FB Mathematik Winterterm 2004/05, Set 3
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# Algebraic Geometry I

Due date: Friday, 18/11/2005, 18:00 Uhr

**Exercise 1:** Consider the variety $V = V(xy - 1) \subset \mathbb{A}^2_K$, the map $\varphi : V \to \mathbb{A}^1_K : (x, y) \mapsto x$, and the set $X = \operatorname{Im}(\varphi) \subset \mathbb{K}^1_K$. What is $I(X)$? Is $X = V\big(I(X)\big)$? Draw a schematic image for $K = \mathbb{R}$.

Note, your answer will depend on the field $K$! Consider the cases $K$ finite or infinite, or, if you prefer, $K = \mathbb{Z}/2\mathbb{Z}$ and $K = \mathbb{R}$.

**Exercise 2:**

a. Consider the surface $V(f) \subset \mathbb{C}^3$ defined by the polynomials $f = x^2 + y^2 + xyz$ and consider the planes $H_c = V(x - c) \subset \mathbb{C}^3$ for $c \in \mathbb{C}$ arbitrary. Determine $I(X_c)$ for $X_c = H_c \cap V(f)$.

b. Draw image of the surface $V(x^2 + y^2 + xyz) \subset \mathbb{R}^3$ and its intersecions with the planes $H_c = V(x - c)$ for $c \in \{-1, 0, 1\}$. You may use Surf.

Note that if you want to show that a certain ideal is a prime ideal it makes sense to consider the quotient ring by the ideal. – You may use Eisenstein's Criterion for irreducibility without proof. – As shown in the example class, if you call Surf from SINGULAR via `plot(f);` and you edit the script in Surf, then you may add at the end of the script `plane=x-1;cut_with_plane;` and execute the script once more. What you get is the surface and a curve, which is the intersection of the surface with the plane $V(x - 1)$. If you have problems with that, just come along and ask for help.

**Exercise 3:** Let $K = \overline{K}$ be an algebraically closed field and $I \trianglelefteq K[\underline{x}]$ an ideal. Use Hilbert's Nullstellensatz to show that:

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{m} \lhd \cdot K[\underline{x}]} \mathfrak{m},$$

where $\mathfrak{m} \lhd \cdot K[\underline{x}]$ means that $\mathfrak{m}$ is a maximal ideal in $K[\underline{x}]$. Interprete this equality geometrically!

**Exercise 4:** Let $I \trianglelefteq K[x_1, \ldots, x_n]$ and $G$ a Groebner basis of $I$ w.r.t. the lexicographical ordering. Show that $V(I)$ is a finite set if for $i = 1, \ldots, n$ there exists an $f \in G$ such that $\operatorname{lead}(f) = x_i^{n_i}$ for some $n_i \geq 0$.

**Proposition (Eisenstein's Criterion)**

*Let* $R$ *be an integral domain and* $0 \neq f = \sum_{i=0}^{n} a_i y^i \in R[y]$. *If there is a prime ideal* $P$ *such that* $a_0, \ldots, a_{n-1} \in P$, $a_n \notin P$ *and* $a_0 \notin P^2$, *then* $f$ *has no non-constant divisor in* $R[y]$ *of degree strictly less than* $\deg(f)$.

*In particular, if* $R$ *is a unique factorisation domain,* $p \in R$ *prime,* $\gcd(a_0, \ldots, a_n) = 1$, $p \mid a_0, \ldots, a_{n-1}$, $p^2 \not| a_0$, *then* $f$ *is irreducible in* $R[y]$.

**Beweis:** Let $f = g \cdot h$ for polynomials $g = \sum_{k=0}^{\mu} g_k y^k, h = \sum_{k=0}^{\nu} h_k y^k \in R[y]$. Since $R$ is an integral domain we know $\deg(g) + \deg(h) = \deg(f)$.

Suppose now that $0 < \deg(g) < \deg(f)$, then also $0 < \deg(h) < \deg(f)$. Consider the ringhomomorphism

$$R[y] \longrightarrow (R/P)[y] : F = \sum_{k=0}^{m} b_k y^k \mapsto \overline{F} = \sum_{k=0}^{m} \overline{b_k} y^k$$

induced by taking the coefficients modulo $P$. By assumption we have

$$\overline{a_n} \cdot y^n = \overline{f} = \overline{g} \cdot \overline{h} = \overline{g_\mu} \cdot \overline{h_\nu} \cdot y^n + \text{terms of lower degree.} \tag{1}$$

In particular $g_\mu, h_\nu \notin P$. Let $\mu'$ and $\nu'$ be minimal such that $g_{\mu'}, h_{\nu'} \notin P$ then

$$\overline{g} \cdot \overline{h} = \overline{g_{\mu'}} \cdot \overline{h_{\nu'}} \cdot y^{\mu'+\nu'} + \text{terms of higher degree,}$$

and since $P$ is prime thus $g_{\mu'} \cdot h_{\nu'} \notin P$, i.e. $\overline{g_{\mu'}} \cdot \overline{h_{\nu'}} \neq 0$. Taking (1) into account we deduce that $\mu' = \mu = \deg(g) > 0$ and $\nu' = \nu = \deg(h) > 0$. In particular, $g_0 \in P$ and $h_0 \in P$. But then

$$a_0 = g_0 \cdot h_0 \in P^2,$$

in contradiction to our assumption.

Let us now proof the in particular part. By the above proof we know that if $f = g \cdot h$ then we may assume that $\deg(g) = 0$ and $\deg(h) = \deg(f)$. However, if $\deg(g) = 0$ then $g$ is a constant and thus divides every coefficient of $f$. Thus by assumption it is a unit since $\gcd(a_0, \ldots, a_n) = 1$. This shows that $f$ is irreducible. □