \sum_{i=1}^{l} (u' \cdot q_i + q_i') \cdot g_i + \sum_{i=l+1}^{k} u' \cdot q_i \cdot g_i + r'', \tag{14}$$

with $r'' = (r_1', \ldots, r_{s-1}', r_s)$, satisfies (ID1) and (SID2).
Since $u, u' \in S_>$ have leading terms 1 leading terms do not change by multiplication with u or u'. Moreover, since (11) and (13) both satisfy (ID1) and taking (12) into account we have

$$\mathrm{lm}_>(f) \geq \mathrm{lm}_>(r) > \mathrm{lm}_>\Big(\iota\big(\pi(r)\big)\Big) \geq \mathrm{lm}_>\Big(q_i' \cdot \iota\big(\pi(g_i)\big)\Big) = \mathrm{lm}_>\big(q_i' \cdot g_i\big)$$

for $i = 1, \ldots, l$. Thus by (ID1) for (11) we have

$$\mathrm{lm}_>(u \cdot u' \cdot f) \geq \mathrm{lm}_>\big((u' \cdot q_i + q_i') \cdot g_i\big),$$

for $i = 1, \ldots, l$ and

$$\mathrm{lm}_>(u \cdot u' \cdot f) \geq \mathrm{lm}_>(u' \cdot q_i \cdot g_i\big)$$

for $i = l+1, \ldots, k$, which shows that (14) satisfies (ID1).
Moreover, by (SID2) for (13) we know that, unless $r_j' = 0$,

$$\mathrm{lm}_{>_*}\big(\pi(g_i)\big) \;\not|\; \mathrm{lm}_{>_*}\big((0, \ldots, r_j', \ldots, 0)\big)$$

for $j = 1, \ldots, s-1$ and $i = 1, \ldots, l$, and hence unless $r_j' = 0$,

$$\mathrm{lm}_>(g_i) = \mathrm{lm}_>\Big(\iota\big(\pi(g_i)\big)\Big) \;\not|\; \mathrm{lm}_>(r_j' \cdot e_j)$$

for $j = 1, \ldots, s-1$ and $i = 1, \ldots, l$. And since $\mathrm{lm}_>(g_i)$ involves e_s for $i = l+1, \ldots, k$ but $r_j' \cdot e_j$ does not for $j = 1, \ldots, s-1$, we have

$$\mathrm{lm}_>(g_i) \;\not|\; \mathrm{lm}_>(r_j' \cdot e_j)$$

for any $i = 1, \ldots, k$ and $j = 1, \ldots, s-1$. And since by (Id2) for (11) we also have that

$$\operatorname{lm}_>(g_i) \not| \operatorname{lm}_>(r) = \operatorname{lm}_>(r_s \cdot e_s)$$

for any $i = 1, \ldots, k$, we are done, i.e. (14) satisfies (SID2). □

Corollary 3.7
Let $>$ be a $\underline{t}$-local monomial ordering on $\operatorname{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in R[\underline{x}]^s_>$. Then any $f \in R[\underline{x}]^s_>$ has a division with remainder with respect to $g_1, \ldots, g_k$ satisfying (SID2).

Proof: Let $f = \frac{f'}{u}$ and $g_i = \frac{g_i'}{u_i}$, $i = 1, \ldots, k$, with $f', g_1', \ldots, g_k' \in R[\underline{x}]^s$ and $u, u_1, \ldots, u_k \in S_>$. Consider

$$v = u \cdot u_1 \cdot \ldots \cdot u_k \in S_>$$

and

$$f'' = v \cdot f,\ g_1'' = v \cdot g_1,\ \ldots,\ g_k'' = v \cdot g_k \in R[\underline{x}].$$

By Corollary 3.6 there exists a weak division with remainder

$$u'' \cdot f'' = q_1'' \cdot g_1'' + \ldots + q_k'' \cdot g_k'' + r'', \tag{15}$$

satisfying (ID1) and (SID2) with $u'' \in S_>$, $q_1'', \ldots, q_k'' \in R[\underline{x}]$ and $r \in R[\underline{x}]^s$. Setting

$$q_1 = \frac{q_1''}{u''},\ \ldots,\ q_k = \frac{q_k''}{u''} \in R[\underline{x}]_>$$

and

$$r = \frac{1}{u'' \cdot v} \cdot r'' \in R[\underline{x}]^s_>,$$

then

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r$$

and this representation satisfies (ID1) and (SID2) since by definition the leading monomials of the elements (including those of the components $r_\nu \cdot e_\nu$) involved in this representation are the same as those in (15). □

4. Standard Bases in $K[[\underline{t}]][\underline{x}]^s$

Definition 4.1
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $I \leq R[\underline{x}]^s$ and $J \leq R[\underline{x}]^s_>$ be submodules.
A *standard basis of* I is a finite subset $G \subset I$ such that

$$L_>(I) = L_>(G).$$

A *standard basis of* J is a finite subset $G \subset J$ such that

$$L_>(J) = L_>(G).$$

A finite subset $G \subseteq R[\underline{x}]^s_>$ is called a *standard basis* with respect to $>$ if G is a standard basis of $\langle G\rangle \leq R[\underline{x}]^s_>$.

The existence of standard bases is immediate from Hilbert's Basis Theorem.

Proposition 4.2
If $>$ is a $\underline{t}$-local monomial ordering then every submodule of $R[\underline{x}]^s$ and of $R[\underline{x}]^s_>$ has a standard basis.

Proof: Follows since $K[\underline{t},\underline{x}]^s$ is noetherian. □

Standard bases are so useful since they are generating sets for submodules of $R[\underline{x}]^s_>$ and since submodule membership can be tested by division with remainder.

Proposition 4.3
Let $>$ be $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $I, J \leq R[\underline{x}]^s_>$ submodules, $G = (g_1,\ldots,g_k) \subset J$ a standard basis of J and $f \in R[\underline{x}]^s_>$ with division with remainder

$$f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r. \tag{16}$$

Then:

(a) *$f \in J$ if and only if $r = 0$.*
(b) *$J = \langle G\rangle$.*
(c) *If $I \subseteq J$ and $L_>(I) = L_>(J)$, then $I = J$.*

Proof: (a) If $r = 0$ then obviously $f \in \langle G\rangle \subseteq J$. If conversely $f \in J$ then

$$r = f - q_1 \cdot g_1 - \ldots - q_k \cdot g_k \in J,$$

and therefore $\mathrm{lm}_>(r) \in L_>(J) = L_>(G)$. But then (ID2) implies $r = 0$.

(b) If $f \in J$ then by Corollary 3.7 f has a division with remainder as in (16), but by (a) $r = 0$ and thus $f \in \langle G\rangle$ since u is a unit.

(c) By Proposition 4.2 there exists a standard basis $G' \subseteq I \subseteq J$ of I. But then G' is a standard basis of J, since

$$L_>(G') = L_>(I) = L_>(J),$$

and thus G' generates both, I and J, by (b).

□

In order to work, even theoretically, with standard bases it is vital to have a good criterion to decide whether a generating set is standard basis or not. In order to formulate Buchberger's Criterion it is helpful to have the notion of an *s-polynomial*.

Definition 4.4
Let $>$ be a $\underline{t}$-local monomial ordering on $R[\underline{x}]^s$ and $f, g \in R[\underline{x}]^s$. We define the *s-polynomial* of f and g as

$$\mathrm{spoly}(f,g) := \frac{\mathrm{lcm}\big(\mathrm{lm}_>(f), \mathrm{lm}_>(g)\big)}{\mathrm{lt}_>(f)} \cdot f - \frac{\mathrm{lcm}\big(\mathrm{lm}_>(f), \mathrm{lm}_>(g)\big)}{\mathrm{lt}_>(g)} \cdot g.$$

Theorem 4.5 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $J \leq R[\underline{x}]^s_>$ a submodule and $g_1, \ldots, g_k \in J$. The following statements are equivalent:

(a) *$G = (g_1, \ldots, g_k)$ is a standard basis of J.*
(b) *Every normal form with respect to G of any element in J is zero.*
(c) *Every element in J has a standard representation with respect to G.*
(d) *$J = \langle G\rangle$ and $\mathrm{spoly}(g_i, g_j)$ has a standard representation for all $i < j$.*

Proof: In Proposition 4.3 we have shown that (a) implies (b), and the implication (b) to (c) is trivially true. And, finally, if $f \in J$ has a standard representation with respect to G, then $\text{lm}_>(f) \in L_>(G)$, so that (c) implies (a). Since $\text{spoly}(g_i, g_j) \in J$ condition (d) follows from (c), and the hard part is to show that (d) implies actually (c). This is postponed to Theorem 5.3. $\square$

Since for $G \subset R[\underline{x}]^s$ we have $L_>\big(\langle G\rangle_{R[\underline{x}]}\big) = L_>\big(\langle G\rangle_{R[\underline{x}]_>}\big)$ we get the following corollary.

