FB Mathematik | Winterterm 2004/05, Set 4
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# Algebraic Geometry I

Due date: Friday, 25/11/2005, 18:00 Uhr

**Exercise 1:** Let $R$ be a ring, $I \trianglelefteq R$ an ideal in $R$. If $J \trianglelefteq R$ is an ideal of $R$ such that $I \subset J$, then obviously $J/I = \{\overline{a} \mid a \in J\}$ is an ideal of $R/I$. Show that

$$\{P \in \operatorname{Spec}(R) \mid I \subseteq P\} \longrightarrow \operatorname{Spec}(R/I) : P \mapsto P/I$$

is a bijection.
What is the relation between $\operatorname{Spec}(R/I)$ and $\operatorname{Spec}\left(R/\operatorname{rad}(I)\right)$?

Recall that $\operatorname{Spec}(S)$ is the set of prime ideals of a ring $S$.

**Exercise 2:** Let $K$ be an algebraically closed field and let $V \subseteq \mathbb{A}^n_K$ be an affine variety. Show that the following statements are equivalent:

a. $|V| = r < \infty$.

b. $I(V) = \bigcap_{i=1}^r \mathfrak{m}_i$, where the $\mathfrak{m}_i \subset K[x_1, \ldots, x_n]$ are maximal ideals.

c. $K[V] \cong K^r$, as a $K$-algebras.

d. $\dim_K \left(K[V]\right) = r < \infty$.

Hint, the Chinese Remainder Theorem might be useful!

**Exercise 3:** Let $X = V\left(yz^2 - yz, xz^2 - xz, xyz + xz - y^2z - yz, y^3z - yz\right) \subseteq \mathbb{C}^3$. Compute:

a. the irreducible componemts of the variety $X$,

b. $\dim(X)$, and

c. the dimension of each irreducible component of $X$,

without using a computer algebra system. Check your result by using SINGULAR. How many connected components does $X$ have? Draw by hand a picture of $X$.

**Exercise 4:** Is the following affine variety

$$V = V(xz - y^2, z - xy) \subseteq \mathbb{A}^3_{\mathbb{C}}$$

irreducible?