FB Mathematik — Winterterm 2004/05, Set 7
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Algebraic Geometry I

Due date: Friday, 16/12/2005, 18:00 Uhr

Exercise 1: Show that every isomorphism $f : \mathbb{A}^1_K \to \mathbb{A}^1_K$ is of the form $f(x) = ax + b$ for some $a, b \in K$ with $a \neq 0$.

Exercise 2: Let K be an algebraically closed field and let $X \subset \mathbb{A}^2_K$ be a conic, i.e. $X = V(f)$ for some $f \in K[x, y]$ with $\deg(f) = 2$. Show that if X is irreducible, then X is *either* isomorphic to $V(y - x^2)$ *or* isomorphic $V(1 - xy)$.

Exercise 3: Let K be an algebraically closed field, $X \subseteq \mathbb{A}^k_K$, $Y \subseteq \mathbb{A}^m_K$, $Z \subseteq \mathbb{A}^n_K$ be affine varieties, $f \in \text{Mor}(X, Y)$, $g \in \text{Mor}(Y, Z)$, and $\varphi \in \text{Hom}_{K-alg}\big(A[Z], A[Y]\big)$, $\psi \in \text{Hom}_{K-alg}\big(A[Y], A[X]\big)$. Show:

a. $\big(\text{id}_X\big)^* = \text{id}_{A[X]}$ and $\big(\text{id}_{A[X]}\big)^\# = \text{id}_X$.

b. $(g \circ f)^* = f^* \circ g^*$ and $(\psi \circ \varphi)^\# = \varphi^\# \circ \psi^\#$.

c. $(f^*)^\# = f$ and $(\varphi^\#)^* = \varphi$.

Exercise 4: Let K be an algebraically closed field, and $f : X \to Y$ be a morphism of affine varieties over K. Show:

a. f is a closed embedding if and only if f^* is surjective.

b. f is dominant if and only if f^* is injective.

Recall, by definition f is a closed embedding if $f(X)$ is closed in Y and $X \to f(X) : x \mapsto f(x)$ is an isomorphism; f is dominant if $f(X)$ is dense in Y.