Corollary 4.6 (Buchberger Criterion)
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ and $g_1, \ldots, g_k \in I \leq R[\underline{x}]^s$. Then the following statements are equivalent:

(a) *$G = (g_1, \ldots, g_k)$ is a standard basis of I.*
(b) *Every weak normal form with respect to G of any element in I is zero.*
(c) *Every element in I has a weak standard representation with respect to G.*
(d) *$\langle I\rangle_{R[\underline{x}]_>} = \langle G\rangle_{R[\underline{x}]_>}$ and $\text{spoly}(g_i, g_j)$ has a weak standard representation for all $i < j$.*

Proof: If G is a standard basis of I then it is a standard basis of $J = \langle I\rangle_{R[\underline{x}]_>}$, since $L_>(I) = L_>(J)$ by Remark 1.15. Suppose now that $f \in I$ has a weak division with remainder

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k + r,$$

then

$$f = \frac{q_1}{u} \cdot g_1 + \ldots + \frac{q_k}{u} \cdot g_k + \frac{1}{u} \cdot r$$

is a division with remainder of $f \in I \subseteq J$, and thus $r = 0$ by Theorem 4.5. Therefore (a) implies (b), and it is obvious that (b) implies (c). Moreover, if (c) holds and $f = \frac{f'}{u'} \in J$ with $f' \in I$ and $u' \in S_>$ then by assumption there exists a weak standard representation

$$u \cdot f' = q_1 \cdot g_1 + \ldots + q_k \cdot g_k$$

with $u \in S_>$ and $q_1, \ldots, q_k \in R[\underline{x}]$. But then

$$f = \frac{q_1}{u \cdot u'} \cdot g_1 + \ldots + \frac{q_k}{u \cdot u'} \cdot g_k$$

is a standard representation of f, and Theorem 4.5 implies that G generates J and that for each $i < j$ there is standard representation

$$\text{spoly}(g_i, g_j) = \frac{q_1}{u_1} \cdot g_1 + \ldots + \frac{q_k}{u_1} \cdot g_k$$

with $q_i \in R[\underline{x}]$ and $u_1, \ldots, u_k \in S_>$. Setting $u = u_1 \cdots u_k \in S_>$ and $q_i' = \frac{q_i \cdot u}{u_i} \in R[\underline{x}]$ we get the weak standard representation

$$u \cdot \text{spoly}(g_i, g_j) = q_1' \cdot g_1 + \ldots + q_k' \cdot g_k,$$

which shows that (d) holds true.
Finally, if (d) holds true then every weak standard representation

$$u \cdot \text{spoly}(g_i, g_j) = q_1 \cdot g_1 + \ldots + q_k \cdot g_k,$$

gives rise to a standard representation

$$\text{spoly}(g_i, g_j) = \frac{q_1}{u} \cdot g_1 + \ldots + \frac{q_k}{u} \cdot g_k,$$

so that by Theorem 4.5 G is a standard basis of J. But by Remark 1.15 $L_>(I) = L_>(J)$, so that G is also a standard basis of I. □

When working with polynomials in $\underline{x}$ as well as in $\underline{t}$ we can actually compute divisions with remainder and standard bases (see Remark 3.5), and they are also standard bases of the corresponding submodules considered over $R[\underline{x}]$ by the following corollary.

Corollary 4.7
Let $>$ be a $\underline{t}$-local monomial ordering on $\text{Mon}^s(\underline{t}, \underline{x})$ and let $G \subset K[\underline{t}, \underline{x}]^s$ be finite. Then G is a standard basis of $\langle G \rangle_{K[\underline{t},\underline{x}]}$ if and only if G is a standard basis of $\langle G \rangle_{R[\underline{x}]}$.

Proof: Let $G = (g_1, \ldots, g_k)$. By Theorem 3.3 and Remark 3.5 each $\text{spoly}(g_i, g_j)$ has a weak division with remainder with respect to G such that the coefficients and remainders involved are polynomials in $\underline{x}$ as well as in $\underline{t}$. But by Corollary 4.6 G is a standard basis of either of $\langle G \rangle_{K[\underline{t},\underline{x}]}$ and $\langle G \rangle_{R[\underline{x}]}$ if and only if all these remainders are actually zero. □

And thus it makes sense to formulate the classical standard basis algorithm also for the case $R[\underline{x}]$.

Algorithm 4.8 (STD – Standard Basis Algorithm)
INPUT: $(f_1,\ldots,f_k)\in\big(R[\underline{x}]^s\big)^k$ and $>$ a $\underline{t}$-local monomial ordering.

OUTPUT: $(f_1,\ldots,f_l)\in\big(R[\underline{x}]^s\big)^l$ a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$.

INSTRUCTIONS:

- $G=(f_1,\ldots,f_k)$
- $P=\big((f_i,f_j)\bigm| 1\leq i<j\leq k\big)$
- WHILE $P\neq\emptyset$ DO
  - Choose some pair $(f,g)\in P$
  - $P=P\setminus\{(f,g)\}$
  - $(u,\underline{q},r)=\mathrm{DwR}\big(\mathrm{spoly}(f,g),G\big)$
  - IF $r\neq 0$ THEN
    - $P=P\cup\{(f,r)\mid f\in G\}$
    - $G=G\cup\{r\}$

Proof: Since in each step when G is enlarged the leading module of G is strictly enlarged and since $K[\underline{t},\underline{x}]^s$ is noetherian the algorithm will terminate. Moreover, by Buchberger's Criterion G will be a standard basis. □

Remark 4.9
If the input of STD are polynomials in $K[\underline{t},\underline{x}]$ then the algorithm works in practice due to Remark 3.5, and it computes a standard basis G of $\langle f_1,\ldots,f_k\rangle_{K[\underline{t},\underline{x}]}$ which due to Corollary 4.7 is also a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]}$, since G still contains the generators $f_1,\ldots,f_k$.

5. Schreyer's Theorem for $K[[t_1,\ldots,t_m]][x_1,\ldots,x_n]^s$

In this section we want to prove Schreyer's Theorem for $R[\underline{x}]^s$ which proves Buchberger's Criterion and shows at the same time that a standard basis of a submodule gives rise to a standard basis of the syzygy module defined by it with respect to a special ordering.

Definition 5.1 (Schreyer Ordering)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k\in R[\underline{x}]^s_>$. We define a *Schreyer ordering* with respect to $>$ and $(g_1,\ldots,g_k)$, say $>_S$, on $\mathrm{Mon}^k(\underline{t},\underline{x})$ by

$$\underline{t}^\alpha\cdot\underline{x}^\beta\cdot\varepsilon_i\;>_S\;\underline{t}^{\alpha'}\cdot\underline{x}^{\beta'}\cdot\varepsilon_j$$

if and only if

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \mathrm{lm}_>(g_i) \ > \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \mathrm{lm}_>(g_j)$$

or

$$\underline{t}^\alpha \cdot \underline{x}^\beta \cdot \mathrm{lm}_>(g_i) \ = \ \underline{t}^{\alpha'} \cdot \underline{x}^{\beta'} \cdot \mathrm{lm}_>(g_j) \ \text{ and } \ i < j,$$

where $\varepsilon_i = (\delta_{ij})_{j=1,\ldots,k}$ is the canonical basis with i-th entry one and the rest zero.

Moreover, we define the *syzygy module of* $(g_1,\ldots,g_k)$ to be

$$\mathrm{syz}(g_1,\ldots,g_k) := \{(q_1,\ldots,q_k) \in R[\underline{x}]_>^k \mid q_1 \cdot g_1 + \ldots + q_k \cdot g_k = 0\},$$

and we call the elements of $\mathrm{syz}(g_1,\ldots,g_k)$ *syzygies* of $g_1,\ldots,g_k$.

Remark 5.2

Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$ and $g_1,\ldots,g_k \in R[\underline{x}]_>^s$. Let us fix for each $i < j$ a division with remainder of $\mathrm{spoly}(g_i,g_j)$, say

$$\mathrm{spoly}(g_i,g_j) = \sum_{\nu=1}^k q_{i,j,\nu} \cdot g_\nu + r_{ij}, \tag{17}$$

and define

$$m_{ji} := \frac{\mathrm{lcm}\big(\mathrm{lm}_>(g_i), \mathrm{lm}_>(g_j)\big)}{\mathrm{lm}_>(g_i)},$$

so that

$$\mathrm{spoly}(g_i,g_j) = \frac{m_{ji}}{\mathrm{lc}_>(g_i)} \cdot g_i - \frac{m_{ij}}{\mathrm{lc}_>(g_j)} \cdot g_j.$$

Then

$$s_{ij} := \frac{m_{ji}}{\mathrm{lc}_>(g_i)} \cdot \varepsilon_i - \frac{m_{ij}}{\mathrm{lc}_>(g_j)} \cdot \varepsilon_j - \sum_{\nu=1}^k q_{i,j,\nu} \cdot \varepsilon_\nu \in R[\underline{x}]_>^k$$

has the property

$$s_{ij} \in \mathrm{syz}(g_1,\ldots,g_k) \quad \Longleftrightarrow \quad r_{ij} = 0.$$

Moreover, the leading monomial of s_{ij} with respect to the Schreyer ordering on $\mathrm{Mon}^k(\underline{t},\underline{x})$ induced by $>$ and $(g_1,\ldots,g_k)$ is

$$\mathrm{lm}_{>_S}(s_{ij}) = m_{ji} \cdot \varepsilon_i, \tag{18}$$

since $\mathrm{lm}_>\left(\frac{m_{ji}}{\mathrm{lc}_>(g_i)} \cdot g_i\right) = \mathrm{lm}_>\left(\frac{m_{ij}}{\mathrm{lc}_>(g_j)} \cdot g_j\right)$ but $i < j$ and since (17) satisfies (ID1).

