TROPICAL GEOMETRY

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Tropical geometry still is a rather young field of mathematics which nevertheless over the past couple of years has proved to be rather powerful for tackling questions in enumerative geometry (see e.g. [Mik05], [GM08], [IKS04]). It also has been used to provide alternative algorithmic means to eliminate variables (see [StY08]). People from applied mathematics such as optimisation or control theory are interested in it as a kind of algebraic geometry over the max-plus algebra (see e.g. [CGQ99]). Even though all these people talk about *tropical varieties* in some way or the other, their ideas of what a tropical variety should be differ quite a bit. For enumerative questions it is most helpful to consider tropical curves as parametrised objects, while for elimination one has to take an implicit point of view, and for many other questions a purely combinatorial description seems best. In general the classes of objects considered do not completely coincide, but they share a sufficiently large overlap. For the purpose of this paper we will mainly consider the implicit approach. We will explain some of the combinatorial structure which is inherent in all the different approaches, and we will focus on certain computational questions in tropical geometry.

One could think of tropical geometry as being a shadow of classical algebraic geometry, which carries enough information to shed some light on the classical objects but which at the same time is light enough to be easier to deal with, or better to allow the application of tools from other areas of mathematics. The base field over which the classical objects live should be algebraically closed and carry a non-trivial nonarchimedean valuation into the real numbers. The prototype of such a field are the *Puiseux series*

$$a = c_0 \cdot t^{q_0} + c_1 \cdot t^{q_1} + c_2 \cdot t^{q_2} + \ldots$$

where the $c_i \in \mathbb{C}$ are complex numbers and $q_0 < q_1 < q_2 < \ldots$ is a strictly ascending sequence of rational numbers whose denominators are bounded for each series separately. t is just an indeterminate, and the order of a Puiseux series

$$\operatorname{ord}(a) = q_0$$

is a nonarchimedean valuation on the field $\mathbb{K}$ of all Puiseux series. For later use we call

$$\operatorname{lc}(a) = c_0$$

the *leading coefficient* of the Puiseux series a. The valuation is a map $\operatorname{ord} : \mathbb{K}^* \longrightarrow \mathbb{R}$ which naturally extends to a map

$$\operatorname{ord} : (\mathbb{K}^*)^n \longrightarrow \mathbb{R}^n$$

by

$$(a_1, \ldots, a_n) \mapsto (\operatorname{ord}(a_1), \ldots, \operatorname{ord}(a_n)),$$

whose image being $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

An ideal

$$0 \neq I \trianglelefteq \mathbb{K}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$$

in the ring of Laurent polynomials over $\mathbb{K}$ defines a classical algebraic variety

$$V(I) = \left\{\underline{a} \in \left(\mathbb{K}^*\right)^n \mid f(\underline{a}) = 0 \;\forall\; f \in I\right\}$$

in the torus $\left(\mathbb{K}^*\right)^n$, and its image under the valuation is basically what we call the *tropical variety* associated to the ideal I. Unfortunately, being a subset of $\mathbb{Q}^n$ this would not be a closed subset of $\mathbb{R}^n$. One way around this is to pass to the topological closure and define the *tropical variety* associated to I as

$$\operatorname{Trop}(I) = \overline{\operatorname{ord}\left(V(I)\right)}.$$

Another way around the problem would be to enlarge the base field so as to get a surjective valuation onto $\mathbb{R}$ (see e.g. [Mar07]), but for computational reasons we prefer to stick with the Puiseux series.

Let us have a look at the line in $(\mathbb{K}^*)^2$ given by

$$x + y + 1 = 0.$$

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If a point with coordinates

$$(c_0 \cdot t^{\omega_1} + h.o.t.\ ,\ d_0 \cdot t^{\omega_2} + h.o.t.)$$

lies on this line, then in particular the lowest terms of

$$(c_0 \cdot t^{\omega_1} + h.o.t.) + (d_0 \cdot t^{\omega_2} + h.o.t.) + 1 \tag{1}$$

have to vanish. This shows that one of the following cases occurs:

$$\omega_1 = \omega_2 \leq 0, \quad \omega_1 = 0 \leq \omega_2 \quad \text{or} \quad \omega_2 = 0 \leq \omega_1.$$

The tropical curve associated to $\langle x+y+1\rangle$ therefore is the following piecewise linear graph:

