A NEW INVARIANT FOR PLANE CURVE SINGULARITIES

THOMAS KEILEN AND CHRISTOPH LOSSEN

Abstract. In [GLS01] the authors gave a general sufficient numerical condition for the T-smoothness (smoothness and expected dimension) of equisingular families of plane curves. This condition involves a new invariant γ^* for plane curve singularities, and it is conjectured to be asymptotically proper. In [Kei04], similar sufficient numerical conditions are obtained for the T-smoothness of equisingular families on various classes surfaces. These conditions involve a series of invariants γ_α^*, $0 \leq \alpha \leq 1$, with $\gamma_1^* = \gamma^*$. In the present paper we compute (respectively give bounds for) these invariants for semiquasihomogeneous singularities.

When studying numerical conditions for the T-smoothness of equisingular families of curves, new invariants of plane curve singularities $V(f) \subset (\mathbb{C}^2, 0)$ turn up. These invariants are defined as the maximum of a function depending on the codimension of complete intersection ideals containing the Tjurina ideal, respectively the equisingularity ideal, of f, and on the intersection multiplicity of f with elements of the complete intersection ideals. In Section 1 we will define these invariants, and we will calculate them for several classes of singularities, the main results being Proposition 11, Proposition 12 and Proposition 13. It is the upper bound in Lemma 8 which ensures that the conditions for T-smoothness with these new conditions (see [GLS00], [GLS01], [Kei04]) improve than the previously known ones (see [GLS97]). In the remaining sections we introduce some notation and we gather some necessary, though mainly well-known technical results used in the proofs of Section 1.
We should like to point out that the definition of the invariant γ_1^* below is a modification of the invariant "γ^*" defined in [GLS01], and it is always bound from above by the latter. Moreover, the latter can be replaced by it in the conditions of [GLS01] Proposition 2.2.

Notation

Throughout this paper $R = \mathbb{C}\{x, y\}$ will be the ring of convergent power series in the variables x and y, and $\mathfrak{m} = \langle x, y \rangle \lhd R$ will be its maximal ideal.

Contents

Date: September, 2003.
1991 *Mathematics Subject Classification.* Primary 14H20, 14H15, 14H10.
Key words and phrases. Algebraic geometry, singularity theory.
The first author was supported by the European mathematical network EAGER. The second author was supported by DFG grant no. Lo 864/1.

1. The γ_α^*-Invariants

For the definition of the γ_α^*-invariants the Tjurina ideal, respectively the equisingularity ideal in the sense of [Wah74], play an essential role. For the convenience of the reader we recall their definitions.

Definition 1
Let $f \in \mathfrak{m}$ be a reduced power series. The *Tjurina ideal* of f is defined as

$$I^{ea}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle,$$

and the *equisingularity ideal* of f is defined as

$$I^{es}(f) = \big\{ g \in R \;\big|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \big\} \supseteq I^{ea}(f).$$

Their codimensions

$$\tau(f) = \dim_{\mathbb{C}} R/I^{ea}(f),$$

respectively

$$\tau^{es}(f) = \dim_{\mathbb{C}} R/I^{es}(f),$$

are analytical, respectively topological, invariants of the singularity type defined by f. Note that $\tau^{es}(f)$ is the codimension of the μ-constant stratum in the equisingular deformation of the plane curve singularity defined by f. It can be computed in terms of multiplicities of the strict transform of f at essential infinitely near points in the resolution tree of $\big(V(f), 0\big)$ (cf. [Shu91]).

Definition 2
Let $f \in \mathfrak{m}$ be a reduced power series, and let $0 \leq \alpha \leq 1$ be a rational number.
If I is a zero-dimensional ideal in R with $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ and $g \in I$, we define

$$\lambda_\alpha(f; I, g) := \frac{\big(\alpha \cdot i(f,g) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)},$$

and

$$\gamma_\alpha(f; I) := \max\big\{(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\big|\; g \in I, i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)\big\},$$

where $i(f,g)$ denotes the intersection multiplicity of f and g. Note that, by Lemma 3, $i(f,g) > \dim_{\mathbb{C}}(R/I)$ for all $g \in I$. Thus $\gamma_\alpha(f;I)$ is a well-defined positive rational number.
We then set

$$\gamma_\alpha^{ea}(f) := \max\big\{0,\ \gamma_\alpha(f;I) \;\big|\; I \supseteq I^{ea}(f) \text{ is a complete intersection ideal}\big\}$$

and

$$\gamma_\alpha^{es}(f) := \max\big\{0,\ \gamma_\alpha(f;I) \;\big|\; I \supseteq I^{es}(f) \text{ is a complete intersection ideal}\big\}$$

Note that if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $I^{ea}(f) = I^{es}(f) = R$ and there is no zero-dimensional complete intersection ideal containing any of those two, hence $\gamma_\alpha^{ea}(f) = \gamma_\alpha^{es}(f) = 0$.

Lemma 3
Let $f \in \mathfrak{m}^2$ be reduced, and let I be an ideal such that $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$.
Then, for any $g \in I$, we have

$$\dim_{\mathbb{C}}(R/I) < \dim_{\mathbb{C}} \big(R/\langle f, g\rangle\big) = i(f,g).$$

Proof: Cf. [Shu97] Lemma 4.1; the idea is mainly to show that not both derivatives of f can belong to $\langle f, g\rangle$. $\square$

Up to embedded isomorphism the Tjurina ideal only depends on the analytical type of the singularity. More precisely, if $f \in R$ any power series, $u \in R$ a unit and $\phi : R \to R$ an isomorphism, then $I^{ea}(u \cdot f \circ \phi) = \{g \circ \phi \mid g \in I^{ea}(f)\}$. Thus the following definition makes sense.

Definition 4
Let $\mathcal{S}$ be an analytical, respectively topological, singularity type, and let $f \in R$ be a representative of $\mathcal{S}$. We then define

$$\gamma_\alpha^{ea}(\mathcal{S}) := \gamma_\alpha^{ea}(f),$$

respectively

$$\gamma_\alpha^{es}(\mathcal{S}) := \max\{\gamma_\alpha^{es}(g) \mid g \text{ is a representative of } \mathcal{S}\}.$$

Since $i(f, g) > \dim_{\mathbb{C}}(R/I)$ in the above situation, we deduce the following lemma.

Lemma 5
Let $f \in \mathfrak{m}^2$ be reduced, $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ be a zero-dimensional ideal, and $0 \le \alpha < \beta \le 1$, then $\gamma_\alpha(f; I) < \gamma_\beta(f; I)$.
In particular, for any analytical, respectively topological, singularity type

$$\gamma_\alpha^{ea}(\mathcal{S}) < \gamma_\beta^{ea}(\mathcal{S}) \qquad \textit{respectively} \qquad \gamma_\alpha^{es}(\mathcal{S}) < \gamma_\beta^{es}(\mathcal{S}).$$

For reasons of comparison let us also recall the definition of τ_{ci}^{ea}, τ_{ci}^{es}, κ and δ.

Definition 6
For $f \in R$ we define

$$\tau_{ci}^{ea}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{ea}(f) \text{ a complete intersection}\},$$

and

$$\tau_{ci}^{es}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{es}(f) \text{ a complete intersection}\}.$$

Again, for analytically equivalent singularities the values coincide, so that for an analytical singularity type $\mathcal{S}$, choosing some representative $f \in R$, we may define

$$\tau_{ci}^{ea}(\mathcal{S}) := \tau_{ci}(f).$$

For a topological singularity type we set

$$\tau_{ci}^{es}(\mathcal{S}) := \max\{\tau_{ci}^{es}(g) \mid g \text{ a representative of } \mathcal{S}\}.$$

Note that obviously

$$\tau_{ci}^{ea}(\mathcal{S}) \le \tau(\mathcal{S}) \quad \text{and} \quad \tau_{ci}^{es}(\mathcal{S}) \le \tau^{es}(\mathcal{S}),$$

where $\tau(\mathcal{S})$ is the Tjurina number of $\mathcal{S}$ and $\tau^{es}(\mathcal{S})$ is as defined in Definition 1.

Definition 7
For $f \in R$ and $\mathcal{O} = R/\langle f\rangle$, we define the δ-invariant

$$\delta(f) = \dim_{\mathbb{C}} \widetilde{\mathcal{O}}/\mathcal{O}$$

where $\mathcal{O} \subset \widetilde{\mathcal{O}}$ is the normalisation of $\mathcal{O}$, and the κ-invariant

$$\kappa(f) = i\left(f, \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial x}\right),$$

where $(\alpha : \beta) \in \mathbb{P}^1_{\mathbb{C}}$ is generic.
δ and κ are topological (thus also analytical) invariants of the singularity defined by f so that for the topological, respectively analytical, singularity type $\mathcal{S}$ given by f we can set

$$\delta(\mathcal{S}) = \delta(f) \quad \text{and} \quad \kappa(\mathcal{S}) = \kappa(f).$$

> Throughout this article we will sometimes treat topological and analytical singularities at the same time. Whenever we do so, we will write $I^*(f)$ for $I^{ea}(f)$ respectively for $I^{ea}(f)$, and analogously we will use the notation γ^*_α, τ^*_{ci} and τ^*.

The following lemma is again obvious from the definition of $\gamma_\alpha(f;I)$, once we take into account that $\kappa(f) = i(f,g)$ for a generic element $g \in I^{ea}(f)$ of f and that for a fixed value of $d = \dim_{\mathbb{C}}(R/I)$ the function $i \mapsto \frac{(\alpha i+(1-\alpha)\cdot d)^2}{i-d}$ takes its maximum on $[d+1, 2d]$ for the minimal possible value $i = d+1$.

Lemma 8
Let $f \in \mathfrak{m}^2$ be reduced, and let I be an ideal in R such that $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$. Then

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \gamma_\alpha(f;I) \leq \big(\dim_{\mathbb{C}}(R/I) + \alpha\big)^2.$$

Moreover, if $\kappa(f) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$, then

$$\gamma_\alpha(f;I) \geq \frac{\big(\alpha \cdot \kappa(f) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{\kappa(f) - \dim_{\mathbb{C}}(R/I)}.$$

In particular, for any analytical, respectively topological, singularity type $\mathcal{S}$

$$(1+\alpha)^2 \cdot \tau^*_{ci}(\mathcal{S}) \leq \gamma^*_\alpha(\mathcal{S}) \leq \big(\tau^*_{ci}(\mathcal{S}) + \alpha\big)^2,$$

and if $\kappa(\mathcal{S}) \leq 2 \cdot \tau^*_{ci}(\mathcal{S})$, then

$$\gamma^*_\alpha(\mathcal{S}) \geq \frac{\big(\alpha \cdot \kappa(\mathcal{S}) + (1-\alpha)\cdot \tau^*_{ci}(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau^*_{ci}(\mathcal{S})}.$$

In order to make the conditions for T-smoothness in [Kei04] as sharp as possible, it is useful to know under which circumstances the term $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ involved in the definition of $\gamma^*_\alpha(f)$ is actually exceeded.

Lemma 9
If $\mathcal{S}$ is a topological or analytical singularity type such that $\kappa(\mathcal{S}) < 2 \cdot \tau^*_{ci}(\mathcal{S})$, then

$$(1+\alpha)^2 \cdot \tau^*_{ci}(\mathcal{S}) < \gamma^*_\alpha(\mathcal{S}).$$

This is in particular the case, if $\mathcal{S} \neq A_1$ and $\tau^*_{ci}(\mathcal{S}) = \tau^*(\mathcal{S})$, i. e. if the Tjurina ideal, respectively the equisingularity ideal, of some representative is a complete intersection.