Theorem 5.3 (Schreyer)
Let $>$ be a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t},\underline{x})$, $g_1,\ldots,g_k \in R[\underline{x}]_>^s$ and suppose that $\mathrm{spoly}(g_i,g_j)$ has a weak standard representation with respect to $G=(g_1,\ldots,g_k)$ for each $i<j$.
Then G is a standard basis, and with the notation in Remark 5.2 $\{s_{ij} \mid i<j\}$ is a standard basis of $\mathrm{syz}(g_1,\ldots,g_k)$ with respect to $>_S$.

Proof: Let $I=\langle G\rangle_{R[\underline{x}]_>}$ and consider the $R[\underline{x}]_>$-linear map

$$\phi : R[\underline{x}]_>^k \longrightarrow R[\underline{x}]_>^s : (q_1,\ldots,q_k) \mapsto q_1\cdot g_1+\ldots+q_k\cdot g_k.$$

For $f\in I$ there is a $q:=(q_1,\ldots,q_k)\in R[\underline{x}]_>^k$ such that $f=\sum_{i=1}^k q_i\cdot g_i$, and by Corollary 3.7 there is a division with remainder of q with respect to $(s_{ij} \mid i<j)$ and $>_S$, say

$$q=\sum_{i<j} a_{ij}\cdot s_{ij}+r \tag{19}$$

with $a_{ij}\in R[\underline{x}]_>$ and $r=\sum_{\nu=1}^k r_\nu\cdot\varepsilon_\nu \in R[\underline{x}]_>^k$, which satisfies (ID1) and (SID2). By (SID2)

$$m_{ji}\cdot\varepsilon_i=\mathrm{lm}_{>_S}(s_{ij}) \not\mid \mathrm{lm}_{>_S}(r_\nu\cdot\varepsilon_\nu)$$

whenever $r_\nu\neq 0$, and hence

$$m_{j\nu} \not\mid r_\nu, \tag{20}$$

whenever $r_\nu\neq 0$.
Note that

$$f=\phi(q)=\phi(r)=\sum_{\nu=1}^k r_\nu\cdot g_\nu, \tag{21}$$

since $s_{ij}\in\ker(\phi)$, and we claim that

$$\mathrm{lm}_>(f)\geq \mathrm{lm}_>(r_\nu\cdot g_\nu). \tag{22}$$

For this it suffices to show that

$$\mathrm{lm}_>(r_\nu)\cdot\mathrm{lm}_>(g_\nu)\neq \mathrm{lm}_>(r_\mu)\cdot\mathrm{lm}_>(g_\mu)$$

for $\nu<\mu$, whenever $r_\nu\neq 0\neq r_\mu$. Suppose the contrary, then

$$0\neq m_{\mu\nu}\cdot\mathrm{lm}_>(g_\nu)=\mathrm{lcm}\big(\mathrm{lm}_>(g_\nu),\mathrm{lm}_>(g_\mu)\big)$$

divides

$$\mathrm{lm}_>(r_\mu)\cdot\mathrm{lm}_>(g_\mu)=\mathrm{lm}_>(r_\nu)\cdot\mathrm{lm}_>(g_\nu),$$

since both $\mathrm{lm}_>(g_\nu)$ and $\mathrm{lm}_>(g_\mu)$ divide the latter. But this contradicts (20).

It follows from (22) that (21) is a standard representation of f with respect to $(g_1, \ldots, g_k)$ and $>$, and since $f \in I$ was arbitrary it follows from Theorem 4.5 "(c)$\Longrightarrow$(a)", which we have already proved, that G is actually a standard basis of $\langle G \rangle_{R[\underline{x}]_>}$.

Moreover, $q \in \mathrm{syz}(g_1, \ldots, g_k)$ if and only if $\phi(q) = f = 0$, and by (21) and (22) this is the case if and only if $r = 0$. Thus by (19) every element in $\mathrm{syz}(g_1, \ldots, g_k)$ has a standard representation with respect to $\{s_{ij} \mid i < j\}$ and $>_S$, and therefore, as before, $\{s_{ij} \mid i < j\}$ is a standard basis of $\mathrm{syz}(g_1, \ldots, g_k)$ with respect to $>_S$ by Theorem 4.5 "(c)$\Longrightarrow$(a)". This finishes the proof. □

6. Algorithms Relying on Standard Bases

Having division with remainder, standard bases and Buchberger's Criterion at hand one can, from a theoretical point of view, basically derive all the standard results from computer algebra also for free modules over $R[\underline{x}]$ respectively $R[\underline{x}]_>$. We will gather here some of these results which are explicitly needed for the Lifting Algorithm for tropical varieties.

The simplest algorithm is the one for testing submodule membership.

Algorithm 6.1 (MEMBERSHIP)

Input: $f, f_1, \ldots, f_k \in R[\underline{x}]^s$ and $>$ a $\underline{t}$-local monomial ordering.

Output: N, where $N = 1$ if $f \in \langle f_1, \ldots, f_k \rangle_{R[\underline{x}]_>}$, and $N = 0$ else.

Instructions:

- $(u, \underline{q}, r) = \mathrm{DwR}\big(f, (f_1, \ldots, f_k), >\big)$
- IF $r = 0$ THEN $N = 1$ ELSE $N = 0$

In order to do more complicated computations one needs elimination. Recall that a monomial ordering on $\mathrm{Mon}(y_1, \ldots, y_k)$ is said to be *global* if $y_i > 1$ for all $i = 1, \ldots, k$.

Definition 6.2

Divide the variables $\underline{x} = (x_1, \ldots, x_n)$ into two disjoint subsets $\underline{x}_0$ and $\underline{x}_1$. We call a $\underline{t}$-local monomial ordering $>$ on $\mathrm{Mon}^s(\underline{t}, \underline{x})$ an *elimination* *ordering with respect to* $\underline{x}_1$ if for $f \in R[\underline{x}]$

$$\mathrm{lm}_>(f) \in K[\underline{t}, \underline{x}_1]^s \quad \Longrightarrow \quad f \in R[\underline{x}_1]^s.$$

Typical examples of elimination orderings are *block orderings* like the one defined by

$$\underline{t}^\alpha \cdot \underline{x}_0{}^\beta \cdot \underline{x}_1{}^\gamma \cdot e_i \;>\; \underline{t}^{\alpha'} \cdot \underline{x}_0{}^{\beta'} \cdot \underline{x}_1{}^{\gamma'} \cdot e_j$$

if and only if

$$\underline{x}_0{}^\beta \;>_0\; \underline{x}_0{}^{\beta'}$$

or

$$\underline{x}_0{}^\beta \;=\; \underline{x}_0{}^{\beta'} \quad \text{and} \quad \underline{t}^\alpha \cdot \underline{x}_1{}^\gamma \cdot e_i \;>_1\; \underline{t}^{\alpha'} \cdot \underline{x}_1{}^{\gamma'} \cdot e_j,$$

where $>_0$ is a global monomial ordering on $\mathrm{Mon}(\underline{x}_0)$ and $>_1$ is a $\underline{t}$-local monomial ordering on $\mathrm{Mon}^s(\underline{t}, \underline{x}_1)$. Denote $>$ by $(>_0, >_1)$.

Proposition 6.3
Let $>$ be a $\underline{t}$-local elimination ordering with respect to $\underline{x}_0$ on $\mathrm{Mon}^s(\underline{t}, \underline{x})$, $I \leq R[\underline{x}]^s_>$, and G be a standard basis of I with respect to $>$.
Then $\big(g \in G \;\big|\; \mathrm{lm}_>(g) \in K[t, \underline{x} \setminus \underline{x}_0]^s\big)$ is a standard basis of $I \cap R[\underline{x} \setminus \underline{x}_0]^s_>$.

Proof: $G' = \big(g \in G \;\big|\; \mathrm{lm}_>(g) \in K[t, \underline{x} \setminus \underline{x}_0]^s\big)$ is contained in $I \cap R[\underline{x} \setminus \underline{x}_0]^s$ since $>$ is an elimination ordering with respect to $\underline{x}_0$. Moreover, if $f \in I \cap R[\underline{x} \setminus \underline{x}_0]^s_> \subseteq I$ then there is there is a $g \in G$ such that $\mathrm{lm}_>(g) \mid \mathrm{lm}_>(f) \in K[t, \underline{x} \setminus \underline{x}_0]^s$, since G is a standard basis of I. However, this forces $\mathrm{lm}_>(g) \in K[t, \underline{x} \setminus \underline{x}_0]^s$ and thus $g \in G'$. This shows that G' is a standard basis of $I \cap R[\underline{x} \setminus \underline{x}_0]^s_>$. □

This leads to the following elimination algorithm.

Algorithm 6.4 (ELIMINATE)
INPUT: $f_1, \ldots, f_k \in R[\underline{x}]^s_>$, $\underline{x}_0 \subseteq \underline{x}$, and $>$ a $\underline{t}$-local elimination ordering with respect to $\underline{x}_0$.
OUTPUT: $G \subset R[\underline{x} \setminus \underline{x}_0]^s$ a standard basis of $\langle f_1, \ldots, f_k \rangle_{R[\underline{x}]_>} \cap R[\underline{x} \setminus \underline{x}_0]_>$.

