FB Mathematik Winterterm 2005/06, Set 9
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Algebraic Geometry I

Due date: Friday, 13/01/2006, 18:00 Uhr

Exercise 1: Let (U_i, φ_i), $i = 0, \dots, n$ be the canonical charts of $\mathbb{P}^n$. Recall that $U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\} \subset \mathbb{P}^n$ and $\varphi_i : U_i \to \mathbb{A}^n$ is given by

$$U_i \ni (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \in \mathbb{A}^n.$$

Show that φ_i, $i = 0, \dots, n$ are homeomorphisms in the Zariski topology. Deduce that the Zariski topology on $\mathbb{P}^n$ coincides with the Z-topology defined by the canonical charts, *i. e.*, $A \subseteq \mathbb{P}^n$ is a projective variety if and only if $\varphi_i(A \cap U_i) \subseteq \mathbb{A}^n$ is an affine variety for $i = 0, \dots, n$.

Exercise 2: In view of Exercise 1 we may identify (and we often do) $\mathbb{A}^n$ with the subset $U_0 \subset \mathbb{P}^n$ via the map $\varphi_0^{-1} : \mathbb{A}^n \to U_0$. A subset $V \subseteq \mathbb{A}^n$ is identified with $\varphi_0^{-1}(V) \subset \mathbb{P}^n$. The closure of $\varphi_0^{-1}(V) \subset \mathbb{P}^n$ in $\mathbb{P}^n$ is called the *projective closure* of V and denoted $\overline{V}$. Let $I \trianglelefteq K[x_1, \dots, x_n]$ be an ideal, $V = V_{\mathbb{A}^n}(I) \subseteq \mathbb{A}^n \subset \mathbb{P}^n$. Show that

a. $\overline{V} = V_{\mathbb{P}_n}(I^h)$, where $I^h \trianglelefteq K[x_0, \dots, x_n]$ is the homogenization of I with respect to the variable x_0.

b. $I_{\mathbb{P}^n}(\overline{V}) = I_{\mathbb{A}^n}(V)^h$.

Exercise 3: Prove the following statements.

a. Every projective variety $X \subseteq \mathbb{P}^n$ has a unique decomposition into irreducible components $X = X_1 \cup \dots \cup X_r$, where X_i, $i = 1, \dots, r$ are irreducible projective varieties.

b. $X \subseteq \mathbb{P}^n$ is irreducible if and only if $I(X) \trianglelefteq K[x_0, \dots, x_n]$ is a (homogeneous) prime ideal.

c. If $V = V_1 \cup \dots \cup V_r$ is the minimal irreducible decomposition of $V \subseteq \mathbb{A}^n$, then $\overline{V} = \overline{V_1} \cup \dots \cup \overline{V_r}$ is the minimal irreducible decomposition of $\overline{V} \subset \mathbb{P}^n$.

d. The map $V_{\mathbb{A}^n}(I) \mapsto V_{\mathbb{P}_n}(I^h)$ induces bijections

$$\begin{array}{ccc}
\{\text{affine varieties in } \mathbb{A}^n\} & \leftrightarrow & \left\{\begin{array}{c}\text{projective varieties in } \mathbb{P}^n \text{ without irreducible}\\ \text{components contained in } V_{\mathbb{P}^n}(x_0)\end{array}\right\} \\
\cup & & \cup \\
\left\{\begin{array}{c}\text{irreducible affine}\\ \text{varieties in } \mathbb{A}^n\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{irreducible projective varieties in } \mathbb{P}^n\\ \text{not contained in } V_{\mathbb{P}^n}(x_0)\end{array}\right\}
\end{array}$$

Exercise 4: Let $V \subseteq \mathbb{P}^n$ be a projective variety. Show that

$$\dim V = \max\{\dim \varphi(V \cap U_i) \mid i = 0, \dots, n\}.$$