Proof: Lemma 8 gives

$$\gamma^*_\alpha(\mathcal{S}) \geq \frac{\big(\alpha \cdot \kappa(\mathcal{S}) + (1-\alpha)\cdot \tau^*_{ci}(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau^*_{ci}(\mathcal{S})}.$$

If we consider the right-hand side as a function in $\kappa(\mathcal{S})$, it is strictly decreasing on the interval $[0, 2\cdot\tau_{ci}^*(\mathcal{S})]$ and takes its minimum thus at $2\cdot\tau_{ci}^*(\mathcal{S})$. By the assumption on $\kappa(\mathcal{S})$ we, therefore, get

$$\gamma_\alpha^*(\mathcal{S}) > (1+\alpha)^2 \cdot \tau_{ci}^*(\mathcal{S}).$$

Suppose now that $\tau_{ci}^*(\mathcal{S}) = \tau^*(\mathcal{S})$ and $\mathcal{S} \neq A_1$. By Lemma 10 we know $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S}) \leq \tau(\mathcal{S})$. On the other hand we have $\kappa(\mathcal{S}) \leq 2\cdot\delta(\mathcal{S})$ (see [GLS05]). Therefore, $\kappa(\mathcal{S}) < 2\cdot\tau_{ci}^*(\mathcal{S})$. □

Lemma 10
If $\mathcal{S} \neq A_1$ is any analytical or topological singularity type, then $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S})$.

Proof: If (C,z) is a representative of $\mathcal{S}$ and if $\mathcal{T}^*(C,z)$ is the essential subtree of the complete embedded resolution tree of (C,z), then

$$\delta(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C)\cdot(\mathrm{mult}_p(C)-1)}{2}$$

and

$$\tau^{es}(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\mathrm{mult}_p(C)\cdot(\mathrm{mult}_p(C)+1)}{2} - \#\text{ free points in }\mathcal{T}^*(C,z) - 1,$$

where $\mathrm{mult}_p(C)$ denotes the multiplicity of the strict transform of C at p (see [GLS05]). Setting $\varepsilon_p = 0$ if p is satellite, $\varepsilon_p = 1$ if $p \neq z$ is free, and $\varepsilon_z = 2$, then $\mathrm{mult}_p(C) \geq \varepsilon_p$ and therefore

$$\tau^{es}(\mathcal{S}) = \delta(\mathcal{S}) + \sum_{p\in\mathcal{T}^*(C,z)} \big(\mathrm{mult}_p(C) - \varepsilon_p\big) \geq \delta(\mathcal{S}).$$

Moreover, we have equality if and only if $\mathrm{mult}_z(C) = 2$, $\mathrm{mult}_p(C) = 1$ for all $p \neq z$ and there is no satellite point, but this implies that $\mathcal{S} = A_1$. □

For some classes of singularities we can calculate the γ_α^*-invariant concretely, and for some others we can at least give an upper bound, which in general is much better than the one derived from Lemma 8. We restrict our attention to singularities having a convenient semi-quasihomogeneous representative $f \in R$ (see Definition 31). Throughout the following proofs we will frequently make use of monomial orderings, see Section 2.

Proposition 11 ((Simple Singularities))
Let α be a rational number with $0 \leq \alpha \leq 1$. Then we obtain the following values for $\gamma_\alpha^{es}(\mathcal{S}) = \gamma_\alpha^{ea}(\mathcal{S})$, where $\mathcal{S}$ is a simple singularity type.

$\mathcal{S}$		$\gamma_\alpha^{ea}(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$
A_k,	$k \geq 1$	$(k+\alpha)^2$
D_k,	$4 \leq k \leq 4+\sqrt{2}\cdot(2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
D_k,	$k \geq 4+\sqrt{2}\cdot(2+\alpha)$	$(k-2+\alpha)^2$
E_k,	$k = 6,7,8$	$\frac{(k+2\alpha)^2}{2}$

Proof: Let $\mathcal{S}_k$ be one of the simple singularity types A_k, D_k or E_k, and let $f \in R$ be a representative of $\mathcal{S}_k$. Note that the Tjurina ideal $I^{ea}(f)$ and the equisingularity ideal $I^{es}(f)$ coincide, and hence so do the γ_α^*-invariants, i. e.

$$\gamma_\alpha^{ea}(\mathcal{S}_k) = \gamma_\alpha^{es}(\mathcal{S}_k).$$

Moreover, in the considered cases the Tjurina ideal is indeed a complete intersection ideal with $\dim_{\mathbb{C}}\big(R/I^{ea}(f)\big) = k$, so that in particular the given values are upper bounds for $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ for any complete intersection ideal I containing the Tjurina ideal. By Lemma 8 we know

$$\frac{(\alpha \cdot \kappa(\mathcal{S}_k) + (1-\alpha)\cdot k)^2}{\kappa(\mathcal{S}_k) - k} \leq \gamma_\alpha(\mathcal{S}_k) \leq (k+\alpha)^2.$$

Note that $\kappa(A_k) = k+1$, $\kappa(D_k) = k+2$ and $\kappa(E_k) = k+2$, which in particular gives the result for $\mathcal{S}_k = A_k$. Moreover, it shows that for $\mathcal{S}_k = D_k$ or $\mathcal{S}_k = E_k$ we have

$$\gamma_\alpha(\mathcal{S}_k) \geq \frac{(k+2\alpha)^2}{2}.$$

If we fix a complete intersection ideal I with $I^{ea}(f) \subseteq I$, then

$$\lambda_\alpha(f;I,g) = \frac{\big(\alpha \cdot i(f,g) + (1-\alpha)\cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)},$$

with $g \in I$ such that $i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$, considered as a function in $i(f,g)$ is maximal, when $i(f,g)$ is minimal. If $i(f,g) - \dim_{\mathbb{C}}(R/I) \geq 2$, then

$$\lambda_\alpha(f;I,g) \leq \frac{(k+2\alpha)^2}{2}.$$

It therefore remains to consider the case where

$$i(f,g) - \dim_{\mathbb{C}}(R/I) = 1 \tag{1.1}$$

for some I and some $g \in I$, and to maximise the possible $\dim_{\mathbb{C}}(R/I)$.
We claim that for $\mathcal{S}_k = D_k$ with $f = x^2y - y^{k-1}$ as representative, $\dim_{\mathbb{C}}(R/I) \leq k-2$, and thus $I = \langle x, y^{k-2}\rangle$ and $g = x$ are suitable with

$$\lambda_\alpha(f;I,x) = (k-2+\alpha)^2,$$

which is greater than $\frac{(k+2\alpha)^2}{2}$ if and only if $k \geq 4 + \sqrt{2}\cdot(2+\alpha)$. Suppose, therefore, $\dim_{\mathbb{C}}(R/I) = k-1$. Then $y^{k-1}, x^3 \in I^{ea}(f) = \langle xy, x^2 - (k-1)\cdot y^{k-2}\rangle \subset I$, the leading ideal $L_{<_{ls}}\big(I^{ea}(f)\big) = \langle x^3, xy, y^{k-2}\rangle \subset L_{<_{ls}}(I)$, and since by Proposition 18 $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_{<_{ls}}(I)\big)$, either $L_{<_{ls}}(I) = \langle x^3, xy, y^{k-3}\rangle$ or $L_{<_{ls}}(I) = \langle x^2, xy, y^{k-2}\rangle$. In the first case there is a power series $g \in I$ such that $g \equiv y^{k-3} + ax + bx^2 \pmod{I}$, and hence $I \ni yg \equiv y^{k-2} \pmod{I}$, i. e. $y^{k-2} \in I$. But then $x^2 \in I$ and $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case, similarly, there is a $g \in I$ such that $g \equiv x^2 \pmod{I}$, and hence $x^2 \in I$ which in turn implies that $y^{k-2} \in I$. Thus $I = \langle x^2, xy, y^{k-2}\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ which by Remark 25 contradicts the fact that I is a complete intersection.
If $\mathcal{S}_k = E_6$, then $f = x^3 - y^4$ is a representative and $I^{ea}(f) = \langle x^2, y^3\rangle$. Suppose that $\dim_{\mathbb{C}}(R/I) = k-1 = 5$, then $L_{<_{ds}}(I) = \langle x^2, y^3, xy^2\rangle$ and $H^0_{R/I} = H^0_{R/L_{<_{ds}}(I)}$, in contradiction to Lemma 24, since $H^0_{R/L_{<_{ds}}(I)}(2) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(3) = 0$. Thus $\dim_{\mathbb{C}}(R/I) \leq 4$ and $\lambda_\alpha(f;I,g) \leq (4+\alpha)^2 \leq \frac{(6+2\alpha)^2}{2}$.

If $\mathcal{S}_k = E_7$, then $f = x^3 - xy^3$ is a representative and $I^{ea}(f) = \langle 3x^2 - y^3, xy^2\rangle \ni x^3, y^5$. If $\dim_{\mathbb{C}}(R/I) \leq 4$, then $\lambda_\alpha(f; I, g) \leq (4+\alpha)^2 \leq \frac{(7+2\alpha)^2}{2}$, and we are done. It thus remains to exclude the cases where $\dim_{\mathbb{C}}(R/I) \in \{5, 6\}$. For this we note first that if there is a $g \in I$ such that $L_{<_{ls}}(g) = y^2$, then

$$g \equiv y^2 + ax + bx^2 + cxy + dx^2y \pmod{I}, \tag{1.2}$$

and therefore $y^2 g \equiv y^4 \pmod{I}$, which implies $y^4 \in I$ and hence $x^2y \in I$. Analogously, if there is a $g \in I$ such that $L_{<_{ls}}(g) = x^2y$, then $g \equiv x^2y \pmod{I}$ and again $x^2y, y^4 \in I$. Suppose now that $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^2, x^3\rangle$ or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2y, x^3\rangle$. In both cases we thus have $x^2y, y^4 \in I$. However, in the first case then $x^2y \in L_{<_{ls}}(I)$, in contradiction to the assumption. While in the second case we find $I = \langle xy^2, x^2y, 3x^2 - y^3\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts the fact that I is a complete intersection by Lemma 25. Suppose, therefore, that $\dim_{\mathbb{C}}(R/I) = 5$. Then $L_{<_{ls}}(I) = \langle y^2, x^2y, x^3\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy^2, x^2\rangle$, or $L_{<_{ls}}(I) = \langle y^3, xy, x^3\rangle$. In the first case, we know already that $y^4, x^2y \in I$. Looking once more on (1.2) we consider the cases $a = 0$ and $a \neq 0$. If $a = 0$, then $yg \equiv y^3 \pmod{I}$, and thus $y^3 \in I$, which in turn implies $x^2 \in I$. Similarly, if $a \neq 0$, then $xg \equiv ax^2 \pmod{I}$ implies $x^2 \in I$. But then also $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case there is a $g \in I$ such that $g \equiv x^2 + ax^2y \pmod{I}$, and thus $yg \equiv x^2y \in I$. But then also $x^2 \in I$ and $y^3 \in I$, so that $I = \langle y^3, xy^2, x^2\rangle$. However, $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ contradicts again the fact that I is a complete intersection. Finally in the third case there is a $g \in I$ with $g \equiv xy + ax^2 + bx^2y \pmod{I}$, and thus $xg \equiv x^2y \pmod{I}$ implies $x^2y \in I$ and then $xy + ax^2 \in I$. Therefore, $I = \langle xy + ax^2, 3x^2 - y^3\rangle$, and for for $h \in I$ and for generic $b, c \in \mathbb{C}$ we have $i(f, h) \geq i(x, h) + i\big(x^2 - y^3, b\cdot(xy + ax^2) + c\cdot(3x^2 - y^3)\big) \geq 3 + 5 = 8$, in contradiction to (1.1).