INSTRUCTIONS:

- $G' = \mathrm{STD}(f_1, \ldots, f_k, >)$
- $G = \big(g \in G \;\big|\; \mathrm{lm}_>(g) \in K[\underline{t}, \underline{x} \setminus \underline{x}_0]^s\big)$

Proposition 6.5
Let $>$ *be a* $\underline{t}$*-local monomial ordering on* $\operatorname{Mon}^s(\underline{t},\underline{x})$*, and let* $I=\langle f_1,\ldots,f_k\rangle, J=\langle g_1,\ldots,g_l\rangle \le R[\underline{x}]^s_>$*, then*

$$I\cap J=\langle \tau\cdot f_1,\ldots,\tau\cdot f_k,(1-\tau)\cdot g_1,\ldots,(1-\tau)\cdot g_l\rangle_{R[\underline{x}]_>[\tau]}\cap R[\underline{x}]^s_>.$$

Proof: If $f=\sum_{i=1}^k a_i\cdot f_i=\sum_{j=1}^l b_j\cdot g_j$ with $a_i,b_j\in R[\underline{x}]_>$ then

$$f=\tau\cdot f+(1-\tau)\cdot f=\sum_{i=1}^k a_i\cdot\tau\cdot f_i+\sum_{j=1}^l b_j\cdot(1-\tau)\cdot g_j$$

is in the right-hand side. Conversely, if

$$f=\sum_{i=1}^k a_i\cdot\tau\cdot f_i+\sum_{j=1}^l b_j\cdot(1-\tau)\cdot g_j$$

is in the right-hand side with $a_i,b_j\in R[\underline{x}]_>[\tau]$, then

$$f=f_{|\tau=0}=\sum_{i=1}^k a_{i|\tau=0}\cdot f_i\in I$$

and

$$f=f_{|\tau=1}=\sum_{j=1}^l b_{j|\tau=1}\cdot g_j\in J,$$

so that $f\in I\cap J$. □

This leads to the following algorithm for computing intersections of submodules.

Algorithm 6.6 (INTERSECTION)
INPUT: $f_1,\ldots,f_k,g_1,\ldots,g_l\in R[\underline{x}]^s$ and $>$ a $\underline{t}$-local ordering.

OUTPUT: $G\subset R[\underline{x}]^s$ a standard basis of $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]_>}\cap\langle g_1,\ldots,g_l\rangle_{R[\underline{x}]_>}$.

INSTRUCTIONS:

- Let $>'=(>_0,>)$ be the block ordering with respect to the unique global ordering $>_0$ on $\operatorname{Mon}(\tau)$
- $G=$ ELIMINATE $\Big(\big(\tau f_1,\ldots,\tau f_k,(1-\tau)g_1,\ldots,(1-\tau)g_l\big),\tau,>'\Big)$

Proposition 6.7
Let $>$ *be a* $\underline{t}$*-local monomial ordering on* $\operatorname{Mon}^s(\underline{t},\underline{x})$*, let* $I=\langle f_1,\ldots,f_k\rangle\le R[\underline{x}]^s_>$ *and* $0\ne f\in R[\underline{x}]^s_>$.

If $I \cap \langle f\rangle = \langle g_1 \cdot f, \ldots, g_l \cdot f\rangle$, *then* $I : \langle f\rangle = \langle g_1, \ldots, g_l\rangle_{R[\underline{x}]_>}$.

Proof: It is clear that $g_1, \ldots, g_l \in I : \langle f\rangle$, and we may thus suppose that we have some $h \in I : \langle f\rangle \trianglelefteq R[\underline{x}]_>$. By assumption $h \cdot f \in I \cap \langle f\rangle$ and thus there are $a_1, \ldots, a_k \in R[\underline{x}]_>$ such that

$$h \cdot f = \left(\sum_{i=1}^{k} a_i \cdot g_i\right) \cdot f,$$

and since $R[\underline{x}]_>$ has no zero divisors this implies $h = \sum_{i=1}^{k} a_i \cdot g_i$. □

We thus get the following algorithm for computing the ideal quotient with respect to a single element.

Algorithm 6.8 (QUOTIENT)
INPUT: $f_1, \ldots, f_k, f \in R[\underline{x}]^s$ and $>$ a $\underline{t}$-local monomial ordering.

OUTPUT: $G \subset R[\underline{x}]$ a standard basis of $\langle f_1, \ldots, f_k\rangle_{R[\underline{x}]_>} : \langle f\rangle_{R[\underline{x}]_>}$.

INSTRUCTIONS:

- $G' = \text{INTERSECTION}\big((f_1, \ldots, f_k), (f)\big)$
- $G = \big(\frac{g}{f} \;\big|\; g \in G\big)$

Finally, this leads to the following algorithm for computing the saturation with respect to a single element.

Algorithm 6.9 (SATURATION)
INPUT: $f_1, \ldots, f_k, f \in R[\underline{x}]^s$ and $>$ a $\underline{t}$-local monomial ordering.

OUTPUT: $G \subset R[\underline{x}]$ a standard basis of $\langle f_1, \ldots, f_k\rangle_{R[\underline{x}]_>} : \langle f\rangle^{\infty}_{R[\underline{x}]_>}$.

INSTRUCTIONS:

- $N = 0$
- WHILE $N = 0$ DO
  - $G' = \text{QUOTIENT}\big(G, f, >\big)$
  - Test using MEMBERSHIP if $G' \subset \langle G\rangle$.
  - IF so THEN $N = 1$

Proof: The procedure constructs an ascending sequence of modules generated by the sets G, and since $R[\underline{x}]_>$ is noetherian the algorithm must stop after a finite number of steps. Moreover, once the procedure stops then

$$\langle G\rangle : \langle f\rangle = \langle G'\rangle = \langle G\rangle,$$

which shows that $\langle G\rangle$ is actually saturated with respect to f. $\square$

We will use the existence of these procedures at the end of the next section to show that generators for $\langle f_1,\ldots,f_k\rangle_{R[\underline{x}]} : \langle t\rangle^\infty_{R[\underline{x}]}$ can be computed over $K[t,\underline{x}]$ when the $f_i \in K[t,\underline{x}]$ are polynomials.

7. Application to t-Initial Ideals

In this section we want to show that for an ideal J over the field of Puiseux series which is generated by elements in $K[[t^{\frac{1}{N}}]][\underline{x}]$ respectively in $K[t^{\frac{1}{N}},\underline{x}]$ the t-initial ideal (a notion we will introduce further down) with respect to $w \in \mathbb{Q}_{<0}\times\mathbb{Q}^n$ can be computed from a standard basis of the generators.

Definition 7.1
We consider for $0 \neq N \in \mathbb{N}$ the discrete valuation ring

$$R_N = K\big[\big[t^{\frac{1}{N}}\big]\big] = \left\{\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \;\middle|\; a_\alpha \in K\right\}$$

of power series in the unknown $t^{\frac{1}{N}}$ with *discrete valuation*

$$\mathrm{val}\left(\sum_{\alpha=0}^{\infty} a_\alpha\cdot t^{\frac{\alpha}{N}}\right) = \mathrm{ord}_t\left(\sum_{\alpha=0}^{\infty} a_\alpha\cdot t^{\frac{\alpha}{N}}\right) = \min\left\{\frac{\alpha}{N} \;\middle|\; a_\alpha \neq 0\right\} \in \frac{1}{N}\cdot\mathbb{Z},$$

and we denote by

$$L_N = \mathrm{Quot}(R_N)$$

its quotient field. If $N \mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L = K\{\{t\}\} = \lim_{\longrightarrow} L_N = \bigcup_{N\geq 0} L_N$$

the *field of (formal) Puiseux series* over K.

Remark 7.2
If $0 \neq N \in \mathbb{N}$ then

$$S_N = \left\{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, t^{\frac{2}{N}}, \ldots\right\}$$

is a multiplicative subset of R_N, and obviously

$$L_N = S_N^{-1} R_N = \left\{ t^{\frac{-\alpha}{N}} \cdot f \;\middle|\; f \in R_N, \alpha \in \mathbb{N} \right\},$$

since

$$R_N^* = \left\{ \sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \;\middle|\; a_0 \neq 0 \right\}.$$

The valuations of R_N extend to L_N, and thus L, by

$$\mathrm{val}\left(\frac{f}{g}\right) = \mathrm{val}(f) - \mathrm{val}(g)$$

for $f, g \in R_N$ with $g \neq 0$.

