FB Mathematik — Winterterm 2004/05, Set 2
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Algebraic Geometry I

Due date: Friday, 11/11/2005, 18:00 Uhr

Exercise 1: Let K be a field and

$$\begin{array}{ccc} K[x_1,\ldots,x_n] & \xrightarrow{\Phi} & K^K = \{f : K^n \to K \mid f \text{ is a map}\} \\ \Psi & & \Psi \\ f & \longmapsto & \big(\widetilde{f} : K^n \to K : (k_1,\ldots,k_n) \mapsto f(k_1,\ldots,k_n)\big) \end{array}$$

be the map which associates to a polynomial the corresponding polynomial map from K^n to K. Show that Φ is injective if and only if K is infinite.

Hint, for the interesting direction do induction on n.

Exercise 2: Let X be a topological space and $\emptyset \neq Y \subseteq X$.

a. If Y is irreducible, then the closure $\overline{Y}$ in X is irreducible.

b. If X is irreducible and Y is open in X, then Y is dense in X, i.e. $\overline{Y} = X$.

c. If Y is irreducible, there is a maximal irreducible subspace Y' in X containing Y, i.e. there is a $Y' \subseteq X$ irreducible such that $Y \subseteq Y'$ and for all irreducible $Y'' \subseteq X$ with $Y' \subseteq Y''$ we have $Y' = Y''$.

   We call these maximal irreducible subsets of X its *irreducible components*.

Exercise 3: Find a parametrisation of the curve $C = \{y^2 - x^2 - x^3 = 0\} \subset \mathbb{R}^2$, i.e. find a map

$$\varphi : \mathbb{A}^1_K \to \mathbb{A}^2_K : t \mapsto \big(f(t), g(t)\big)$$

whose image is C, where $f, g \in K[t]$ are polynomials.

Hint, draw the curve $C = \{y^2 - x^2 - x^3 = 0\}$ with Surf and you will find that it has a double point in $(x, y) = (0, 0)$. Then consider the lines in $\mathbb{R}^2$ through this point. Each such line cuts the curve in precisely one more point. Now consider the line L parallel to the y-axis through the point $(x, y) = (-1, 0)$. Each line through the origin also cuts L in precisely one point. That way you can define a map from L to C which is a parametrisation as those considered in Exercise 3.

Exercise 4: Consider the three plane curves C_i in $\mathbb{C}^2$ given by the equations $f_i = 0$, $i = 1, 2, 3$, where

$$f_1 = y^2 - 5x^2 - x^3, \quad f_2 = x^4 + y^4 - 2, \quad \text{respectively} \quad f_3 = y^2 + 5x^2 + x^3.$$

How many intersection points have C_1 and C_2 respectively C_1 and C_3? How many of these points are real? You may use SINGULAR for the calculations. Verify the real points by drawing the curves using Surf.

Hint, have a look at the SINGULAR example sing-02 at the web page of Prof. Greuel.