Finally, if $\mathcal{S}_k = E_8$ with representative $f = x^3 - y^5$ and $I^{ea}(f) = \langle x^2, y^4\rangle$, we get for $\dim_{\mathbb{C}}(R/I) \leq 5$ that $\lambda_\alpha(f; I, g) \leq (5+\alpha)^2 \leq \frac{(8+2\alpha)^2}{2}$. It therefore remains to exclude the cases $\dim_{\mathbb{C}}(R/I) \in \{6, 7\}$. If $\dim_{\mathbb{C}}(R/I) = 7$ then $L_{<_{ds}}(I) = \langle x^2, y^4, xy^3\rangle$. But then $H^0_{R/L_{<_{ds}}(I)}(3) = 2$ and $H^0_{R/L_{<_{ds}}(I)}(4) = 0$ are in contradiction to Lemma 24. And if $\dim_{\mathbb{C}}(R/I) = 6$, then $L_{<_{ls}}(I) = \langle y^3, x^2\rangle$ or $L_{<_{ls}}(I) = \langle y^4, xy^2, x^2\rangle$. In the first case there is some $g \in I$ such that $g \equiv y^3 + ax + bxy + cxy^2 + dxy^3 \pmod{I}$, and thus $xg \equiv xy^3 \pmod{I}$ and $xy^3 \in I$. But then $yg \equiv axy + bxy^2 \pmod{I}$ and hence $axy + bxy^2 \in I$. Since neither $xy \in L_{<_{ls}}(I)$ nor $xy^2 \in L_{<_{ls}}(I)$, we must have $a = 0 = b$. Therefore, $g \equiv y^3 + cxy^2 \pmod{I}$ and $I = \langle x^2, y^3 + cxy^2\rangle$, which for $h \in I$ and $a, b \in \mathbb{C}$ generic gives $i(f, g) \geq i\big(x^3 - y^4, ax^2 + b\cdot(y^3 + cxy^2)\big) \geq 8$, in contradiction to (1.1). In the second case, there is $g \in I$ such that $g \equiv xy^2 + axy^3 \pmod{I}$, therefore $yg \equiv xy^3 \pmod{I}$ and $xy^3 \in I$. But then $xy^2 \in I$ and $I = \langle y^4, xy^2, x^2\rangle$. This, however, is not a complete intersection, since $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$, in contradiction to the assumption.

This finishes the proof. □

Proposition 12 ((Ordinary Multiple Points))
Let α be a rational number with $0 \leq \alpha \leq 1$, and let M_k denote the topological singularity type of an ordinary k-fold point with $k \geq 3$. Then

$$\gamma_\alpha^{es}(M_k) = 2\cdot(k - 1 + \alpha)^2.$$

In particular

$$\gamma_\alpha^{es}(M_k) > (1+\alpha)^2 \cdot \tau_{ci}^{es}(M_k).$$

Proof: Note that for any representative f of M_k we have

$$I^{es}(f) = I^{ea}(f) + \mathfrak{m}^k = \left\langle \frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y} \right\rangle + \mathfrak{m}^k,$$

where f_k is the homogeneous part of degree k of f, so that we may assume f to be homogeneous of degree k.
If I is a complete intersection ideal with $\mathfrak{m}^k \subset I^{es}(f) \subseteq I$, then by Lemma 28

$$\dim_{\mathbb{C}}(R/I) \leq \big(k - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

We note moreover that for any $g \in I$

$$i(f,g) \geq \mathrm{mult}(f) \cdot \mathrm{mult}(g) \geq k \cdot \mathrm{mult}(I),$$

and that for a fixed I we may attain an upper bound for $\lambda_\alpha(f;I,g)$ by replacing $i(f,g)$ by a lower bound for $i(f,g)$.
Hence, if $\mathrm{mult}(I) \geq 2$, we have

$$\lambda_\alpha(f;I,g) \leq \frac{\big(k-(1-\alpha)\cdot(\mathrm{mult}(I)-1)\big)^2 \cdot \mathrm{mult}(I)^2}{\mathrm{mult}(I)\cdot\big(\mathrm{mult}(I)-1\big)} \leq 2\cdot(k-1+\alpha)^2, \quad (1.3)$$

while $\dim_{\mathbb{C}}(R/I) \leq k-1$ for $\mathrm{mult}(I)=1$ and the above inequality (1.3) is still satisfied. To see $\dim_{\mathbb{C}}(R/I) \leq k-1$ for $\mathrm{mult}(I)=1$ note that the ideal I contains an element g of order 1 with $g_1 = ax+by$ as homogeneous part of degree 1 and the partial derivatives of f; applying a linear change of coordinates we may assume $g_1 = x$ and $f = \prod_{i=1}^k (x - a_i y)$ with pairwise different a_i, and we may consider the negative degree lexicographical monomial ordering $>$ giving preference to y; if some $a_i = 0$, then $L_>\big(\frac{\partial f}{\partial x}\big) = y^{k-1}$, while otherwise $L_>\big(\frac{\partial f}{\partial y}\big) = y^{k-1}$, so that in any case $\langle x, y^{k-1}\rangle \subseteq L_>(I)$, and by Proposition 18 therefore $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_>(I)\big) \leq \dim_{\mathbb{C}}(R/\langle x,y^{k-1}\rangle) = k-1$.
Equation (1.3) together with Lemma 28 shows

$$\gamma_\alpha^{es}(M_k) \leq 2\cdot(k-1+\alpha)^2.$$

On the other hand, considering the representative $f = x^k - y^k$, we have

$$I^{es}(f) = \langle x^{k-1}, y^{k-1}, x^a y^b \mid a+b=k\rangle,$$

and $I = \langle y^{k-1}, x^2\rangle$ is a complete intersection ideal containing $I^{es}(f)$. Moreover, $i\big(f,x^2\big) = 2k$, $\dim_{\mathbb{C}}(R/I) = 2\cdot(k-1)$, thus

$$\gamma_\alpha^{es}(M_k) \geq \frac{\big(\alpha\cdot i(f,x^2) + (1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{i\big(f,x^2\big) - \dim_{\mathbb{C}}(R/I)} = 2\cdot(k-1+\alpha)^2.$$

The "in particular" part then follows right away from Corollary 29. □

Since a convenient semi-quasihomogeneous power series of multiplicity 2 defines an A_k-singularity and one with a homogeneous leading term defines an ordinary multiple point, the following proposition together with the previous two gives upper bounds for all singularities defined by a convenient semi-quasihomogeneous representative.

Proposition 13 ((Semiquasihomogeneous Singularities))
Let $\mathcal{S}_{p,q}$ be a singularity type with a convenient semi-quasihomogeneous representative $f \in R$, $q > p \geq 3$.
Then $\gamma_\alpha^{es}(\mathcal{S}_{p,q}) \geq \frac{\left(q-(1-\alpha)\cdot\left\lfloor\frac{q}{p}\right\rfloor\right)^2}{\left\lfloor\frac{q}{p}\right\rfloor} \geq \frac{q\cdot(p-1+\alpha)^2}{p}$ and we obtain the following upper bound for $\gamma_\alpha^{es}(f)$:

p, q	$\gamma_\alpha^{es}(f)$
$q \geq 39$	$\leq 3\cdot(q-2+\alpha)^2$
$\frac{q}{p} \in (1,2)$	$\leq 3\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [2,4)$	$\leq 2\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [4,\infty)$	$\leq (q-1+\alpha)^2$

Proof: To see the claimed lower bound for $\gamma_\alpha^{es}(\mathcal{S}_{p,q})$ recall that (see [GLS05])

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, x^\alpha y^\beta \;\middle|\; \alpha p + \beta q \geq pq \right\rangle. \tag{1.4}$$

In particular, $I^{es}(f) \subseteq \left\langle y, x^{q-\lfloor\frac{q}{p}\rfloor}\right\rangle$, $\dim_{\mathbb{C}}(R/I) = q - \left\lfloor\frac{q}{p}\right\rfloor$ and $i(f,y) = q$, which implies the claim.
Let now I be a complete intersection ideal with $I^{es}(f) \subseteq I$. Applying Lemma 28 and $\mathrm{d}(I) \leq q$, we first of all note that

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2\cdot(q+1)^2}{4} \leq 2\cdot(q-1+\alpha)^2.$$

Moreover, if $\frac{q}{p} \geq 3$, then

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2\cdot\left(q^2+4q+3\right)}{6} \leq (q-1+\alpha)^2.$$

since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}}\left(R/I^{es}(f)\right) \leq \frac{(p+1)\cdot(q+1)}{2}$ by (1.4).
It therefore suffices to show

$$\lambda_\alpha(f;I,g) \leq \begin{cases} 3\cdot(q-2+\alpha)^2, & \text{if } q \geq 39,\\ 3\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p} \in (1,2),\\ 2\cdot(q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [2,4),\\ (q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [4,\infty), \end{cases} \tag{1.5}$$

where $g \in I$ with $i(f,g) \leq 2\cdot\dim_{\mathbb{C}}(R/I)$. Recall that

$$\lambda_\alpha(f;I,g) = \frac{\left(\alpha\cdot i(f,g) + (1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\right)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)}.$$

Fixing I and considering $\lambda_\alpha(f;I,g)$ as a function in $i(f,g)$, where due to (1.12) the latter takes values between $\dim_{\mathbb{C}}(R/I)+1$ and $2\cdot\dim_{\mathbb{C}}(R/I)$, we note that the function is monotonously decreasing. In order to calculate an upper bound for $\lambda_\alpha(f;I,g)$ we may therefore replace $i(f,g)$ by some lower bound, which still exceeds $\dim_{\mathbb{C}}(R/I)+1$. Having done this we may then replace $\dim_{\mathbb{C}}(R/I)$ by an upper bound in order to find an upper bound for $\lambda(f;I,g)$.

Note that for $q \geq 39$ we have

$$\frac{54}{19} \cdot (q-1+\alpha)^2 \leq 3 \cdot (q-2+\alpha)^2. \tag{1.6}$$

Fix I and g, and let $L_{(p,q)}(g) = x^A y^B$ be the leading term of g w. r. t. the weighted ordering $<_{(p,q)}$ (see Definition 16). By Remark 32 we know

$$i(f,g) \geq Ap + Bq. \tag{1.7}$$

Working with this lower bound for $i(f,g)$ we reduce the problem to find suitable upper bounds for $\dim_{\mathbb{C}}(R/I)$. For this purpose we may assume that $L_{(p,q)}(g)$ is minimal, and thus, in particular, $B \leq \operatorname{mult}(I)$.
If $A = 0$, in view of Remark 26 we therefore have

$$B = \operatorname{mult}(I) \leq \frac{\mathrm{d}(I)+1}{2} \leq \frac{q+1}{2},$$

and thus by Lemma 28 then

$$\dim_{\mathbb{C}}(R/I) \leq B \cdot (q - B + 1). \tag{1.8}$$

Moreover, for $A = 0$ Lemma 34 applies with $h = g$ and we get

$$\dim_{\mathbb{C}}(R/I) \leq B \cdot q - 1 - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor \leq B \cdot q - 1 - \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B \cdot (B-1)}{2}. \tag{1.9}$$

Since $x^\alpha y^\beta \in I$ for $\alpha p + \beta q \geq pq$, we may assume $Ap + Bq \leq pq$. But then, since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}} R/\big\langle \frac{\partial f}{\partial y}, g, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \big\rangle$, we may apply Lemma 35 with $h = \frac{\partial f}{\partial y}$ and $C = p - 1$. This gives

$$\dim_{\mathbb{C}}(R/I) \leq Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \min\left\{A, \left\lceil \tfrac{q}{p} \right\rceil \right\}, \tag{1.10}$$

and if $B = 0$ we get in addition

$$\dim_{\mathbb{C}}(R/I) \leq A \cdot (p-1). \tag{1.11}$$

Finally note that by Lemma 3

$$i(f,g) > \dim_{\mathbb{C}}(R/I). \tag{1.12}$$

Let us now use the inequalities (1.6)-(1.12) to show (1.5). For this we have to consider several cases for possible values of A and B.

Case 1: $A = 0$, $B \geq 1$.
If $B = 1$, then by (1.9) and (1.12) we have $\lambda_\alpha(f; I, g) \leq (q-1+\alpha)^2$.
We may thus assume that $B \geq 2$. By (1.7) and (1.8)

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha) \cdot (B-1)\big)^2}{B \cdot (B-1)} \leq 2 \cdot (q-1+\alpha)^2.$$

If, moreover, $\frac{q}{p} \geq 3$, then we may apply (1.9) to find

$$\lambda_\alpha(f; I, g) \leq \frac{B^2 \cdot \big(q - (1-\alpha) \cdot (B-1)\big)^2}{\left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B \cdot (B-1)}{2} + 1} \leq (q-1+\alpha)^2.$$

Taking (1.6) into account, this proves (1.5) in the case $A = 0$ and $B \geq 1$.