Definition 7.3
For $0 \neq N \in \mathbb{N}$ if we consider $t^{\frac{1}{N}}$ as a variable, we get the set of monomials

$$\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) = \left\{ t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta \mid \alpha \in \mathbb{N}, \beta \in \mathbb{N}^n \right\}$$

in $t^{\frac{1}{N}}$ and $\underline{x}$. If $N \mid M$ then obviously

$$\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) \subset \mathrm{Mon}\left(t^{\frac{1}{M}}, \underline{x}\right).$$

Remark and Definition 7.4
Let $0 \neq N \in \mathbb{N}$, $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.
We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta.$$

As a K-vector space the formal power series ring $K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right]$ is just

$$K\left[\left[t^{\frac{1}{N}}, \underline{x}\right]\right] = \prod_{q \in \mathbb{R}} V_{q,w,N},$$

and we can thus write any power series $f \in K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$ in a unique way as

$$f = \sum_{q\in\mathbb{R}} f_{q,w} \quad \text{with} \quad f_{q,w} \in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f \in K\big[\big[t^{\frac{1}{N'}},\underline{x}\big]\big]$ for some other $0 \neq N' \in \mathbb{N}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.

Moreover, if $0 \neq f \in R_N[\underline{x}] \subset K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$, then there is a maximal $\hat{q} \in \mathbb{R}$ such that $f_{\hat{q},w} \neq 0$ and

$$f_{q,w} \in W_{q,w,N} \quad \text{for all} \quad q \in \mathbb{R},$$

since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w}) = q \in \mathbb{R}$,

$$\operatorname{in}_w(f) = f_{\hat{q},w} \in K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial form* of f or the *initial form of f w.r.t. w*, and

$$\operatorname{ord}_w(f) = \hat{q} = \max\{\deg_w(f_{q,w}) \mid f_{q,w} \neq 0\}$$

the *w-order* of f.

For $I \subseteq R_N[\underline{x}]$ we call

$$\operatorname{in}_w(I) = \big\langle \operatorname{in}_w(f) \;\big|\; f \in I\big\rangle \trianglelefteq K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial ideal* of I. Note that its definition depends on N!

Moreover, we call

$$\text{t-in}_w(f) = \operatorname{in}_w(f)(1,\underline{x}) = \operatorname{in}_w(f)_{|t=1} \in K[\underline{x}]$$

the *t-initial form of f w.r.t. w*, and if $f = t^{\frac{-\alpha}{N}} \cdot g \in L[\underline{x}]$ with $g \in R_N[\underline{x}]$ we set

$$\text{t-in}_w(f) := \text{t-in}_w(g).$$

This definition does not depend on the particular representation of f, since $t^{\frac{-\alpha}{N}} \cdot g = t^{\frac{-\alpha'}{N'}} \cdot g'$ implies that $t^{\frac{\alpha'}{N'}} \cdot g = t^{\frac{\alpha}{N}} \cdot g'$ in $R_{N\cdot N'}$ and thus

$$t^{\frac{\alpha'}{N'}} \cdot \operatorname{in}_w(g) = \operatorname{in}_w\big(t^{\frac{\alpha'}{N'}} \cdot g\big) = \operatorname{in}_w\big(t^{\frac{\alpha}{N}} \cdot g'\big) = t^{\frac{\alpha}{N}} \cdot \operatorname{in}_w(g'),$$

which shows that $\text{t-in}_w(g) = \text{t-in}_w(g')$.

If $I \subseteq L[\underline{x}]$ is an ideal, then

$$\text{t-in}_w(I) = \langle \text{t-in}_w(f) \mid f \in I\rangle \lhd K[\underline{x}]$$

is the *t-initial ideal* of I, which does not depend on any N.
Note also that the product of two w-quasihomogeneous elements

$$f_{q,w} \cdot f_{q',w} \in V_{q+q',w,N},$$

and thus

$$W_{q,w,N} \cdot W_{q,w,N} \subseteq W_{q+q',w,N}$$

and

$$V_{q,w,N} \cdot V_{q',w,N} \subseteq V_{q+q',w,N}.$$

In particular,

$$\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$$

for $f, g \in R_N[\underline{x}]$, and for $f, g \in L[\underline{x}]$

$$\text{t-in}_w(f \cdot g) = \text{t-in}_w(f) \cdot \text{t-in}_w(g).$$

An immediate consequence of this is the following lemma.

Lemma 7.5
If $0 \neq f = \sum_{i=1}^{k} g_i \cdot h_i$ *with* $f, g_i, h_i \in R_N[\underline{x}]$ *and* $\mathrm{ord}_w(f) \geq \mathrm{ord}_w(g_i \cdot h_i)$ *for all* $i = 1, \ldots, k$*, then*

$$\mathrm{in}_w(f) \in \big\langle\, \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k) \big\rangle \lhd K\big[t^{\frac{1}{N}}, \underline{x}\big].$$

Proof: Due to the direct product decomposition we have that

$$\mathrm{in}_w(f) = f_{\hat{q},w} = \sum_{i=1}^{k} (g_i \cdot h_i)_{\hat{q},w}$$

where $\hat{q} = \mathrm{ord}_w(f)$. By assumption $\mathrm{ord}_w(g_i) + \mathrm{ord}_w(h_i) = \mathrm{ord}_w(g_i \cdot h_i) \leq \mathrm{ord}_w(f) = \hat{q}$ with equality if and only if $(g_i \cdot h_i)_{\hat{q},w} \neq 0$. In that case necessarily

$$(g_i \cdot h_i)_{\hat{q},w} = \mathrm{in}_w(g_i) \cdot \mathrm{in}_w(h_i),$$

which finishes the proof. □

In order to be able to apply standard bases techniques we need to fix a t-local monomial ordering which refines a given weight vector w.

Definition 7.6
Fix any *global* monomial ordering, say $>$, on $\mathrm{Mon}(\underline{x})$, i.e. $x_i > 1$ for all $i = 1, \ldots, n$, and let $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$.

We define a t-local monomial ordering, say $>_w$, on $\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$ by

$$t^{\frac{\alpha}{N}} \cdot \underline{x}^{\beta} >_w t^{\frac{\alpha'}{N}} \cdot \underline{x}^{\beta'}$$

if and only if

$$w \cdot \left(\frac{\alpha}{N}, \beta\right) > w \cdot \left(\frac{\alpha'}{N}, \beta'\right)$$

or

$$w \cdot \left(\frac{\alpha}{N}, \beta\right) = w \cdot \left(\frac{\alpha'}{N}, \beta'\right) \text{ and } \underline{x}^{\beta} > \underline{x}^{\beta'}.$$

Note that this ordering is indeed t-local since $w_0 < 0$, and that it depends on w and on $>$, but assuming that $>$ is fixed we will refrain from writing $>_{w,>}$ instead of $>_w$.

Remark 7.7
If $N \mid M$ then $\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right) \subset \mathrm{Mon}\left(t^{\frac{1}{M}}, \underline{x}\right)$, as already mentioned. For $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we may thus consider the ordering $>_w$ on both $\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$ and on $\mathrm{Mon}\left(t^{\frac{1}{M}}, \underline{x}\right)$, and let us call them for a moment $>_{w,N}$ respectively $>_{w,M}$. It is important to note, that the restriction of $>_{w,M}$ to $\mathrm{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$ coincides with $>_{w,N}$. We therefore omit the additional subscript in our notation.

We now fix some global monomial ordering $>$ on $\mathrm{Mon}(\underline{x})$, and given a vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we will throughout this section always denote by $>_w$ the monomial ordering from Definition 7.6.

Proposition 7.8
If $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and $f \in R_N[\underline{x}]$ with $\mathrm{lt}_{>_w}(f) = 1$, then $\mathrm{in}_w(f) = 1$.

Proof: Suppose this is not the case then there exists a monomial of f, say $1 \neq t^{\alpha} \cdot \underline{x}^{\beta} \in \mathcal{M}_f$, such that

$$w \cdot (\alpha, \beta) \geq w \cdot (0, \ldots, 0) = 0,$$

and since $\mathrm{lm}_{>_w}(f) = 1$ we must necessarily have equality. But since $>$ is global $\underline{x}^{\beta} > 1$, which implies that also $t^{\alpha} \cdot \underline{x}^{\beta} >_w 1$, in contradiction to $\mathrm{lm}_{>_w}(f) = 1$. □

Proposition 7.9
Let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, $I \trianglelefteq R_N[\underline{x}]$ be an ideal, and let $G = \{g_1, \ldots, g_k\}$ be a standard basis of I with respect to $>_w$ then

$$\mathrm{in}_w(I) = \big\langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k) \big\rangle \trianglelefteq K\big[t^{\frac{1}{N}}, \underline{x}\big],$$

and in particular,

$$\text{t-in}_w(I) = \big\langle \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k) \big\rangle \trianglelefteq K[\underline{x}].$$

Proof: If G is standard basis of I then by Corollary 4.6 every element $f \in I$ has a weak standard representation of the form

$$u \cdot f = q_1 \cdot g_1 + \ldots + q_k \cdot g_k,$$

where $\mathrm{lt}_{>_w}(u) = 1$ and

$$\mathrm{lm}_{>_w}(u \cdot f) \geq \mathrm{lm}_{>_w}(q_i \cdot g_i).$$

The latter in particular implies that

$$\mathrm{ord}_w(u \cdot f) = \deg_w\big(\mathrm{lm}_{>_w}(u \cdot f)\big) \geq \deg_w\big(\mathrm{lm}_{>_w}(q_i \cdot g_i)\big) = \mathrm{ord}_w(q_i \cdot g_i).$$

We conclude therefore by Lemma 7.5 and Proposition 7.8 that

$$\mathrm{in}_w(f) = \mathrm{in}_w(u \cdot f) \in \big\langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_k) \big\rangle.$$

