FB Mathematik | Winterterm 2004/05, Set 6
Gert-Martin Greuel | Thomas Markwig

Algebraic Geometry I

Due date: Friday, 09/12/2005, 18:00 Uhr

Exercise 1: Let $I = \left\langle -x^2 - y^2 + z^2 + x^2z + y^2z - z^3, -x^3 - xy^2 + xz^2 + x^3y + xy^3 - xyz^2, -x^3 - xy^2 + xz^2 + x^4 + x^2y^2 - x^2z^2 \right\rangle \trianglelefteq \mathbb{C}[x,y,z]$. Compute the dimension of $X = V(I)$ at $p = (0,0,0)$, $q = (0,0,1)$ and $r = (1,1,1)$. Compute the irreducible components of X in $\mathbb{A}^3_{\mathbb{C}}$ (not only in a neighbourhood of 0) and draw pictures of them to confirm your computations geometrically.

Exercise 2: Let X be a topological space, $\mathcal{F}$ a presheaf of abelian groups on X, $U \subseteq X$ open, and $s \in \mathcal{F}(U)$.
Show that the support $\operatorname{supp}(s) = \{p \in U \mid s_p \neq 0\}$ of s is a closed subset of U, and find an example which shows that the support $\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$ of $\mathcal{F}$ in X need not be closed.

Exercise 3: Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be ringed spaces and let $f : X \to Y$ be a continous map. Then the following are equivalent:

a. f is a morphism, i.e. $\forall\, U \subseteq Y$ open $f^*\big(\mathcal{R}_Y(U)\big) \subseteq \mathcal{R}_X\big(f^{-1}(U)\big)$.

b. For all $p \in X$ we have $f^*_p\big(\mathcal{R}_{Y,f(p)}\big) \subseteq \mathcal{R}_{X,p}$.

Note that the map $f^* : \mathcal{A}_{Y/K}(U) \to \mathcal{A}_{X/K}\big(f^{-1}U\big) : \varphi \mapsto \varphi \circ f$ induces a map $f^*_p : \mathcal{A}_{Y/K,f(p)} \to \mathcal{A}_{X/K,p} : \big[(V,\varphi)\big] \mapsto \big[(f^{-1}(V), \varphi \circ f)\big]$ for $p \in U$, and that in a natural way $\mathcal{R}_{X,p} \subseteq \mathcal{A}_{X/K,p}$.

Exercise 4: Let K be any field and let $X = V\big(x^2 - y^3\big) \subset \mathbb{A}^2_K$ be the cuspidal cubic.

a. Show that the map $\varphi : \mathbb{A}^1_K \to \mathbb{A}^2_K : t \mapsto \big(t^3, t^2\big)$ induces a homeomorphism from $\mathbb{A}^1_K$ to X which is a morphism, but NOT an isomorphism.

b. Consider the map $\psi : X \to \mathbb{A}^1_K : (x,y) \mapsto x$. Is this map an isomorphism when $K = \mathbb{R}$?

Recall, a homeomorphism is a bijective map f such that f and f^{-1} are continous.