Case 2: $A = 1$, $B \geq 1$.

From (1.10) we deduce

$$\dim_{\mathbb{C}}(R/I) \leq B\cdot(q-1)+(p-1)-\left\lfloor\tfrac{q}{p}\right\rfloor\cdot\tfrac{B\cdot(B-1)}{2}.$$

Since $\frac{p-1+\alpha}{q-1+\alpha}\leq\frac{p}{q}$ we thus get

$$\lambda_\alpha(f;I,g)\leq\frac{\left(B+\frac{p-1+\alpha}{q-1+\alpha}\right)^2}{B+\left\lfloor\frac{q}{p}\right\rfloor\cdot\frac{B\cdot(B-1)}{2}+1}\cdot(q-1+\alpha)^2$$

$$\leq\begin{cases}\frac{(B+\frac{1}{3})^2}{\frac{3B^2}{2}-\frac{B}{2}+1}\cdot(q-1+\alpha)^2 & \leq(q-1+\alpha)^2, \quad \text{if } \frac{q}{p}\geq 3,\\ \frac{(B+\frac{1}{2})^2}{B^2+1}\cdot(q-1+\alpha)^2 & \leq\frac{5}{4}\cdot(q-1+\alpha)^2, \quad \text{if } \frac{q}{p}\geq 2,\\ 2\cdot\frac{(B+1)^2}{B^2+B+2}\cdot(q-1+\alpha)^2 & \leq\frac{16}{7}\cdot(q-1+\alpha)^2, \quad \text{if } \frac{q}{p}>1.\end{cases}$$

Once more we are done, since $\frac{16}{7}\leq\frac{54}{19}$.

Case 3: $A\geq 2$, $B\geq 1$.

Note that $\lfloor r\rfloor\geq r-1$ for any rational number r, and set $s=\frac{q}{p}$, then by (1.10)

$$\dim_{\mathbb{C}}(R/I)\leq Ap+Bq-(A-1)\cdot(B-1)-\frac{A\cdot(A-1)}{2s}-\frac{s\cdot B\cdot(B-1)}{2}-1-\min\left\{A,\lceil s\rceil\right\}.$$

This amounts to

$$\lambda_\alpha(f;I,g)\leq$$

$$\frac{\Big(Ap+Bq-(1-\alpha)\cdot\big((A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+1+\min\{A,\lceil s\rceil\}\big)\Big)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}$$

$$\leq\frac{\big(A\cdot(p-1+\alpha)+B\cdot(q-1+\alpha)\big)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}\leq\varphi(A,B)\cdot(q-1+\alpha)^2,$$

where

$$\varphi(A,B)=\frac{\left(\frac{A}{s}+B\right)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}.$$

For the last inequality we just note again that $\frac{p-1+\alpha}{q-1+\alpha}\leq\frac{p}{q}=\frac{1}{s}$, while for the second inequality a number of different cases has to be considered. We postpone this for a moment.

In order to show (1.5) in the case $A\geq 2$ and $B\geq 1$ it now suffices to show

$$\varphi(A,B)\leq\begin{cases}\frac{54}{19}, & \text{if } s\geq 1,\\ 2, & \text{if } s\geq 2,\\ 1, & \text{if } s\geq 4.\end{cases}\tag{1.13}$$

Elementary calculus shows that for $B\geq 1$ fixed the function $[2,\infty)\to\mathbb{R}:A\mapsto\varphi(A,B)$ takes its maximum at

$$A=\max\left\{2,\frac{16-3B}{2+\frac{1}{s}}\right\}.$$

If $B\leq 3$, then the maximum is attained at $A=\frac{16-3B}{2+\frac{1}{s}}$, and

$$\varphi(A,B)\leq\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)=\frac{8sB-8B+64}{4s^2B-4s^2-4sB+28s-1}.$$

Again elementary calculus shows that the function $B\mapsto\varphi\left(\frac{16-3B}{2+\frac{1}{s}},B\right)$ is monotonously decreasing on $[1,3]$ and, therefore,

$$\varphi(A,B)\leq\varphi\left(\frac{13}{2+\frac{1}{s}},1\right)=\frac{8s+56}{24s-1}=:\psi_1(s).$$

Since also the function ψ_1 is monotonously decreasing on $[1,\infty)$ and $\psi_1(1)=\frac{64}{23}\leq\frac{54}{19}$, $\psi_1(2)=\frac{72}{47}\leq 2$ and $\psi_1(4)=\frac{88}{95}\leq 1$ Equation (1.13) follows in this case.
As soon as $B\geq 4$ the maximum for $\varphi(A,B)$ is attained for $A=2$ and

$$\varphi(A,B)\leq\varphi(2,B)=\frac{2\cdot(sB+2)^2}{s^3B^2-s^3B+2s^2B+4s^2+2s}.$$

Once more elementary calculus shows that the function $B\mapsto\varphi(2,B)$ is monotonously decreasing on $[4,\infty)$. Thus

$$\varphi(A,B)\leq\varphi(2,4)=\frac{4\cdot(1+2s)^2}{6s^3+6s^2+s}=:\psi_2(s).$$

Applying elementary calculus again, we find that the function ψ_2 is monotonously decreasing on $[1,\infty)$, so that we are done since $\psi_2(1)=\frac{36}{13}\leq\frac{54}{19}$, $\psi_2(2)=\frac{50}{37}\leq 2$ and $\psi_2(4)=\frac{81}{121}\leq 1$.
Let us now come back to proving the missing inequality above. We have to show

$$A+B\leq(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+1+\min\big\{A,\lceil s\rceil\big\},$$

or equivalently

$$\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+2+\min\big\{A,\lceil s\rceil\big\}+AB-2A-2B\geq 0.$$

If $B\geq 2$, then $AB\geq 2A$ and $\frac{s\cdot B\cdot(B-1)}{2}+2+\min\big\{A,\lceil s\rceil\big\}\geq 2B$, so we are done. It remains to consider the case $B=1$, and we have to show

$$A^2-A-2sA+2s\cdot\min\big\{A,\lceil s\rceil\big\}\geq 0.$$

If $A\leq\lceil s\rceil$ or $A=2$ this is obvious. We may thus suppose that $A>\lceil s\rceil$ and $A\geq 3$. Since $\frac{A^2}{3}\geq A$ it remains to show

$$\frac{2A^2}{3}-2sA+2s\cdot\lceil s\rceil\geq 0.$$

For this

$$\frac{2A^2}{3}-2sA+2s\cdot\lceil s\rceil\geq\begin{cases}\frac{2A^2}{3}-2sA & \geq 0, \quad \text{if } A\geq 3s,\\ \frac{2A^2}{3}-\frac{4sA}{3} & \geq 0, \quad \text{if } 2s\leq A\leq 3s,\\ \frac{2A^2}{3}-sA & \geq 0, \quad \text{if } \frac{3s}{2}\leq A\leq 2s,\\ \frac{2A^2}{3}-\frac{2sA}{3} & \geq 0, \quad \text{if } \lceil s\rceil\leq A\leq\frac{3s}{2}.\end{cases}$$

Case 4: $A\geq 1$, $B=0$.

Applying (1.10) and (1.11) we get

$$\lambda_\alpha(f;I,g) \leq \begin{cases} \frac{A^2\cdot(p-1+\alpha)^2}{A} \leq \begin{Bmatrix} \frac{A}{s^2}\cdot(q-1+\alpha)^2 \\ A\cdot(q-2+\alpha)^2 \end{Bmatrix} & \text{for any } A, \quad \text{and} \\ \frac{A^2\cdot(p-1+\alpha)^2}{\sum_{i=1}^{A-1}\lfloor\frac{pi}{q}\rfloor+\min\{A,\lceil\frac{q}{p}\rceil\}} \leq \varphi_{\nu,s}(A)\cdot(q-1+\alpha)^2, & \text{if } A\geq 3, \end{cases}$$

where

$$\varphi_{\nu,s}(A) = \frac{\frac{A^2}{s^2}}{\frac{A\cdot(A-1)}{2s}-(A-1)+\nu} = \frac{2A^2}{sA^2-(2s^2+s)\cdot A+2\cdot(\nu+1)\cdot s^2}$$

with $\nu=2$ for $s\in(1,2]$ and $\nu=3$ for $s\in(2,\infty)$.
In particular, due to the first two inequalities we may thus assume that

$$A > \begin{cases} 3, & \text{if } q\geq 39, \\ 3s^2, & \text{if } s\in(1,2), \\ 2s^2, & \text{if } s\in[2,4), \\ s^2, & \text{if } s\in[4,\infty). \end{cases}$$

Note that $\varphi_{3,s}(A)\leq 1$ for $s\geq 4$, since

$$A\geq s^2 = \frac{9s^2}{16}+\frac{7s^2}{16} \geq \frac{s\cdot(1+2s)}{2\cdot(s-2)}+\frac{s}{s-2}\cdot\sqrt{s^2-3s+\tfrac{33}{4}}.$$

This gives (1.5) for $s\geq 4$.
If now $s\in(2,4)$, then $\varphi_{3,s}$ is monotonously decreasing on $\big[2s^2,\infty\big)$, as is $s\mapsto\varphi_{3,s}\big(2s^2\big)$ on $[2,4)$, and thus

$$\varphi_{3,s}(A)\leq\varphi_{3,s}\big(2s^2\big) = \frac{4s^2}{2s^3-2s^2-s+4} \leq \frac{8}{5}\leq 2,$$

while for $s=2$ the function $\varphi_{2,2}$ is monotonously decreasing on $[8,\infty)$ and thus $\varphi_{2,2}(A)\leq\frac{16}{9}\leq 2$. This finishes the case $s\in[2,4)$.
Let's now consider the case $s\in(1,2)$ and $q\geq 39$ parallel. Applying elementary calculus, we find that $\varphi_{2,s}$ takes its maximum on $[3,\infty)$ at $A=\frac{12s}{1+2s}$ and is monotonously decreasing on $\big[\frac{12s}{1+2s},\infty\big)$. Moreover, the function $s\mapsto\varphi_{2,s}\big(\frac{12s}{1+2s}\big)$ is monotonously decreasing on $(1,2)$. If $s\geq\frac{7}{6}$, then

$$\varphi_{2,s}(A)\leq\varphi_{2,s}\big(\tfrac{12s}{1+2s}\big)\leq\varphi_{2,\frac{7}{6}}\big(\tfrac{21}{5}\big) = \frac{54}{19}.$$

Due to (1.6) it thus remains to consider the case $s\in\big(1,\frac{7}{6}\big)$ and $A>3$. If $A\geq 8$, then

$$\varphi_{2,s}(A)\leq\varphi_{2,1}(8) = \frac{64}{23}\leq\frac{54}{19},$$

since the function $s\mapsto\varphi_{2,s}(8)$ is monotonously decreasing on $[1,2)$.
So, we are finally stuck with the case $A\in\{4,5,6,7\}$ and $1\leq\frac{q}{p}=s\leq\frac{7}{6}$. We want to apply Lemma 28. For this we note first that by Lemma 36 in our situation $\mathrm{d}(I)\leq p+1$ and $A=\mathrm{mult}(I)\leq\frac{p+2}{2}$. But then

$$\dim_{\mathbb{C}}(R/I)\leq A\cdot(p-A+2)$$

and thus,

$$\lambda_\alpha(f;I,g) \leq \frac{A^2\cdot\big(p-(1-\alpha)\cdot(A-2)\big)^2}{A\cdot(A-2)} \leq \frac{A}{(A-2)}\cdot(q-2+\alpha)^2 \leq 2\cdot(q-2+\alpha)^2.$$

This finishes the proof. □

Remark 14

In the proof of the previous proposition we achieved for almost all cases $\lambda_\alpha(f;I,g) \leq \frac{54}{19}\cdot(q-1+\alpha)^2$, apart from the single case $L_{<_{(p,q)}}(g) = x^3$. The following example shows that indeed in this case we cannot, in general, expect any better coefficient than 3. More precisely, the example shows that the bound

$$3\cdot(q-2+\alpha)^2$$

is sharp for the family of singularities given by $x^q - y^{q-1}$, $q \geq 39$. A closer investigation should allow to lower the bound on q, but we cannot get this for all $q \geq 4$, as the example of E_6 and E_8 show.