For the part on the t-initial ideals just note that if $f \in I$ then by the above

$$\mathrm{in}_w(f) = \sum_{i=1}^{k} h_i \cdot \mathrm{in}_w(g_i)$$

for some $h_i \in K\big[t^{\frac{1}{N}}, \underline{x}\big]$, and thus

$$\text{t-in}_w(f) = \sum_{i=1}^{k} h_i(1, \underline{x}) \cdot \text{t-in}_w(g_i) \in \langle \text{t-in}_w(g_1), \ldots, \text{t-in}_w(g_k) \rangle_{K[\underline{x}]}.$$

□

Theorem 7.10
Let $J \trianglelefteq L[\underline{x}]$ and $I \trianglelefteq R_N[\underline{x}]$ be ideals with $J = \langle I \rangle_{L[\underline{x}]}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of I with respect to $>_w$.
Then

$$\text{t-in}_w(J) = \text{t-in}_w(I) = \big\langle \text{t-in}_w(G) \big\rangle \lhd K[\underline{x}].$$

Proof: Since $R_N[\underline{x}]$ is noetherian, we may add a finite number of elements of I to G so as to assume that $G = (g_1, \ldots, g_k)$ generates I. Since by Proposition 7.9 we already know that the t-initial forms of any standard basis of I with respect to $>_w$ generate $\text{t-in}_w(I)$ this does not change the right-hand side. But then by assumption

$$J = \langle G \rangle_{L[\underline{x}]},$$

and given an element $f \in J$ we can write it as

$$f = \sum_{i=1}^{k} t^{\frac{-\alpha}{N \cdot M}} \cdot a_i \cdot g_i$$

for some $M >> 0$, $a_i \in R_{N \cdot M}$ and $\alpha \in \mathbb{N}$. It follows that

$$t^{\frac{\alpha}{N \cdot M}} \cdot f = \sum_{i=1}^{k} a_i \cdot g_i \in \langle G \rangle_{R_{N \cdot M}[\underline{x}]}.$$

Since G is a standard basis over $R_N[\underline{x}]$ with respect to $>_w$ on $\text{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$ by Buchberger's Criterion 4.6 $\text{spoly}(g_i, g_j)$, $i < j$, has a weak standard representation

$$u_{ij} \cdot \text{spoly}(g_i, g_j) = \sum_{\nu=1}^{k} q_{ij\nu} \cdot g_\nu$$

with $u_{ij}, q_{ij\nu} \in R_N[\underline{x}] \subseteq R_{N \cdot M}[\underline{x}]$ and $\text{lt}_{>_w}(u_{ij}) = 1$. Taking Remark 7.7 into account these are also weak standard representations with respect to the corresponding monomial ordering $>_w$ on $\text{Mon}(t^{\frac{1}{N \cdot M}}, \underline{x})$, and again by Buchberger's Criterion 4.6 there exists a weak standard representation

$$u \cdot t^{\frac{\alpha}{N \cdot M}} \cdot f = \sum_{i=1}^{k} q_i \cdot g_i.$$

By Lemma 7.5 and Proposition 7.8 this implies that

$$t^{\frac{\alpha}{N \cdot M}} \cdot \text{in}_w(f) = \text{in}_w\left(u \cdot t^{\frac{\alpha}{N \cdot M}} \cdot f\right) \in \left\langle \text{in}_w(G) \right\rangle.$$

Setting $t = 1$ we get

$$\text{t-in}_w(f) = \left(t^{\frac{k}{N \cdot M}} \cdot \text{in}_w(f)\right)_{|t=1} \in \left\langle \text{t-in}_w(G) \right\rangle.$$

$\square$

Corollary 7.11

Let $J = \langle I' \rangle_{L[\underline{x}]}$ with $I' \trianglelefteq K\left[t^{\frac{1}{N}}, \underline{x}\right]$, $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ and G is a standard basis of I' with respect to $>_w$ on $\operatorname{Mon}\left(t^{\frac{1}{N}}, \underline{x}\right)$*, then*

$$\text{t-in}_w(J) = \text{t-in}_w(I') = \left\langle \text{t-in}_w(G) \right\rangle \trianglelefteq K[\underline{x}].$$

Proof: Enlarge G to a finite generating set G' of I', then G' is still a standard basis of I'. By Corollary 4.7 G' is then also a standard basis of

$$I := \langle G' \rangle_{R_N[\underline{x}]} = \langle f_1, \ldots, f_k \rangle_{R_N[\underline{x}]},$$

and Theorem 7.10 applied to I thus shows that

$$\text{t-in}(J) = \left\langle \text{t-in}_w(G') \right\rangle.$$

However, if $f \in G' \subset I'$ is one of the additional elements then it has a weak standard representation

$$u \cdot f = \sum_{g \in G} q_g \cdot g$$

with respect to G and $>_w$, since G is a standard basis of I'. Applying Propositions 7.5 and 7.8 then shows that $\operatorname{in}_w(f) \in \langle \operatorname{in}_w(G) \rangle$, which finishes the proof. □

Remark 7.12

Note that if $I \trianglelefteq R_N[\underline{x}]$ and $J = \langle I \rangle_{L[\underline{x}]}$, then

$$J \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}} \right\rangle^{\infty},$$

but the saturation is in general necessary.

Since $L_N \subset L$ is a field extension Corollary 7.15 implies

$$J \cap L_N[\underline{x}] = \langle I \rangle_{L_N[\underline{x}]},$$

and it suffices to see that

$$\langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}} \right\rangle^{\infty}.$$

If $I \cap S_N \neq \emptyset$ then both sides of the equation coincide with $R_N[\underline{x}]$, so that we may assume that $I \cap S_N$ is empty. Recall that $L_N = S_N^{-1} R_N$, so that if $f \in R_N[\underline{x}]$ with $t^{\frac{\alpha}{N}} \cdot f \in I$ for some α, then

$$f = \frac{t^{\frac{\alpha}{N}} \cdot f}{t^{\frac{\alpha}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}].$$

Conversely, if

$$f = \frac{g}{t^{\frac{k}{N}}} \in \langle I \rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}]$$

with $g \in I$, then $g = t^{\frac{\alpha}{N}} \cdot f \in I$ and thus f is in the right-hand side.

Lemma 7.13
Let $F \subset F'$ be a field extension, and $I = \langle \underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_k} \rangle \trianglelefteq F[\underline{x}]$ be a monomial ideal. Then $I = \langle I \rangle_{F'[\underline{x}]} \cap F[\underline{x}]$.

Proof: It suffices to show that $\langle I \rangle_{F'[\underline{x}]} \cap F[\underline{x}] \subseteq I$. For this we consider an $f \in \langle I \rangle_{F'[\underline{x}]} \cap F[\underline{x}]$. Since $\langle I \rangle_{F'[\underline{x}]} = \langle \underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_k} \rangle_{F'[\underline{x}]}$ is a monomial ideal, for every term, say f_j, of f there is some i such that $\underline{x}^{\alpha_i} \mid f_j$, i.e. $f_j = \underline{x}^{\alpha_i} \cdot f'_j$ with $f'_j \in F'[\underline{x}]$. However, since all coefficients of f are in F so must be all coefficients of f'_j, and thus $f_j = \underline{x}^{\alpha_i} \cdot f'_j \in I$, which implies $f \in I$. □

Lemma 7.14
Let $F \subset F'$ be a field extension, $f_1, \ldots, f_k \in F[\underline{x}]$, and $>$ a global monomial ordering on $\operatorname{Mon}(\underline{x})$*. Then every Gröbner basis G of $\langle f_1, \ldots, f_k \rangle_{F[\underline{x}]}$ with respect to $>$ is also a Gröbner basis of $\langle f_1, \ldots, f_k \rangle_{F'[\underline{x}]}$.*

Proof: If $G = \{g_1, \ldots, g_l\}$ then $g_i \in \langle f_1, \ldots, f_k \rangle_{F[\underline{x}]} \subseteq \langle f_1, \ldots, f_k \rangle_{F'[\underline{x}]}$, and

$$\langle f_1, \ldots, f_k \rangle_{F[\underline{x}]} = \langle G \rangle_{F[\underline{x}]}$$

shows that

$$\langle f_1, \ldots, f_k \rangle_{F'[\underline{x}]} = \langle G \rangle_{F'[\underline{x}]}.$$

If $s_{i,j}$ denotes the S-polynomial of g_i and g_j, then by Buchberger's Criterion 4.5 there exists a standard representation

$$s_{i,j} = q_{1,i,j} \cdot g_1 + \ldots + q_{l,i,j} \cdot g_l$$

with $q_{s,i,j} \in F[\underline{x}] \subseteq F'[\underline{x}]$. But then these same representations together with Buchberger's Criterion imply that G is a Gröbner basis of $\langle f_1, \ldots, f_k \rangle_{F'[\underline{x}]}$. □

Corollary 7.15
Let $F \subset F'$ be a field extension and $I \trianglelefteq F[\underline{x}]$. Then $I = \langle I \rangle_{F'[\underline{x}]} \cap F[\underline{x}]$.