In general, the definition is not at all helpful when it comes down to understanding the geometric structure of a tropical variety or finally to compute it in some way. But the example shows an important feature which points $\underline{\omega} \in \mathrm{Trop}(I)$ have to satisfy with respect to the polynomials $f \in I$. If we consider the point (ω_1, ω_2) as a weight vector on the monomials of f (e.g. the monomial $x^i \cdot y^j$ has weighted degree $i \cdot \omega_1 + j \cdot \omega_2$), then the lowest term of $x+y+1$ with respect to that weight vector will either be $x+y$ or $x+1$ or $y+1$ or $x+y+1$, but it will certainly *not* be a monomial, since otherwise no cancellation of the terms of lowest order in (1) would be possible. This, of course, generalises: given a point $\underline{\omega} = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ we define the *t-initial form*

$$\text{t-in}_{\underline{\omega}}(f) = \sum_{\substack{\mathrm{ord}(a_{\underline{\alpha}}) + \langle \underline{\alpha}, \underline{\omega}\rangle \\ \text{minimal}}} \mathrm{lc}(a_{\underline{\alpha}}) \cdot \underline{x}^{\underline{\alpha}}$$

of

$$f = \sum_{\alpha} a_\alpha \cdot \underline{x}^{\underline{\alpha}},$$

using the usual multi-index notation, and we call

$$\text{t-in}_{\underline{\omega}}(I) = \left\langle \text{t-in}_{\underline{\omega}}(f) \mid f \in I \right\rangle \trianglelefteq \mathbb{C}\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$$

the *t-initial ideal* of f. For a point $\underline{\omega}$ to belong to the tropical variety $\mathrm{Trop}(I)$ it is then obviously necessary that no t-initial form of an element in I is a monomial. However, the converse is true as well, and this is known as the *Lifting Lemma*.

Theorem 1 (Lifting Lemma)**.**

$$\mathrm{Trop}(I) = \left\{ \underline{\omega} \mid \text{t-in}_{\underline{\omega}}(I) \textit{ is monomial-free} \right\}.$$

In the case that I is a principal ideal the proof basically goes back to Newton and was formulated for more general valuation fields in [EKL06]. A constructive proof can be found in [Tab05]. The general case was proven in [SpS04], but the proof contained a gap which led to a series of papers repairing the proof using different methods and applying to various types of nonarchimedean valued fields – in [Dra08] affinoid algebras are used, in [Kat06] flat deformations over valuation rings are used, and recently in [Pay07] the general problem is reduced to the hypersurface case using intersections with and projections to tori and the last proof works for any algebraically closed nonarchimedean valued field.

In [JMM08] we give a *constructive* proof of the Lifting Lemma over the Puiseux series field reducing the general case to the zero dimensional case and using a space curve version of the Newton-Puiseux algorithm proposed in [Mau80]. "Constructive" here means that given a point $\underline{\omega}$ in the right hand side in Theorem 1 which has only rational entries, then we are able to construct a point p in $V(I)$ with $\mathrm{ord}(p) = \underline{\omega}$. The algorithms deduced from the proof are implemented using the computer algebra system SINGULAR and the program `gfan` for computing tropical varieties in the SINGULAR library `tropical.lib` (see [JMM07]). Of course, the input data for SINGULAR procedures have to be restricted to polynomials in the ring $\mathbb{Q}(t)[x_1, \dots, x_n]$ instead of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and we can only construct p up to a finite number of terms, but this is sufficient for most purposes. Whenever it is necessary, field extensions of $\mathbb{Q}$ will be computed.

The algorithm basically consists of two steps. If $\dim(I) = d$ then in a first step we choose d generic hyperplanes in $\mathbb{K}^n$ whose tropicalisation passes through $\underline{\omega}$ and add the corresponding linear forms to I thus reducing to the zero-dimensional case. Then considering t as a variable we have the germ at the origin of a space curve and we can use a space curve version of the Newton-Puiseux algorithm. For this second step one chooses a zero

$$\underline{u} \in V\big(\text{t-in}_{\underline{\omega}}(I)\big) \cap \big(\mathbb{C}^*\big)^n$$

and transforms the ideal I by

$$x_i \mapsto t^{\omega_i} \cdot (u_i + x_i)$$

into an ideal I'. Choosing a point in the tropical variety of I' one can go on in the same way and construct a solution of the form

$$(u_1 \cdot t^{\omega_1} + h.o.t., \dots, u_n \cdot t^{\omega_n} + h.o.t.)\,.$$

There are of course a couple of technical issues one has to take care of.