Moreover, we give series of examples for which the bound $(q-1+\alpha)^2$ is sharp, respectively for which $2\cdot(q-1+\alpha)^2$ is a lower bound.

Example 15

Throughout these examples $q > p \geq 3$ are integers.

(a) Let $f = x^q - y^{q-1}$, then $\gamma_\alpha^{es}(f) \geq 3\cdot(q-2+\alpha)^2$. In particular, for $q \geq 39$,

$$\gamma_\alpha^{es}(f) = 3\cdot(q-2+\alpha)^2.$$

For this we note that $I = \langle x^3, y^{q-2}\rangle$ is a complete intersection ideal in R with $I^{es}(f) = \big\langle x^{q-1}, y^{q-2}, x^\alpha y^\beta \;\big|\; \alpha\cdot(q-1)+\beta q \geq q\cdot(q-1)\big\rangle \subseteq I$, since $2\cdot(q-1)+(q-3)\cdot q = q^2-q-2 < q\cdot(q-1)$ and thus $x^2y^{q-3} \notin I^{es}(f)$. This also shows that the monomial x^iy^j with $0 \leq i \leq 2$ and $0 \leq j \leq q-3$ form a $\mathbb{C}$-basis of R/I, so that $\dim_{\mathbb{C}}(R/I) = 3q-6$. Since $i\big(f,x^3\big) = 3q-3$, the claim follows.

(b) Let $\frac{q}{p} < 2$ and $f = x^q - y^p$, then

$$\gamma_\alpha^{es}(f) \geq 2\cdot(q-1+\alpha)^2.$$

By the assumption on p and q we have $(q-2)\cdot p + q < pq$ and hence $x^{q-2}y \notin I^{es}(f)$. Thus $I^{es}(f) = \big\langle x^{q-1}, y^{p-1}, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq\big\rangle \subseteq I = \langle y^2, x^{q-1}\rangle$, and we are done since $\dim_{\mathbb{C}}(R/I) = 2q-2$ and $i\big(f,y^2\big) = 2q$.

(c) Let $f \in R$ be convenient, semi-quasihomogeneous of $\operatorname{ord}_{(p,q)}(f) = pq$, and suppose that in f no monomial x^ky, $k \leq q-2$, occurs (e. g. $f = x^q - y^p$), then $\gamma_\alpha^{es}(f) \geq (q-1+\alpha)^2$. In particular, if $\frac{q}{p} \geq 4$, then

$$\gamma_\alpha^{es}(f) = (q-1+\alpha)^2.$$

By the assumption, $I^{es}(f) \subseteq I = \langle x^{q-1}, y\rangle$, since $\frac{\partial f}{\partial x} \equiv x^{q-1}\cdot u(x) \pmod{y}$ for a unit u and $\frac{\partial f}{\partial y} \equiv 0 \big(\mathrm{mod}\ \langle y, x^{q-1}\rangle\big)$. Hence we are done since $\dim_{\mathbb{C}}(R/I) = q-1$ and $i(f,y) = q$.

(d) Let $f = y^3 - 3x^8y + 3x^{12}$, then f does not satisfy the assumptions of (c), but still $\gamma_\alpha^{es}(f) = (11+\alpha)^2 = (q-1+\alpha)^2$.

For this note that $I = \langle y - x^4, x^{11}\rangle$ contains $I^{es}(f)$, $\dim_{\mathbb{C}}(R/I) = 11$ and $i\big(f, y-x^4\big) = 12$.

(e) Let $f = 7y^3 + 15x^7 - 21x^5y$, then f is semi-quasihomogeneous with weights $(p, q) = (3, 7)$ and convenient, but $\gamma_0^{es}(f) \leq 25 < 36 = (q-1)^2$. This shows that $(q-1)^2$ is not a general lower bound for $\gamma_0^{es}(\mathcal{S}_{p,q})$.

We note first that $I^{es}(f) = \langle x^7, y^2 - x^5, x^6 - x^4y\rangle$ is not a complete intersection and $\dim_{\mathbb{C}}\big(R/I^{es}(f)\big) = 11$. Let now I be a complete intersection ideal with $I^{es}(f) \subset I$ and let $h \in I$ such that $L_{<_{(3,7)}}(h) = x^Ay^B$ is minimal, in particular, $\operatorname{ord}_{(3,7)}(h) = 3A + 7B$ is minimal. Then $\dim_{\mathbb{C}}(R/I) \leq 10$ and $i(f, g) \geq 3A + 7B$ for all $g \in I$.

If, therefore, $3A + 7B \geq 14$, then

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 25.$$

We may thus assume that $3A + 7B \leq 13$, in particular $B < 2$. If $B = 0$, and hence $A \leq 4$, then by Lemma 35 $\dim_{\mathbb{C}}(R/I) \leq 2A$, so that

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq 4A \leq 16.$$

Similarly, if $B = 1$ and $A = 2$, then by the same Lemma $\dim_{\mathbb{C}}(R/I) \leq 9$ and $i(f, g) \geq 13$, so that

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{81}{4}.$$

So it remains to consider the case $B = 1$ and $A \in \{0, 1\}$. That is $h = x^Ay + h'$ with $\operatorname{ord}_{(3,7)}(h') \geq 9 + 3A$. Consider the ideal $J = \big\langle x^\alpha y^\beta \;\big|\; 3\alpha + 7\beta \geq 21\big\rangle \subseteq I$. Then $x^{4-A} \cdot h \equiv x^4y \pmod{J}$, and thus $x^6 - x^4y \equiv x^6 \pmod{\langle h\rangle + J}$, i. e. $\langle h, x^6 - x^4y\rangle + J = \langle h, x^6\rangle + J$. Moreover, $x^6 \notin \langle h\rangle + J$, so that $\dim_{\mathbb{C}}\big(R/\langle g, x^6 - x^4y\rangle + J\big) \leq 6 + A$. If we can show that $\langle g, x^6 - x^4y\rangle + J \subsetneq I$, then

$$\frac{\dim_{\mathbb{C}}(R/I)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)} \leq \frac{(5+A)^2}{3A + 7 - 5 - A} \leq \frac{25}{2}.$$

We are therefore done, once we know that $y^2 - x^5 \notin \langle g, x^6\rangle + J$. Suppose there was a g such that $gh = y^2 - x^5 \,\big(\mathrm{mod}\ \langle x^6\rangle + J\big)$. Then $y^2 = L_{<_{(3,7)}}(g) \cdot L_{<_{(3,7)}}(h)$, which in particular means $A = 0$ and $L_{<_{(3,7)}}(h) = L_{<_{(3,7)}}(g) = y$. But then the coefficients of 1, x and x^2 in h and g must be zero, so that x^5 cannot occur with a non-zero coefficient in the product. This gives the desired contradiction.

2. Local Monomial Orderings

Throughout the proofs of the auxilary statements in Section 4 we make use of some results from computer algebra concerning properties of local monomial orderings. In this section we recall the relevant definitions and results.

Definition 16

A *monomial ordering* is a total ordering $<$ on the set of monomials $\big\{x^\alpha y^\beta \;\big|\; \alpha, \beta \geq 0\big\}$ such that for all $\alpha, \beta, \gamma, \delta, \mu, \nu \geq 0$

$$x^\alpha y^\beta < x^\gamma y^\delta \quad \Longrightarrow \quad x^{\alpha+\mu} y^{\beta+\nu} < x^{\gamma+\mu} y^{\delta+\nu}.$$

A monomial ordering $<$ is called *local* if $1 > x^\alpha y^\beta$ for all $(\alpha, \beta) \neq (0,0)$, and it is a local *degree ordering* if

$$\alpha+\beta > \gamma+\delta \quad \Longrightarrow \quad x^\alpha y^\beta < x^\gamma y^\delta.$$

Finally, if $<$ is any local monomial ordering, then we define the *leading monomial* $L_<(f)$ with respect to $<$ of a non-zero power series $f \in R$ to be the maximal monomial $x^\alpha y^\beta$ such that the coefficient of $x^\alpha y^\beta$ in f does not vanish. For $f = 0$, we set $L_<(f) := 0$.
If $I \trianglelefteq R$ is an ideal in R, then $L_<(I) = \langle L_<(f) \mid f \in I\rangle$ is called its *leading ideal.*

We will give now some examples of local monomial orderings which are used in the proofs.

Example 17
Let $\alpha, \beta, \gamma, \delta \geq 0$ be integers.

(a) The *negative lexicographical ordering* $<_{ls}$ is defined by the relation

$$x^\alpha y^\beta <_{ls} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha > \gamma \text{ or } (\alpha = \gamma \text{ and } \beta > \delta).$$

(b) The *negative degree reverse lexicographical ordering* $<_{ds}$ is defined by the relation

$$x^\alpha y^\beta <_{ds} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha+\beta > \gamma+\delta \text{ or } (\alpha+\beta = \gamma+\delta \text{ and } \beta > \delta).$$

(c) If positive integers p and q are given, then we define the *local weighted degree ordering* $<_{(p,q)}$ with weights (p, q) by the relation

$$x^\alpha y^\beta <_{(p,q)} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \begin{array}{l} \alpha p+\beta q > \gamma p+\delta q \text{ or} \\ (\alpha p+\beta q = \gamma p+\delta q \text{ and } \beta < \delta). \end{array}$$

We note that $<_{ds}$ is a local degree ordering, while $<_{ls}$ is not and $<_{(p,q)}$ is if and only if $p = q$.

Let us finally recall some useful properties of local orderings (see e. g. [GrP02] Corollary 7.5.6 and Proposition 5.5.7).

Proposition 18
Let $<$ be any local monomial ordering, and let I be a zero-dimensional ideal in R.

(a) *The monomials of $R/L_<(I)$ form a $\mathbb{C}$-basis of R/I. In particular*

$$\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big).$$

(b) *If $<$ is a degree ordering, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide (see Definition 19, and see also Remark 21).*

3. The Hilbert Samuel Function

A useful tool in the study of the degree of zero-dimensional schemes and their subschemes is the Hilbert Samuel function of the structure sheaf, that is of the corresponding Artinian ring.

Definition 19
Let $I \triangleleft R$ be a zero-dimensional ideal.

(a) The function

$$H^1_{R/I}:\mathbb{Z}\to\mathbb{Z}:d\mapsto\begin{cases}\dim_{\mathbb{C}}\big(R/(I+\mathfrak{m}^{d+1})\big), & d\geq 0,\\ 0, & d<0,\end{cases}$$

is called the *Hilbert Samuel function* of R/I.

(b) We define the *slope* of the Hilbert Samuel function of R/I to be the function

$$H^0_{R/I}:\mathbb{N}\to\mathbb{N}:d\mapsto H^1_{R/I}(d)-H^1_{R/I}(d-1).$$

Thus

$$H^0_{R/I}(d)=\dim_{\mathbb{C}}\big(\mathfrak{m}^d/((I\cap\mathfrak{m}^d)+\mathfrak{m}^{d+1})\big),$$

is just the number $d+1$ of linearly independent monomials of degree d in $\mathfrak{m}^d$, minus the number of linearly independent monomials of degree d in $\big(I\cap\mathfrak{m}^d\big)+\mathfrak{m}^{d+1}$.

Note that if $\overline{\mathfrak{m}}=\mathfrak{m}/I$ denotes the maximal ideal of R/I and $\mathrm{Gr}_{\mathfrak{m}}(R/I)=\bigoplus_{d\geq 0}\overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}$ the associated graded ring, then

$$H^0_{R/I}(d)=\dim_{\mathbb{C}}\big(\overline{\mathfrak{m}}^d/\overline{\mathfrak{m}}^{d+1}\big)$$

is just the dimension of the graded piece of degree d of $\mathrm{Gr}_{\mathfrak{m}}(R/I)$.