Proof: Fix any global monomial ordering $>$ on $\mathrm{Mon}(\underline{x})$ and set $I^e = \langle I\rangle_{F'[\underline{x}]}$. Since $I \subseteq I^e \cap F[\underline{x}] \subseteq I^e$ we also have

$$L_>(I) \subseteq L_>\big(I^e \cap F[\underline{x}]\big) \subseteq L_>(I^e) \cap F[\underline{x}]. \qquad (23)$$

If we choose a standard basis $G = (g_1, \ldots, g_k)$ of I, then by Lemma 7.14 G is also a Gröbner basis of I^e and thus

$$L_>(I) = \langle \mathrm{lm}_>(g_i) \mid i = 1, \ldots, k\rangle_{F[\underline{x}]}$$

and

$$L_>(I^e) = \langle \mathrm{lm}_>(g_i) \mid i = 1, \ldots, k\rangle_{F'[\underline{x}]} = \big\langle L_>(I)\big\rangle_{F'[\underline{x}]}.$$

Since the latter is a monomial ideal, by Lemma 7.13 we have

$$L_>(I^e) \cap F[\underline{x}] = L_>(I).$$

In view of (23) this shows that

$$L_>(I) = L_>\big(I^e \cap F[\underline{x}]\big),$$

and since $I \subseteq I^e \cap F[\underline{x}]$ this finishes the proof by Proposition 4.3. □

We can actually show more, namely, that for each $I \trianglelefteq R_N[\underline{x}]$ and each $M > 0$ (see Corollary 7.16)

$$\langle I\rangle_{R_{M\cdot N}[\underline{x}]} \cap R_N[\underline{x}] = I,$$

and if I is saturated with respect to $t^{\frac{1}{N}}$ then (see Corollary 7.19)

$$\mathrm{in}_w\left(\langle I\rangle_{R_{M\cdot N}[\underline{x}]}\right) = \big\langle\, \mathrm{in}_w(G)\big\rangle,$$

if G is a standard basis of I with respect to $>_w$. This requires, however, some extra work which is partly outsourced to Section 8.

Corollary 7.16
If $I \trianglelefteq R_N[\underline{x}]$ then $\langle I\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_N[\underline{x}] = I$.

Proof: If $f = g \cdot h \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]} \cap R_N[\underline{x}]$ with $g \in I$ and $h \in R_{N\cdot M}[\underline{x}]$ then by Corollary 8.2 there are uniquely determined $h_i \in R_N$ such that

$$h = \sum_{i=0}^{M-1} h_i \cdot t^{\frac{i}{N\cdot M}},$$

and hence

$$f = \sum_{i=0}^{M-1} (g \cdot h_i) \cdot t^{\frac{i}{N\cdot M}}$$

with $g \cdot h_i \in R_N[\underline{x}]$. By assumption $f \in R_N[\underline{x}] = R_{N\cdot M}[\underline{x}] \cap \langle 1\rangle_{R_N[\underline{x}]}$ and by Corollary 8.2 we thus have

$$g \cdot h_i = 0 \quad \text{for all } \; i = 1, \ldots, M-1.$$

But then $f = g \cdot h_0 \in I$. □

Lemma 7.17
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$, then for any $M \geq 1$

$$\langle I\rangle_{R_{N\cdot M}[\underline{x}]} = \langle I\rangle_{R_{N\cdot M}[\underline{x}]} : \left\langle t^{\frac{1}{N\cdot M}}\right\rangle^\infty.$$

Proof: Let $f, h \in R_{N\cdot M}[\underline{x}]$, $\alpha \in \mathbb{N}$, $g \in I$ such that

$$t^{\frac{\alpha}{N\cdot M}} \cdot f = g \cdot h. \tag{24}$$

We have to show that $f \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. For this purpose do division with remainder in order to get

$$\alpha = a \cdot M + b \quad \text{with} \quad 0 \leq b < M.$$

By Corollary 8.2 there are $h_i, f_i \in R_N[\underline{x}]$ such that $f = \sum_{i=0}^{M-1} f_i \cdot t^{\frac{i}{N\cdot M}}$ and $h = \sum_{i=0}^{M-1} h_i \cdot t^{\frac{i}{N\cdot M}}$. (24) then translates into

$$\sum_{i=0}^{M-1-b} t^{\frac{b+i}{N\cdot M}} \cdot t^{\frac{a}{N}} \cdot f_i + \sum_{i=M-b}^{M-1} t^{\frac{b+i-M}{N\cdot M}} \cdot t^{\frac{a+1}{N}} \cdot f_i = \sum_{i=0}^{M-1} g \cdot h_i \cdot t^{\frac{i}{N\cdot M}},$$

and since $\left\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\right\}$ is $R_N[\underline{x}]$-linearly independent we can compare coefficients to find

$$t^{\frac{a}{N}} \cdot f_i = g \cdot h_{b+i} \in I$$

for $i = 0, \ldots, M-b-1$, and

$$t^{\frac{a+1}{N}} \cdot f_i = g \cdot h_{b+i-M} \in I$$

for $i = M-b, \ldots, M-1$. In any case, since I is saturated with respect to $t^{\frac{1}{N}}$ by assumption we conclude that $f_i \in I$ for all $i = 0, \ldots, M-1$, and therefore $f \in \langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. □

Corollary 7.18
Let $J \trianglelefteq L[\underline{x}]$ be an ideal such that $J = \langle J \cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and let G be a standard basis of $J \cap R_N[\underline{x}]$ with respect to $>_w$.

Then for all $M \geq 1$

$$\operatorname{in}_w\big(J\cap R_{N\cdot M}[\underline{x}]\big)=\big\langle\operatorname{in}_w(G)\big\rangle\lhd K\big[t^{\frac{1}{N\cdot M}},\underline{x}\big]$$

and

$$\text{t-in}_w\big(J\cap R_{N\cdot M}[\underline{x}]\big)=\big\langle\text{t-in}_w(G)\big\rangle=\text{t-in}_w\big(J\cap R_N[\underline{x}]\big)\lhd K[\underline{x}].$$

Proof: Enlarge G to a generating set G' of $I = J \cap R_N[\underline{x}]$ over $R_N[\underline{x}]$ by adding a finite number of elements of I. Then

$$\big\langle L_{>_w}(G')\big\rangle\subseteq\big\langle L_{>_w}(I)\big\rangle=\big\langle L_{>_w}(G)\big\rangle\subseteq\big\langle L_{>_w}(G')\big\rangle$$

shows that G' is still a standard basis of I with respect to $>_w$. So we can assume that $G = G'$.

By Proposition 7.9 it suffices to show that G is also a standard basis of $J \cap R_{N\cdot M}[\underline{x}]$. Since by assumption

$$J=\langle I\rangle_{L[\underline{x}]}=\langle G\rangle_{L[\underline{x}]},$$

Corollary 7.15 implies that

$$J\cap L_{N\cdot M}[\underline{x}]=\langle G\rangle_{L_{N\cdot M}[\underline{x}]}=S_{N\cdot M}^{-1}\langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Moreover, by Remark 7.12 the ideal $I = \langle G\rangle_{R_N[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N}}$ and by Lemma 7.17 therefore also $\langle G\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, which implies that

$$J\cap R_{N\cdot M}[\underline{x}]=S_{N\cdot M}^{-1}\langle G\rangle_{R_{N\cdot M}[\underline{x}]}\cap R_{N\cdot M}[\underline{x}]=\langle G\rangle_{R_{N\cdot M}[\underline{x}]}.$$

Since $G = (g_1, \ldots, g_k)$ is a standard basis of I every $\operatorname{spoly}(g_i, g_j)$, $i < j$, has a weak standard representation with respect to G and $>_w$ over $R_N[\underline{x}]$ by Buchberger's Criterion 4.6, and these are of course also weak standard representations over $R_{N\cdot M}[\underline{x}]$, so that again by Buchberger's Criterion G is a standard basis of $\langle G\rangle_{R_{N\cdot M}[\underline{x}]} = J \cap R_{N\cdot M}[\underline{x}]$. □

Corollary 7.19
Let $I \trianglelefteq R_N[\underline{x}]$ *be an ideal such that* $I = I : \big\langle t^{\frac{1}{N}}\big\rangle^\infty$, *let* $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, *and let* G *be a standard basis of* I *with respect to* $>_w$.
Then for all $M \geq 1$

$$\operatorname{in}_w\big(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\big)=\big\langle\operatorname{in}_w(G)\big\rangle\lhd K\big[t^{\frac{1}{N\cdot M}},\underline{x}\big]$$

and

$$\text{t-in}_w\big(\langle I\rangle_{R_{N\cdot M}[\underline{x}]}\big)=\big\langle\,\text{t-in}_w(G)\big\rangle=\text{t-in}_w(I)\lhd K[\underline{x}]$$

Proof: If we consider $J=\langle I\rangle_{L[\underline{x}]}$ then by Remark 7.12 $J\cap R_N[\underline{x}]=I$, and moreover, by Lemma 7.17 also $\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$ is saturated with respect to $t^{\frac{1}{N\cdot M}}$, so that applying Remark 7.12 once again we also find $J\cap R_{N\cdot M}[\underline{x}]=\langle I\rangle_{R_{N\cdot M}[\underline{x}]}$. The result therefore follows from Corollary 7.18. □

