FB Mathematik
Gert-Martin Greuel

Winterterm 2004/05, Set 5
Thomas Markwig

Algebraic Geometry I

Due date: Friday, 02/12/2005, 18:00 Uhr

Exercise 1:

a. Let $X = V(I) \subseteq \mathbb{A}^n_K$ and suppose the $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$. Show that the map

$$\varphi : X \to \mathbb{A}^k_K : p \mapsto \big(f_1(p), \ldots, f_k(p)\big)$$

is continous with respect to the Zariski topology on X and on $\mathbb{A}^k_K$.

b. Consider the set

$$X = \big\{(s,t) \in \mathbb{A}^2_K \mid t = s^2, t-1 \neq 1\big\}.$$

For which of the fields $K \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$ is X an affine subvariety of $\mathbb{A}^2_K$?

Recall that a map is continous if the preimage of any closed set is closed.

Exercise 2: Let X be an affine variety, $X = X_1 \cup \cdots \cup X_r$ its decomposition into irreducible components, $U \subseteq X$ open, and $f \in A[X]$.

a. U is dense in X if and only if $U \cap X_i$ is dense in X_i for all i.

b. Show that the following are equivalent:

   (a) f is a zero-divisor in $A[X]$.

   (b) $D(f)$ is not dense in X.

   (c) There is an irreducible component Y of X such that $f_{|Y} \equiv 0$, i.e. $Y \subseteq V(f)$.

Exercise 3: Let X be an affine variety.

a. Show that each open subset is a finite union of principal open subsets.

b. Show that each subset of X is quasi-compact, i.e. each open covering contains a finite subcovering.

c. If $U \subset X$ is open and $\varphi \in \mathcal{O}_X(U)$, then there are $g_i, h_i \in A[X]$, $i = 1, \ldots, k$, such that $U = \bigcup_{i=1}^k D(h_i)$ and $\varphi_{|D(h_i)} \equiv \frac{g_i}{h_i}$.

Exercise 4: Let $X = V(xy - zw) \subseteq \mathbb{A}^4_K$, $U = X \setminus \big(V(y) \cap V(z)\big)$ and

$$\varphi : U \longrightarrow K : (x,y,z,w) \mapsto \begin{cases} \frac{x}{z}, & \text{if } z \neq 0, \\ \frac{w}{y}, & \text{if } z = 0. \end{cases}$$

Show that φ is regular on U, but that there are no polynomials $f, g \in K[x,y,z,w]$ such that $g(q) \neq 0$ for all $q \in U$ and $\varphi \equiv \frac{f}{g}$ on U.

Hint, suppose that $\varphi \equiv \frac{f}{g}$ on some open $V \subset U$ for some $f, g \in A[X]$, then show that $zf - xg$ on X and consider the points $(0,y,0,w)$ resp. $(0,y,z,0)$ resp. $(x,0,0,w)$.