FB Mathematik
Gert-Martin Greuel

Winterterm 2005/06, Set 10

Oleksandr Manzyuk

Algebraic Geometry I

Due date: Friday, 20/01/2006, 18:00 Uhr

Exercise 1: Prove that every quasiaffine variety is quasiprojective.

Exercise 2: Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasiprojective varieties, $i : Y \hookrightarrow \mathbb{P}^m$ the natural embedding, $U_i = \{x \in \mathbb{P}^n \mid x_i \neq 0\}$, $i = 0, \dots, n$, and $V_j = \{y \in \mathbb{P}^m \mid y_j \neq 0\}$, $j = 0, \dots, m$, the standard charts.

a. A map $\varphi : X \to K$ is regular if and only if the restrictions $\varphi : X \cap U_i \to K$ are regular for $i = 0, \dots, n$ (as functions on quasiaffine varieties).

b. A map $f : X \to Y$ is a morphism if and only if the composition $i \circ f : X \to Y \hookrightarrow \mathbb{P}^m$ is a morphism.

c. A map $f : X \to \mathbb{P}^m$ is a morphism if and only if for each $j = 0, \dots, m$ the restriction $f : f^{-1}(V_j) \to V_j \cong \mathbb{A}^m$ is a morphism.

d. A map $f : X \to \mathbb{A}^m$ is a morphism if and only if the k-th coordinate function $f_k = x_k \circ f : X \to \mathbb{A}^m \to K$ is regular for each $k = 1, \dots, m$.

Exercise 3: Let $I = \{(i_0, \dots, i_n) \in \mathbb{N}^{n+1} \mid \sum_{\nu=0}^{n} i_\nu = d\}$. Note that I indexes the monomials of degree d in $n+1$ variables. It has $\binom{n+d}{n}$ elements. Write $N = \binom{n+d}{n} - 1$, and consider the projective space $\mathbb{P}^N$ whose coordinates are indexed by I; thus a point of $\mathbb{P}^N$ can be written $(\dots : z_{i_0 \dots i_n} : \dots)$. The *Veronese mapping* is defined to be

$$\rho_d : \mathbb{P}^n \to \mathbb{P}^N, \quad (x_0 : \dots : x_n) \mapsto (\dots : z_{i_0 \dots i_n} : \dots),$$

where $z_{i_0 \dots i_n} = x_0^{i_0} \dots x_n^{i_n}$, $(i_0, \dots, i_n) \in I$. Prove that

a. $\operatorname{Im}(\rho_d)$ is a projective variety in $\mathbb{P}^N$ defined by the system of equations:

$$z_{i_0 \dots i_n} z_{j_0 \dots j_n} = z_{k_0 \dots k_n} z_{l_0 \dots l_n}, \quad i_s + j_s = k_s + l_s, \quad s = 0, \dots, n.$$

b. $\rho_d : \mathbb{P}^n \to \operatorname{Im}(\rho_d)$ is an isomorphism.

Exercise 4: Let $n \geq 2$ and $p \in \mathbb{P}^n$. Prove that $\mathbb{P}^n \setminus \{p\}$ is not an affine variety.

Hint. We may assume without loss of generality that $p = (1 : 0 : \dots : 0)$. Denote $U = \mathbb{P}^n \setminus \{p\} = \bigcup_{i=1}^n U_i$ and show that $\mathcal{O}_U(U) = K$. You may proceed as follows. For each $i = 1, \dots, n$ consider the restriction of $f \in \mathcal{O}_U(U)$ to U_i, and show that it has the form $F_i(x_0, \dots, x_n)/x_i^{d_i}$, where F_i is a homogeneous polynomial of degree d_i. Compare these restrictions on the intersection $U_i \cap U_j$.