Corollary 7.20
Let $J\trianglelefteq L[\underline{x}]$ be an ideal such that $J=\langle J\cap R_N[\underline{x}]\rangle_{L[\underline{x}]}$, let $w=(-1,0,\ldots,0)$ and let $M\geq 1$. Then

$$1\in\text{in}_\omega\big(J\cap R_N[\underline{x}]\big)\quad\Longleftrightarrow\quad 1\in\text{in}_\omega\big(J\cap R_{N\cdot M}[\underline{x}]\big).$$

Proof: Suppose that $f\in J\cap R_{N\cdot M}[\underline{x}]$ with $\text{in}_\omega(f)=1$, and let $G=(g_1,\ldots,g_k)$ be standard basis of $J\cap R_N[\underline{x}]$ with respect to $>_w$. By Corollary 7.18

$$1=\text{in}_\omega(f)\in\big\langle\,\text{in}_\omega(g_1),\ldots,\text{in}_\omega(g_k)\big\rangle\lhd K\big[t^{\frac{1}{N\cdot M}},\underline{x}\big],$$

and since this ideal and 1 are w-quasihomogeneous, there exist w-quasihomogeneous elements $h_1,\ldots,h_k\in K\big[t^{\frac{1}{N\cdot M}},\underline{x}\big]$ such that

$$1=\sum_{i=1}^k h_i\cdot\text{in}_\omega(g_i),$$

where each summand on the right-hand side (possibly zero) is w-quasihomogeneous of w-degree zero. Since $w=(-1,0,\ldots,0)$ this forces $h_i\in K[\underline{x}]$ for all $i=1,\ldots,k$ and thus $1\in\text{in}_\omega(J\cap R_N[\underline{x}])$. The converse is clear anyhow. □

We want to conclude the section by a remark on the saturation.

Proposition 7.21
If $f_1,\ldots,f_k\in K[t,\underline{x}]$ and $I=\langle f_1,\ldots,f_k\rangle\trianglelefteq K[t]_{\langle t\rangle}[\underline{x}]$ then

$$\langle I\rangle_{R_1[\underline{x}]}:\langle t\rangle^\infty=\big\langle I:\langle t\rangle^\infty\big\rangle_{R_1[\underline{x}]}.$$

Proof: Let $>_1$ be any global monomial ordering on $\text{Mon}(\underline{x})$ and define a t-local monomial ordering on $\text{Mon}\big(t,\underline{x}\big)$ by

$$t^\alpha\cdot\underline{x}^\beta\;>\;t^{\alpha'}\cdot\underline{x}^{\beta'}$$

if and only if

$$\underline{x}^\alpha >_1 \underline{x}^{\alpha'} \text{ or } \big(\underline{x}^\alpha = \underline{x}^{\alpha'} \text{ and } \alpha < \alpha'\big).$$

Then

$$\{f \in R_1[\underline{x}] \mid \operatorname{lt}_>(f) = 1\} = \{1 + t \cdot p \mid p \in K[t]\},$$

and thus

$$R_1[\underline{x}]_> = R_1[\underline{x}] \quad \text{and} \quad K[t,\underline{x}]_> = K[t]_{\langle t\rangle}[\underline{x}].$$

Using Algorithm 6.9 we can compute at the same time a standard basis of $\langle I\rangle_{R_1[\underline{x}]} : \langle t\rangle^\infty$ and of $\langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]} : \langle t\rangle^\infty$ with respect to $>$. Since a standard basis is a generating set in the localised ring the result follows. □

8. Some Properties of $R_N[\underline{x}]$, $L_N[\underline{x}]$, and $L[\underline{x}]$

In this section we gather some straightforward but useful properties of the rings we are working with and their relations among each other.

Lemma 8.1
$R_{N\cdot M}$ is a finite free R_N-module with basis $\big\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\big\}$.
In particular, $R_{N\cdot M}$ is integral over R_N.

Proof: Note that $f = \sum_{i=0}^\infty a_i \cdot t^{\frac{i}{N\cdot M}} \in R_{N\cdot M}$ can be written as

$$f = \sum_{j=0}^{M-1} t^{\frac{j}{N\cdot M}} \cdot \sum_{i=0}^{\infty} a_{j+i\cdot M} \cdot t^{\frac{i}{N}} \in \Big\langle 1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}} \Big\rangle_{R_N}.$$

Moreover, since no terms can cancel $f = 0$ if and only if

$$\sum_{i=0}^{\infty} a_{j+i\cdot M} \cdot t^{\frac{i}{N}} = 0$$

for all $j = 0, \ldots, M-1$. Thus $R_{N\cdot M}$ is free over R_N with basis $\big\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\big\}$. □

Corollary 8.2
$R_{N\cdot M}[\underline{x}]$ is a free $R_N[\underline{x}]$-module with basis $\big\{1, t^{\frac{1}{N\cdot M}}, \ldots, t^{\frac{M-1}{N\cdot M}}\big\}$.

Proof: If $f=\sum_{|\alpha|=0}^d a_\alpha\cdot\underline{x}^\alpha\in R_{N\cdot M}[\underline{x}]$ with $a_\alpha\in R_{N\cdot M}$ is given, then by Lemma 8.1 there exist $a_{\alpha,i}\in R_N$ such that

$$a_\alpha=\sum_{i=0}^{M-1} a_{\alpha,i}\cdot t^{\frac{i}{N\cdot M}},$$

and thus

$$f=\sum_{i=0}^{M-1} t^{\frac{i}{N\cdot M}}\cdot\sum_{|\alpha|=0}^{d} a_{\alpha,i}\cdot\underline{x}^\alpha\in\left\langle 1,t^{\frac{1}{N\cdot M}},\ldots,t^{\frac{M-1}{N\cdot M}}\right\rangle_{R_N[\underline{x}]}.$$

Suppose now that

$$\sum_{i=0}^{M-1} t^{\frac{i}{N\cdot M}}\cdot f_i=0$$

with

$$f_i=\sum_{|\alpha|=0}^{d} a_{\alpha,i}\cdot\underline{x}^\alpha\in R_N[\underline{x}].$$

Then

$$0=\sum_{|\alpha|=0}^{d}\underline{x}^\alpha\cdot\sum_{i=0}^{M-1} t^{\frac{i}{N\cdot M}}\cdot a_{\alpha,i},$$

and since the $\underline{x}^\alpha$ are linearly independent over R_N it follows that

$$\sum_{i=0}^{M-1} t^{\frac{i}{N\cdot M}}\cdot a_{\alpha,i}=0$$

for all $|\alpha|\leq d$. But then by Lemma 8.1 we have $a_{\alpha,i}=0$ for all i and and all α, which implies that $f_i=0$ for $i=0,\ldots,M-1$. □

Corollary 8.3
L is algebraic over L_N.

Proof: If $A\subset B$ is an integral extension of integral domains, then $\mathrm{Quot}(A)\subset\mathrm{Quot}(B)$ is algebraic. For this consider $0\neq b\in B$ and the integral relation $b^k+a_1\cdot b^{k-1}+\ldots+a_k=0$ with $a_i\in A$ which it fulfils by assumption. Then

$$a_k\cdot\frac{1}{b^k}+\ldots+a_1\cdot\frac{1}{b}+1=0$$

is an algebraic relation of $\frac{1}{b}$ over $\mathrm{Quot}(A)$. This shows that $\mathrm{Quot}(B)$ is algebraic over $\mathrm{Quot}(A)$, since every element of the former is of the

form $\frac{b'}{b}$. In particular, $L_{N\cdot M} = \text{Quot}(R_{N\cdot M})$ is algebraic over $L_N = \text{Quot}(R_N)$.
If $a \in L$, then there is an M such that $a \in L_M \subseteq L_{N\cdot M}$, and therefore a is algebraic over L_N. This shows that L is algebraic over L_N. $\square$

Corollary 8.4
$L[\underline{x}]$ is integral over $L_N[\underline{x}]$.

Proof: If $A \subset B$ is an integral ring extension, then so is $A[x] \subset B[x]$. To see this let $f = \sum_{i=0}^k b_i \cdot x^i \in B[x]$ be given. Then b_i is integral, over A and thus it is integral over $A[x]$. But since $x^i \in A[x]$ we have that

$$f = \sum_{i=0}^{k} b_i x^i \in A[x][b_0, \ldots, b_k]$$

is an element of the integral ring extension $A[x] \subset A[x][b_0, \ldots, b_k]$. This shows that $B[x]$ is integral over $A[x]$. The result follows thus from Corollary 8.3. $\square$

Corollary 8.5
The ring extension $L_N[\underline{x}] \subset L[\underline{x}]$ satisfies the lying-over, going-up and the going-down property.

Proof: See [AtM69, Prop. 5.10, Thm. 5.11 and Thm. 5.16]. $\square$

Corollary 8.6
Let $I \trianglelefteq L[\underline{x}]$ be an ideal then $L[\underline{x}]/I$ is integral over $L_N[\underline{x}]/I \cap L_N[\underline{x}]$. In particular, $\dim(I) = \dim(I \cap L_N[\underline{x}])$.

Proof: See [AtM69, Prop. 5.6] and [Eis96, Prop. 9.2]. $\square$

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