FB Mathematik — Winterterm 2004/05, Set 1

Gert-Martin Greuel — Thomas Markwig

# Algebraic Geometry I

Due date: Tuesday, 08/11/2005, 10:00 Uhr

**Exercise 1:** Let $K$ be any field, $I \trianglelefteq K[x_1,\ldots,x_n] = K[\underline{x}]$ an ideal.
Prove the following statements, or find counterexamples.

a. If $V(I) = \mathbb{A}^n(K)$, then $I = (0)$.

b. If $V(I) = \emptyset$, then $I = K[\underline{x}]$.

c. If $I$ is a prime ideal, then $V(I)$ is irreducible.*

Is it possible to weaken respectively to strengthen the hypotheses such that the results remain respectively become true?

**Exercise 2:** Let $K$ be any field, $I, J \trianglelefteq K[x_1,\ldots,x_n]$. Prove:

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

**Exercise 3:** Let $K$ be an infinite field, $V = V\big(y - x^2\big) \subset \mathbb{A}^2_K$ and $V' = V\big(xy - 1\big) \subset \mathbb{A}^2_K$.
Show that $K[V] := K[x,y]/I(V) \cong K[t]$ and $K\big[V'\big] := K[x,y]/I\big(V'\big) \cong K\big[t, \frac{1}{t}\big]$.

**Exercise 4:** With the aid of the ray tracer `surf` and the library `surf.lib` SINGULAR can draw curves in $A^2_{\mathbb{R}}$ and surfaces in $A^3_{\mathbb{R}}$. Once the library is loaded and the ring and the polynomial are defined, then the command `plot` may be used to draw the zero-set of the polynomial, e. g.

```
LIB "surf.lib";
ring r=0,(x,y,z),lp;
poly f=x2-yz;
plot(f);
```

Draw the curve given by $f = y^2 - x^2 - x^3$ and the surface given by $g = (x+y+z-1) \cdot (x-y-z-1) \cdot (y-x-z-1) \cdot (z-x-y-1) \cdot (x+y+z+1) \cdot (x-y-z+1) \cdot (y-x-z+1) \cdot (z-x-y+1) + \big(x^2+y^2+z^2-1\big) \cdot \big(x^2+y^2+z^2-1\big) \cdot \big(x^2+y^2+z^2-2\big) \cdot \big(x^2+y^2+z^2-2\big)$.†

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*A non-empty topological space $V$ is called *irreducible* if it is not the union of two proper subsets which are both closed.

†For $g$ use the command `plot(g,"scale_x=0.2;scale_y=0.2;");`.