As indicated in the above mentioned algorithm it is necessary to compute again and again points in the tropical varieties of the transformed ideals. This is in general a difficult task. If the input is polynomial in t as well as in $\underline{x}$, then one can simply work in the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}, \underline{x}^{\pm 1}]$ for a start and intersect the resulting tropical variety with the affine plane $t = 1$. From the analogous statement to the Lifting Lemma in the case of polynomials in $\mathbb{C}[t^{\pm 1}, \underline{x}^{\pm 1}]$ one immediately deduces that the tropical variety is contained in the codimension-one skeleton of the Gröbner fan of the ideal, and it even inherits the polyhedral structure.

Theorem 2 ([BiG84],[Stu02]). *If I is a d-dimensional prime ideal, then* $\mathrm{Trop}(I)$ *is a rational polyhedral complex of pure dimension d which is connected in codimension one.*

A possible though rather stupid algorithm for computing the tropical variety would thus be to compute the Gröbner fan and to check all its lower dimensional cones whether they belong to the tropical variety or not. There are, however, better algorithms (see [BJS+07], [Jen07], [HeT07]), and the main idea here is to find a *tropical basis*, i.e. a generating system $f_1, \dots, f_k$ of I such that the tropical variety $\mathrm{Trop}(I)$ is the intersection of the finite number of tropical hypersurfaces $\mathrm{Trop}(f_1), \dots, \mathrm{Trop}(f_k)$. Those used to algebraic geometry might be surprised to find that this does not in general hold true for a generating system of I.

The knowledge of a tropical basis is so helpful since for a hypersurface $V(f)$ the associated tropical hypersurface $\mathrm{Trop}(f)$ can be read off the polynomial f in a rather easy manner. Combinatorially $\mathrm{Trop}(f)$ is completely determined by the subdivision of the Newton polytope of f induced by f. Moreover, since the tropical hypersurface is just the locus of non-differentiability of the piece wise linear function

$$\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R} : \underline{x} \mapsto \min\{\mathrm{ord}(a_{\underline{\alpha}}) + \langle \underline{\alpha}, \underline{x} \rangle\}$$

the subdivision allows to recover the tropical hypersurface completely. Note that the function is non-differentiable precisely if the minimum is attained by at least two of the linear forms.

Let us explain in an example how the above construction can be carried out:

$$f = x^3 + y^3 + \frac{1}{t} \cdot xy + 1,$$

and hence

$$\mathrm{trop}(f) = \min\{3x, 3y, -1 + x + y, 0\}.$$

The Newton polygon is the convex hull of the support of f (see Figure 1, left hand side). One extends the Newton polygon from two to three dimensions by adding to the lattice point $\underline{\alpha}$ the valuation of $a_{\underline{\alpha}}$ as third coordinate. Taking the convex hull of the resulting points we get a convex polytope in $\mathbb{R}^3$ (see Figure 2). Projecting the lower faces into the $\underline{\alpha}$-plane we get the subdivision of the Newton polygon in Figure 1 on the right hand side. The tropical curve $\mathrm{Trop}(f)$

FIGURE 1. Newton Polygon of f

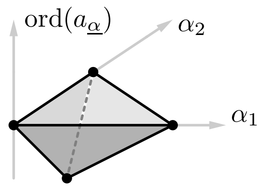
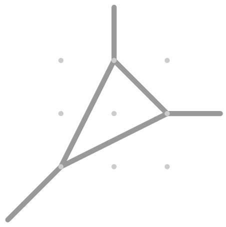


FIGURE 2. Extended Newton Polytope

is now dual to this subdivision in the sense that each two-dimensional polygon in the subdivision corresponds to a vertex in the tropical curve; two vertices are connected by a bounded edge if and only if the corresponding polygons in the subdivision share an edge; the direction vector of the bounded edge in the tropical curve will be perpendicular to the corresponding edge in the Newton subdivision; each edge of a polygon on the boundary corresponds to an unbounded edge of the curve whose direction again is perpendicular to the edge of the polygon. Finally, the vertex of the tropical curve corresponding to the polygon with vertices $(0,0),(1,1),(3,0)$ is determined by

$$0 = -1 + x + y = 3x \leq 3y,$$

and similar for the others. We thus get the tropical curve:



This, of course, leads to a straight forward algorithm for computing tropical hypersurfaces, and for drawing them in the case of plane curves and surfaces in three space. An algorithm for plane curves which produces `latex` output is implemented in the SINGULAR library `tropical.lib` (see [JMM07], Figure 3 has been created using this package). The problem of displaying surfaces in three space in a suitable manner is a more subtle problem. `TropicalSurfaces` by Lars Allermann offers an implementation for this (see [All08]). Figures 4–6 have been created using his package. The algebraic equation for Figure 6 is

$$\begin{aligned} t^2 + \frac{1}{t^8}x + t^3y + t^{14}xy + \frac{1}{16}yz + t^{10}xz \\ + \frac{1}{t^{10}}x^2 + t^2y^2 + \frac{1}{t^{12}}z^2 + \frac{1}{t^{16}}xyz + t^3x^2y \\ + \frac{1}{t^7}xy^2 + y^2z + \frac{1}{t^{14}}yz^2 + x^2z \\ + \frac{1}{t^9}xz^2 + t^8x^3 + t^7y^3 + \frac{1}{t^4}z^3 = 0 \end{aligned}$$

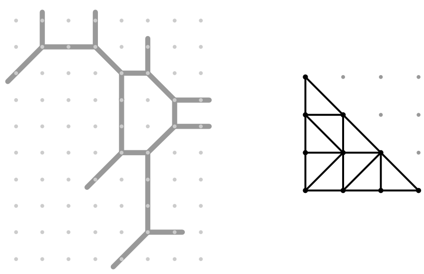


FIGURE 3. A Tropical Cubic

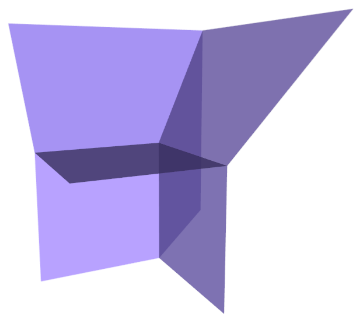


FIGURE 4. A Tropical Plane

FIGURE 5. A Tropical Quadric Surface

Above we have described what tropical varieties are and how computer algebra can be used to compute them respectively to visualise them. We want to end this presentation with an example of how computer algebra can be used to prove

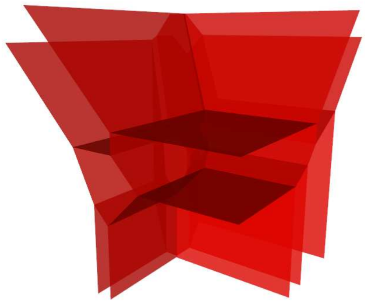


FIGURE 6. A Tropical Cubic Surface

results on the connection between classical algebraic varieties and their tropicalisation.
We have already seen above that tropical varieties carry an interesting geometrical and combinatorial structure, and we have also seen that this structure reflects certain properties of the classical varieties, e.g. the dimension (see Theorem 2). But there are more geometric features which are preserved under sufficiently good circumstances. E.g. if a polynomial f defines a plane curve of genus g, then the tropical curve will be a graph of genus at most g. In the case $g = 1$ the classical curve is an elliptic curve, which up to isomorphism is determined by its j-invariant, an element of the base field $\mathbb{K}$. It so happens that the valuation of the j-invariant is reflected in the geometry of the tropical curve if it has a cycle.

Theorem 3 ([KMM07])**.** *If f is a plane cubic such that* $\mathrm{Trop}(f)$ *is a three-valent graph with a cycle, then the valuation of the j-invariant of $V(f)$ is the negative of the lattice length of the cycle.*

An example is given in Figure 3, where the left part displays the tropical cubic curve and the right part is the correponding Newton subdivision. The equation of the cubic is

$$\begin{aligned} f = t^7\cdot\left(x^3+y^3\right)+t^3x^2+t^2\cdot\left(xy^2+y^2\right) \\ +t\cdot\left(x^2y+x+y+1\right)+xy=0, \end{aligned}$$

and from the equation we can compute the j-invariant which is a quotient of a polynomial in t of degree 48 by a polynomial of degree 60:

$$j(f)=\frac{1-24\cdot t^2+\ldots+2985984\cdot t^{48}}{t^8-5\cdot t^9+\ldots-19683\cdot t^{60}}.$$

However, more interesting than the degree of the polynomials is their order, since the difference of those is the valuation of the j-invariant. In the example we have

$$\mathrm{ord}\left(j(f)\right)=-8.$$

The tropical curve defined by f is a three-valent graph with a cycle. The cycle length in this example can be computed by counting the lattice points on the the cycle and turns out to be 8. Formally the lattice length of the cycle is defined as the sum of the lattice lengths of its edges, and the lattice length of an edge is its Euclidean length normalised by the Euclidean length of its dual edge in the Newton subdivision.
The result of Theorem 3 was proved using the computer algebra systems `polymake`, `topcom` and SINGULAR ([GaJ97], [Ram02], [GPS05]).

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