(c) Finally, we define the *multiplicity* of I to be

$$\mathrm{mult}(I):=\min\big\{\,\mathrm{mult}(f)\;\big|\;0\neq f\in I\big\},$$

and the *degree bound* of I as

$$\mathrm{d}(I):=\min\big\{d\in\mathbb{N}\;\big|\;\mathfrak{m}^d\subseteq I\big\}.$$

Let us gather some straight forward properties of the slope of the Hilbert Samuel function.

Lemma 20

Let $J\subseteq I\lhd R$ be zero-dimensional ideals.

(a) $H^0_{R/I}(d)=d+1$ *for all* $0\leq d<\mathrm{mult}(I)$.
(b) $H^0_{R/I}(d)\leq H^0_{R/I}(d-1)$ *for all* $d\geq\mathrm{mult}(I)$.
(c) $H^0_{R/I}(d)\leq\mathrm{mult}(I)$.
(d) $H^0_{R/I}(d)=0$ *for all* $d\geq\mathrm{d}(I)$ *and* $H^0_{R/I}\neq 0$ *for all* $d<\mathrm{d}(I)$. *In particular*

$$\dim_{\mathbb{C}}(R/I)=\sum_{d=0}^{\mathrm{d}(I)-1}H^0_{R/I}(d).$$

(e) $H^0_{R/I}(d)\leq H^0_{R/J}(d)$ *for all* $d\in\mathbb{N}$.
(f) $\mathrm{d}(I)$ *and* $\mathrm{mult}(I)$ *are completely determined by* $H^0_{R/I}$.

Proof: For (a) we note that $I\subseteq\mathfrak{m}^d$ for all $d\leq\mathrm{mult}(I)$ and thus $H^0_{R/I}(d)=\dim_{\mathbb{C}}\big(\mathfrak{m}^d/\mathfrak{m}^{d+1}\big)=d+1$ for all $0\leq d<\mathrm{mult}(I)$.

By definition we see that $H^0_{R/I}(d)$ is just the number of linearly independent monomials of degree d in $\mathfrak{m}^d$, which is $d+1$, minus the number of linearly independent monomials, say $m_1,\ldots,m_r$, of degree d in $\big(I\cap\mathfrak{m}^d\big)+\mathfrak{m}^{d+1}$. We note that then the set

$$\{xm_1,\ldots,xm_r,ym_1,\ldots,ym_r\}\subseteq\mathfrak{m}\cdot\big((I\cap\mathfrak{m}^d)+\mathfrak{m}^{d+1}\big)\subseteq\big(I\cap\mathfrak{m}^{d+1}\big)+\mathfrak{m}^{d+2}$$

contains at least $r+1$ linearly independent monomials of degree $d+1$, once r was non-zero. However, for $d=\text{mult}(I)$ and $g=g_d+h.o.t\in I$ with homogeneous part $g_d\neq 0$ of degree d, we have $g_d\in\big(I\cap\mathfrak{m}^d\big)+\mathfrak{m}^{d+1}$, that is, $d=\text{mult}(I)$ is the smallest integer d for which there is a monomial of degree d in $\big(I\cap\mathfrak{m}^d\big)+\mathfrak{m}^{d+1}$. Thus for $d\geq\text{mult}(I)-1$

$$H^0_{R/I}(d+1)\leq(d+2)-(r+1)=d+1-r=H^0_{R/I}(d),$$

which proves (b), while (c) is an immediate consequence of (a) and (b).
If $d\geq\text{d}(I)$, then $H^1_{R/I}(d)=\dim_{\mathbb{C}}(R/I)$ is independent of d, and hence $H^0_{R/I}(d)=0$ for all $d\geq\text{d}(I)$. In particular,

$$\sum_{i=0}^{\text{d}(I)-1}H^0_{R/I}(d)=H^1_{R/I}(\text{d}(I)-1)-H^1_{R/I}(-1)=\dim_{\mathbb{C}}(R/I).$$

Moreover, $\mathfrak{m}^{\text{d}(I)-1}+I\neq I=I+\mathfrak{m}^{\text{d}(I)}$, so that $H^0_{R/I}\big(\text{d}(I)-1\big)\neq 0$, and by (b) then $H^0_{R/I}(d)\neq 0$ for all $d<\text{d}(I)$. This proves (d), and (e) and (f) are obvious. □

Remark 21
Let $<$ be a local degree ordering on R, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide by Proposition 18, and hence we have as well

$$H^0_{R/I}=H^0_{R/L_<(I)},\quad \text{d}(I)=\text{d}\left(L_<(I)\right),\quad\text{and}\quad\text{mult}(I)=\text{mult}\left(L_<(I)\right),$$

since by the previous lemma the multiplicity and the degree bound only depend on the slope of the Hilbert Samuel function.

Remark 22
The slope of the Hilbert Samuel function of R/I gives rise to a histogram as the graph of the function $H^0_{R/I}$. By the Lemma 20 we know that up to $\text{mult}(I)-1$ the histogram is just a staircase with steps of height one, and from $\text{mult}(I)-1$ on it can only go down, which it eventually will do until it reaches the value zero for $d=\text{d}(I)$. This means that we get a histogram of form shown in Figure 1.

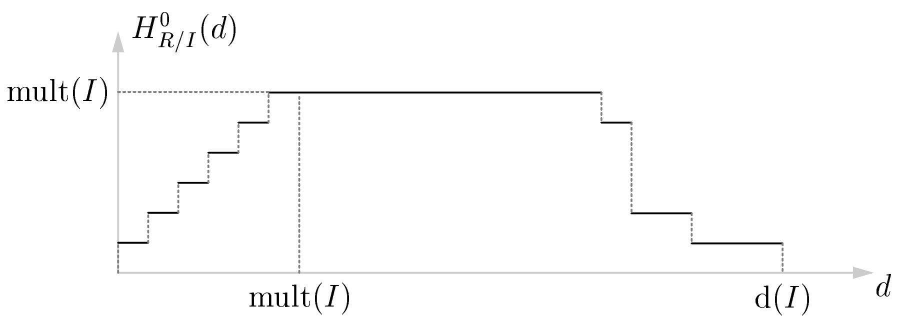

FIGURE 1. The histogram of $H^0_{R/I}$ for a general ideal I.

Note also, that by Lemma 20 (a) the area of the histogram is just $\dim_{\mathbb{C}}(R/I)$!

Example 23
In order to understand the slope of the Hilbert Samuel function better, let us consider some examples.

(a) Let $f=x^2-y^{k+1}$, $k\geq 1$, and let $I=I^{ea}(f)=\langle x,y^k\rangle$ the equisingularity ideal of an A_k-singularity. Then $\text{d}(I)=k$, $\text{mult}(I)=1$ and $\dim_{\mathbb{C}}(R/I)=k$.

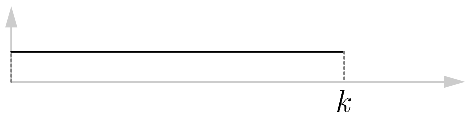

FIGURE 2. The histogram of $H^0_{R/I}$ for an A_k-singularity

(b) Let $f = x^2y - y^{k-1}$, $k \geq 4$, and let $I = I^{ea}(f) = \langle xy, x^2 - (k-1)\cdot y^{k-2}\rangle$ the equisingularity ideal of a D_k-singularity. Then $x^3, xy, y^{k-1} \in I$, and thus $\mathfrak{m}^{k-1} \subset I$, which gives $\mathrm{d}(I) = k-1$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = k$, which shows that the bound in Lemma 28 need not be obtained.

FIGURE 3. The histogram of $H^0_{R/I}$ for a D_k-singularity

(c) Let $f = x^3 - y^4$ and let $I = I^{ea}(f) = \langle x^2, y^3\rangle$ the equisingularity ideal of an E_6-singularity. Then $\mathrm{d}(I) = 4$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 6$.

Let $f = x^3 - xy^3$ and let $I = I^{ea}(f) = \langle 3x^2 - y^3, xy^2\rangle$ the equisingularity ideal of an E_7-singularity. Then $x^3, xy^2, y^5 \in I$, and thus $\mathfrak{m}^5 \subset I$, which gives $\mathrm{d}(I) = 5$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 7$.

Let $f = x^3 - y^5$ and let $I = I^{ea}(f) = \langle x^2, y^4\rangle$ the equisingularity ideal of an E_8-singularity. Then $\mathrm{d}(I) = 6$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 8$.

FIGURE 4. The histogram of $H^0_{R/I}$ for E_6, E_7 and E_8.

(d) Let $I = \langle x^3, x^2y, y^3\rangle$, then $\mathrm{d}(I) = 4$, $\mathrm{mult}(I) = 3$ and $\dim_{\mathbb{C}}(R/I) = 7$.

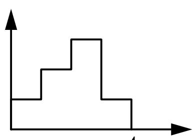

FIGURE 5. The histogram of $H^0_{R/I}$ for $I = \langle x^3, x^2y, y^3\rangle$.

The following result providing a lower bound for the minimal number of generators of a zero-dimensional ideal in R is due to A. Iarrobino.

Lemma 24

Let $I \lhd R$ be a zero-dimensional ideal. Then I cannot be generated by less than $1 + \sup\left\{H^0_{R/I}(d-1) - H^0_{R/I}(d) \;\middle|\; d \geq \mathrm{mult}(I)\right\}$ elements.

In particular, if I is a complete intersection ideal then for $d \geq \mathrm{mult}(I)$

$$H^0_{R/I}(d-1) - 1 \leq H^0_{R/I}(d) \leq H^0_{R/I}(d-1).$$

Proof: See [Iar77] Theorem 4.3 or [Bri77] Proposition III.2.1. □

Moreover, by the Lemma of Nakayama and Proposition 18 we can compute the minimal number of generators for a zero-dimensional ideal exactly.

Lemma 25
Let $I \lhd R$ be zero-dimensional ideal and let $<$ denote any local ordering on R. Then the minimal number of generators of I is

$$\dim_{\mathbb{C}}(I/\mathfrak{m}I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big) - \dim_{\mathbb{C}}\big(R/L_<(\mathfrak{m}I)\big).$$

Remark 26
If we apply Lemma 24 to a zero-dimensional complete intersection ideal $I \lhd R$, i. e. a zero-dimensional ideal generated by two elements, then we know that the histogram of $H^0_{R/I}$ will be as shown in Figure 6; that is, up to the value $d = \mathrm{mult}(I)$ the

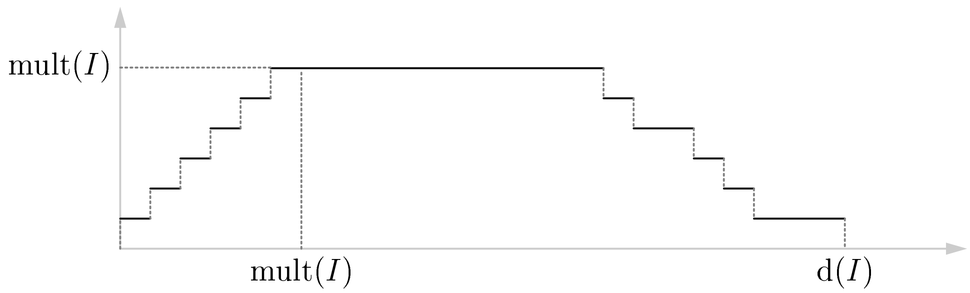

FIGURE 6. The histogram of $H^0_{R/I}$ for a complete intersection.

histogram of $H^0_{R/I}$ is an ascending staircase with steps of height and length one, then it remains constant for a while, and finally it is a descending staircase again with steps of height one, but a possibly longer length. In particular we see that

$$\mathrm{mult}(I) \leq \begin{cases} \frac{\mathrm{d}(I)+1}{2}, & \text{if } \mathrm{d}(I) \text{ is odd,} \\ \frac{\mathrm{d}(I)}{2}, & \text{if } \mathrm{d}(I) \text{ is even.} \end{cases} \tag{3.1}$$

Example 27
Let $I = \mathfrak{m}^k$ for $k \geq 1$. Then $\mathrm{d}(I) = \mathrm{mult}(I) = k$ and $\dim_{\mathbb{C}}(R/I) = \binom{k+1}{2}$.

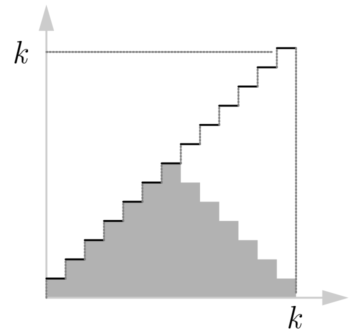

FIGURE 7. The histogram of $H^0_{R/\mathfrak{m}^k}$. The shaded region is the maximal possible value of $\dim_{\mathbb{C}}(R/I)$ for a complete intersection ideal I containing $\mathfrak{m}^k$.

Lemma 28
Let $I \lhd R$ be a zero-dimensional complete intersection ideal, then

$$\dim_{\mathbb{C}}(R/I) \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

In particular

$$\dim_{\mathbb{C}}(R/I) \leq \begin{cases} \frac{(\mathrm{d}(I)+1)^2}{4}, & \textit{if } \mathrm{d}(I) \textit{ odd,} \\ \frac{\mathrm{d}(I)^2+2\,\mathrm{d}(I)}{4}, & \textit{if } \mathrm{d}(I) \textit{ even.} \end{cases}$$

Proof: By Remark 22 we have to find an upper bound for the area A of the histogram of $H^0_{R/I}$. This area would be maximal, if in the descending part the steps had all length one, i. e. if the histogram was as shown in Figure 8. Since the two

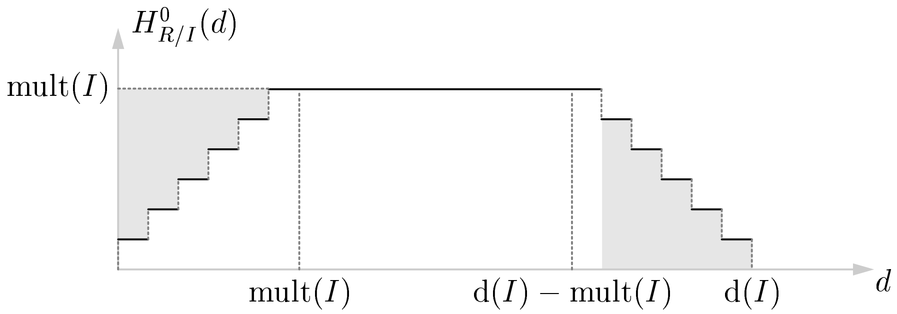

FIGURE 8. Maximal possible area.

shaded regions have the same area, we get

$$A \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

Consider now the function

$$\varphi : \left[\mathrm{mult}(I), \tfrac{\mathrm{d}(I)+1}{2}\right] \longrightarrow \mathbb{R} : x \mapsto \big(\mathrm{d}(I) - x + 1\big) \cdot x,$$

then this function is monotonously increasing, which finishes the proof in view of Equation (3.1). □

Corollary 29
For an ordinary m-fold point M_m we have

$$\tau^{es}_{ci}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \textit{if } m \geq 3 \textit{ odd,} \\ \frac{m^2+2m}{4}, & \textit{if } m \geq 4 \textit{ even,} \\ 1, & \textit{if } m = 2. \end{cases}$$

Proof: Let f be a representative of M_m. Then

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle + \mathfrak{m}^m,$$

and as in the proof of Proposition 12 we may assume that f is a homogeneous of degree m.
In particular, if $m = 2$, then $I^{es}(f) = \mathfrak{m}$ is a complete intersection and $\tau^{es}_{ci}(M_2) = 1$. We may therefore assume that $m \geq 3$.

For any complete intersection ideal I with $\mathfrak{m}^m \subset I^{es}(f) \subseteq I$ we automatically have $\mathrm{d}(I) \leq m$, and by Lemma 28

$$\tau_{ci}^{es}(f) \leq \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \text{ odd,} \\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even.} \end{cases}$$

Consider now the representative $f = x^m - y^m$. If $m = 2k$ is even, then the ideal $I = \langle x^k, y^{k+1} \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau_{ci}^{es}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 + k = \frac{m^2 + 2m}{4}.$$

Similarly, if $m = 2k - 1$ is odd, then the ideal $I = \langle x^k, y^k \rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau_{ci}^{es}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 = \frac{m^2 + 2m + 1}{4}.$$

□

Remark 30
Let $I \lhd R$ be any zero-dimensional ideal, not necessarily a complete intersection, then still

$$\dim_{\mathbb{C}}(R/I) \leq \left(\mathrm{d}(I) - \frac{\mathrm{mult}(I) - 1}{2} \right) \cdot \mathrm{mult}(I).$$

Proof: The proof is the same as for the complete intersection ideal, just that we cannot ensure that the histogram goes down to zero at $\mathrm{d}(I)$ with steps of size one. The dimension is thus bounded by the region of the histogram in Figure 9. □

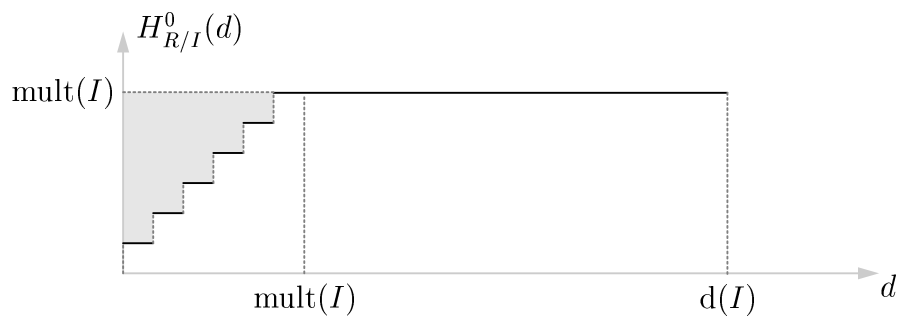

FIGURE 9. Maximal possible area.

4. SEMI-QUASIHOMOGENEOUS SINGULARITIES

Definition 31
A non-zero polynomial of the form $f = \sum_{\alpha \cdot p + \beta \cdot q = d} a_{\alpha,\beta} x^\alpha y^\beta$ is called *quasihomogeneous* of (p,q)-degree d. Thus the Newton polygon of a quasihomogeneous polynomial has just one side of slope $-\frac{p}{q}$.
A quasihomogeneous polynomial is said to be *non-degenerate* if it is reduced, that is if it has no multiple factors, and it is said to be *convenient* if $\frac{d}{p}, \frac{d}{q} \in \mathbb{Z}$ and $a_{\frac{d}{p},0}$ and $a_{0,\frac{d}{q}}$ are non-zero, that is if the Newton polygon meets the x-axis and the y-axis.
If $f = f_0 + f_1$ with f_0 quasihomogeneous of (p,q)-degree d and for any monomial $x^\alpha y^\beta$ occurring in f_1 with a non-zero coefficient we have $\alpha \cdot p + \beta \cdot q > d$, we say

that f is of (p,q)-*order* d, and we call f_0 the (p,q)-*leading form* of f and denote it by $\operatorname{lead}_{(p,q)}(f)$. We denote the (p,q)-order of f by $\operatorname{ord}_{(p,q)}(f)$.
A power series $f \in R$ is said to be *semi-quasihomogeneous* with respect to the weights (p,q) if the (p,q)-leading form is non-degenerate.

Remark 32
Let $f \in R$ with $\deg_{(p,q)}(f) = pq$ and let f_0 denote its (p,q)-leading form.

(a) If $\gcd(p,q) = r$, then f_0 has r factors of the form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$, $i = 1, \ldots, r$.
If, moreover, f_0 is non-degenerate, then these will all be irreducible and pairwise different, i. e. not scalar multiples of each other.

(b) If f is irreducible, then f_0 has only one irreducible factor, possibly of higher multiplicity.

(c) If f_0 is non-degenerate, then f has $r = \gcd(p,q)$ branches $f_1, \ldots, f_r$, which are all semi-quasihomogeneous with irreducible (p,q)-leading form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ for pairwise distinct points $(a_i : b_i) \in \mathbb{P}^1_{\mathbb{C}}$, $i = 1, \ldots, r$.
The characteristic exponents of f_i are $\frac{q}{r}$ and $\frac{p}{r}$ for all $i = 1, \ldots, r$, and thus f_i admits a parametrisation of the form

$$\big(x_i(t), y_i(t)\big) = \Big(\alpha_i t^{\frac{p}{r}} + h.o.t, \beta_i t^{\frac{q}{r}} + h.o.t\Big).$$

(d) If f_0 is non-degenerate, i. e. f is semi-quasihomogeneous, and $g \in R$, then

$$i(f,g) \geq \operatorname{ord}_{(p,q)}(g).$$

Proof:

(a) If $\alpha p + \beta q = pq$, then $p \mid \beta q$ and hence $p \mid \beta r$, so that $\beta \cdot \frac{r}{p}$ is a natural number. Similarly $\alpha \cdot \frac{r}{q}$ is a natural number. We may therefore consider the transformation

$$f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big) \in \mathbb{C}[x,y]_r$$

which is a homogeneous polynomial of degree r. Thus $f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big)$ factors in r linear factors $a_i x - b_i y$, $i = 1, \ldots, r$, so that f_0 factors as

$$f_0 = \prod_{i=1}^{r} \big(a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}\big). \tag{4.1}$$

Since $\gcd\left(\frac{p}{r}, \frac{q}{r}\right) = 1$, the factors $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ are irreducible once neither a_i nor b_i is zero.
If f_0 is non-degenerate, then the irreducible factors of f_0 are pairwise distinct. So, $a_i = 0$ implies $r = p$ and still $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = b_i y$ irreducible, while $b_i = 0$ similarly gives $r = q$ and $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = a_i x$ irreducible. Thus, in any case the factors in (4.1) are irreducible and, hence, pairwise distinct.

(b) With the notation from Lemma 33 and the factorisation of f_0 from (4.1) we get

$$g = \frac{\prod_{i=1}^{r} a_i u^{\frac{bq}{r}} v^{\frac{pq}{r^2}} - b_i u^{\frac{ap}{r}} v^{\frac{pq}{r^2}}}{u^{ap} v^{\frac{pq}{r}}} = \prod_{i=1}^{r} (a_i u - b_i).$$

By assumption f is irreducible, hence according to Lemma 33 g has at most one, possibly repeated, zero. But thus the factors of f_0 all coincide – up to scalar multiple.

(c) The first assertion is an immediate consequence from (a) and (b), while the "in particular" part follows by Puiseux expansion.

(d) Let g_0 be the (p,q)-leading form of g. Using the notation from (c) we have

$$i(f,g)=\sum_{i=1}^{r} i(f_i,g)=\sum_{i=1}^{r}\operatorname{ord}\big(g(x_i(t),y_i(t))\big)$$
$$=\sum_{i=1}^{r}\operatorname{ord}\Big(g_0\big(\alpha_i t^{\frac{p}{r}},\beta_i t^{\frac{q}{r}}\big)+h.o.t\Big)\geq\sum_{i=1}^{r}\frac{\operatorname{ord}_{(p,q)}(g)}{r}=\operatorname{ord}_{(p,q)}(g).$$

□

Lemma 33

Let $f\in R$ with $\operatorname{ord}_{(p,q)}(f)=pq$ and let f_0 denote its (p,q)-leading form. Let $r=\gcd(p,q)$ and $a,b\geq 0$ such that $qb-pa=r$. Finally set

$$g=\frac{f_0\big(u^b v^{\frac{p}{r}},u^a v^{\frac{q}{r}}\big)}{u^{ap}v^{\frac{pq}{r}}}\in\mathbb{C}[u].$$

Then the number of different zeros of g is a lower bound for the number of branches of f.

Proof: See [BrK86] Remark on p. 480. □

The following investigations are crucial for the proof of Proposition 13.

Lemma 34

Let $f\in R$ be convenient semi-quasihomogeneous with leading form f_0 and $\operatorname{ord}_{(p,q)}(f)=pq$, let $I=\big\langle x^\alpha y^\beta\;\big|\;\alpha p+\beta q\geq pq\big\rangle$, and let $h\in R$. Then

$$\dim_{\mathbb{C}} R/\big(\langle h\rangle+I^{es}(f)\big)<\dim_{\mathbb{C}} R/\big(\langle h\rangle+I\big).$$

In particular, if $L_{(p,q)}(h)=y^B$ with $B\leq p$, then

$$\dim_{\mathbb{C}} R/\langle h\rangle+I^{es}(f)\leq Bq-1-\sum_{i=1}^{B-1}\big\lfloor\tfrac{qi}{p}\big\rfloor.$$

Proof: As

$$I^{es}(f)=\big\langle\tfrac{\partial f}{\partial x},\tfrac{\partial f}{\partial y}\big\rangle+I,$$

it suffices to show that

$$I^{es}(f)\not\subseteq\langle h\rangle+I,$$

which is the same as showing that not both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ belong to $\langle h\rangle+I$. Suppose the contrary, that is, there are $h_x,h_y\in R$ such that

$$\tfrac{\partial f}{\partial x}\equiv h_x\cdot h \pmod I\qquad\text{and}\qquad\tfrac{\partial f}{\partial y}\equiv h_y\cdot h \pmod I.$$

We note that

$$\operatorname{lead}_{(p,q)}\big(\tfrac{\partial f}{\partial x}\big)=\tfrac{\partial f_0}{\partial x}\qquad\text{and}\qquad\operatorname{lead}_{(p,q)}\big(\tfrac{\partial f}{\partial y}\big)=\tfrac{\partial f_0}{\partial y},$$

and none of the monomials involved is contained in I. Therefore

$$\operatorname{lead}_{(p,q)}(h_x)\cdot\operatorname{lead}_{(p,q)}(h)=\tfrac{\partial f_0}{\partial x}\qquad\text{and}\qquad\operatorname{lead}_{(p,q)}(h_y)\cdot\operatorname{lead}_{(p,q)}(h)=\tfrac{\partial f_0}{\partial y},$$

which in particular implies that $\frac{\partial f_0}{\partial x}$ and $\frac{\partial f_0}{\partial y}$ have a common factor. This, however, is then a multiple factor of the quasihomogeneous polynomial f_0, in contradiction to f being semi-quasihomogeneous.

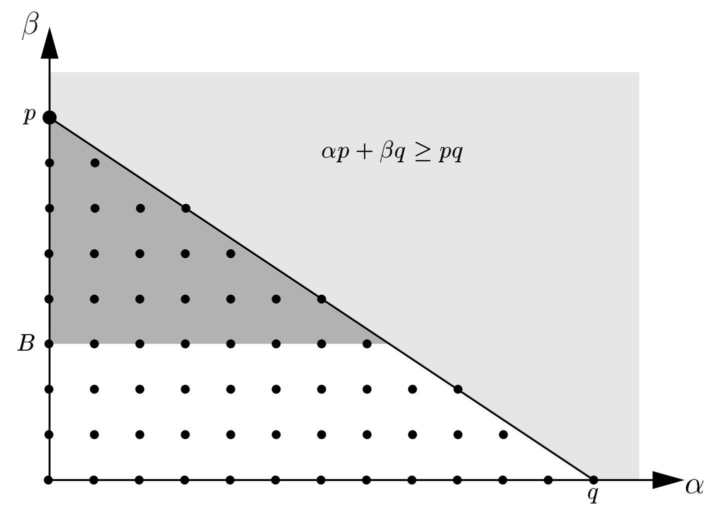

FIGURE 10. A Basis of $R/\langle h\rangle + I$.

For the "in particular" part, we note that by Proposition 18

$$\dim_{\mathbb{C}} R/\langle h\rangle + I = \dim_{\mathbb{C}} R/L_{<_{(p,q)}}\big(\langle h\rangle + I\big) \leq \dim_{\mathbb{C}} R/\big\langle y^B\big\rangle + I,$$

and the monomials $x^\alpha y^\beta$ with $\alpha p + \beta q < pq$ and $\beta < B$ form a $\mathbb{C}$-basis of the latter vector space (see also Figure 10). Hence,

$$\dim_{\mathbb{C}} R/\langle h\rangle + I \leq \sum_{i=0}^{B-1} \left\lceil q - \tfrac{qi}{p}\right\rceil = Bq - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p}\right\rfloor.$$

□

Lemma 35
Let $g, h \in R$ such that $L_{(p,q)}(g) = x^A y^B$ and $L_{(p,q)}(h) = y^C$, and consider the ideals $J = \big\langle x^A y^B, y^C, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq\big\rangle$ and $J' = \big\langle g, h, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq\big\rangle$. Then

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/J,$$

and if $Ap + Bq \leq pq$ and $B \leq C \leq p$, then

$$\dim_{\mathbb{C}} R/J = Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q}\right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p}\right\rfloor - \sum_{i=C}^{p-1} \min\Big\{A, \left\lceil q - \tfrac{Cq}{p}\right\rceil\Big\}.$$

Moreover, if $B = 0$, then $\dim_{\mathbb{C}} R/J \leq A \cdot C$.

Proof: By Proposition 18

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/L_{<_{(p,q)}}(J') \leq \dim_{\mathbb{C}} R/J.$$

Let $I = \big\langle x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq\big\rangle$. Then the monomials $x^\alpha y^\beta$ with $(\alpha,\beta) \in \Lambda = \big\{(\alpha,\beta) \in \mathbb{N}\times\mathbb{N} \;\big|\; \alpha p + \beta q < pq\big\}$ form a basis of R/I. Moreover, the monomials $x^\alpha y^\beta$ with $(\alpha,\beta) \in \Lambda_1 \cup \Lambda_2$ are a basis of J/I, where

$$\Lambda_1 = \big\{(\alpha,\beta) \in \Lambda \;\big|\; \alpha \geq A \text{ and } \beta \geq B\big\}$$

and

$$\Lambda_2 = \big\{(\alpha,\beta) \in \Lambda \setminus \Lambda_1 \;\big|\; \beta \geq C\big\}.$$

(See also Figure 11.) This gives rise to the above values for $\dim_{\mathbb{C}} R/J$.

□

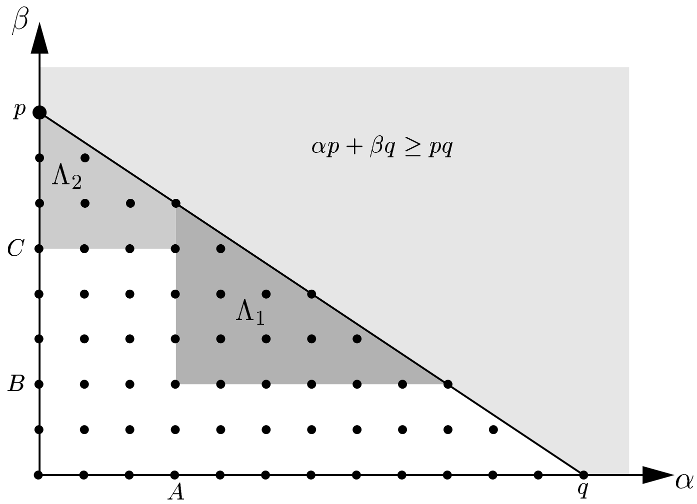

FIGURE 11. A Basis of R/J.

Lemma 36
Let $q > p$ be such that $\frac{q}{p} < \frac{d}{d-1}$ for some integer $d \geq 2$, and let $0 \leq A \leq d$.

(a) *If $L_{(p,q)}(g) = x^A$, then $L_{<_{ds}}(g) = x^A$.*

(b) $\mathfrak{m}^{p+1} \subseteq \langle x^A, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$.

(c) *If I is an ideal such that $g, h, x^\alpha y^\beta \in I$ for $\alpha p+\beta q \geq pq$ and where $L_{<_{(p,q)}}(g) = x^A$ and $L_{<_{(p,q)}}(h) = y^{p-1}$, then $\mathrm{d}(I) \leq p+1$.*

Moreover, if $L_{<_{(p,q)}}(g)$ is minimal among the leading monomials of elements in I w. r. t. $<_{(p,q)}$, then $\mathrm{mult}(I) = A$.

Proof: It suffices to consider the case $A = d$, since this implies the other cases. Note that by assumption $d \leq p$.

(a) Since x^d is less than any monomial of degree at least d with respect to $<_{ds}$, we have to show that in g no monomial of degree less than d can occur with a non-zero coefficient. x^d being the leading monomial of g with respect to $<_{(p,q)}$, it suffices to show that $\alpha+\beta < d$ implies $\alpha p+\beta q < dp$, or alternatively, since $\frac{q}{p} < \frac{d}{d-1}$,

$$\alpha + \beta \cdot \frac{d}{d-1} \leq d.$$

For $\alpha + \beta < d$ the left hand side of this inequality will be maximal for $\alpha = 0$ and $\beta = d-1$, and thus the inequality is satisfied.

(b) We only have to show that $x^\gamma y^{p+1-\gamma} \in \langle x^d, y^{p-1}, x^\alpha y^\beta \mid \alpha p + \beta q \geq pq \rangle$ for $\gamma = 3, \ldots, d-1$, since the remaining generators of $\mathfrak{m}^{p+1}$ definitely are. However, by assumption $\frac{q}{p} < \frac{d}{d-1} \leq \frac{\gamma}{\gamma-1}$, and thus $\gamma \cdot p + (p+1-\gamma) \cdot q \geq pq$.

(c) By the assumption on I we deduce form (a) and (b) that $\mathrm{d}\left(L_{<_{ds}}(I)\right) \leq p+1$. However, by Remark 21 $\mathrm{d}(I) = \mathrm{d}\left(L_{<_{ds}}(I)\right)$, which proves the first assertion.

Suppose now that $\mathrm{mult}(I) < A$, i. e. there is an $f \in I$ such that $\mathrm{mult}(f) \leq A-1$. The considerations for (a) show that then $L_{<_{(p,q)}}(f) < x^A$ in contradiction to the assumption.

□

References

[Bri77] Joel Briancon, *Description of* $\mathrm{Hilb}^n\,\mathbb{C}\{x,y\}$, Inv. math. **41** (1977), 45–89.

[BrK86] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser, 1986.

[GLS97] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices **13** (1997), 595–611.

[GLS00] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Castelnuovo function, zero-dimensional schemes, and singular plane curves*, J. Algebraic Geom. **9** (2000), no. 4, 663–710.

[GLS01] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *The variety of plane curves with ordinary singularities is not irreducible*, Intern. Math. Res. Notes **11** (2001), 542–550.

[GLS05] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Singular algebraic curves*, Springer, 2005.

[GrP02] Gert-Martin Greuel and Gerhard Pfister, *A* SINGULAR *introduction to commutative algebra*, Springer, 2002.

[Iar77] Anthony Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **10** (1977), no. 188, 1–97.

[Kei04] Thomas Keilen, *Smoothness of equisingular families of curves*, To appear in: Trans. Amer. Math. Soc. (2004), http://www.mathematik.uni-kl.de/~keilen/download/Keilen003/Keilen003.ps.gz.

[Shu91] Eugenii Shustin, *On manifolds of singular algebraic curves*, Selecta Math. Sov. **10** (1991), 27–37.

[Shu97] Eugenii Shustin, *Smoothness of equisingular families of plane algebraic curves*, Math. Res. Not. **2** (1997), 67–82.

[Wah74] Jonathan M. Wahl, *Equisingular deformations of plane algebroid curves*, Trans. Amer. Math. Soc. **193** (1974), 143–170.

UNIVERSITY OF WARWICK, MATHEMATICS INSTITUTE, COVENTRY CV4 7AL, UNITED KINGDOM

E-mail address: `keilen@mathematik.uni-kl.de`

URL: `http://www.mathematik.uni-kl.de/~keilen`

UNIVERSITÄT KAISERSLAUTERN, FACHBEREICH MATHEMATIK, ERWIN-SCHRÖDINGER-STRASSE, D – 67663 KAISERSLAUTERN, GERMANY

E-mail address: `lossen@mathematik.uni-kl.de`

URL: `http://www.mathematik.uni-kl.de/~lossen`