FB Mathematik
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Winterterm 2005/06, Set 11

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Algebraic Geometry I

Due date: Friday, 03/02/2006, 18:00 Uhr

Exercise 1: Let X, Y be quasiprojective varieties. Consider the set of pairs (U, f), where $U \subseteq X$ is a dense open subset, $f : U \to Y$ is a morphism of varieties. It is equipped with the following equivalence relation: $(U, f) \sim (V, g)$ if there exists a dense open subset $W \subseteq U \cap V$ such that $f|_W = g|_W$. The corresponding equivalence classes are called *rational maps* from X to Y and conventionally denoted $X \dashrightarrow Y$.

a. Let $\varphi : X \dashrightarrow Y$ be a rational map. Show that there exists a representative (U, f) of φ which is maximal in the following sense: an arbitrary representative (V, g) of φ is the restriction of (U, f), i.e., $V \subseteq U$ and $f|_V = g$. The pair (U, f) is uniquely defined by the above property. The set U is called the *domain* of the rational map φ.

b. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona map. It is given as the equivalence class of the following map:

$$U \to \mathbb{P}^2, \quad (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1),$$

where $U = \mathbb{P}^2 \setminus V_{\mathbb{P}^2}(x_0x_1x_2)$. Compute the domain of φ.

Exercise 2: Let $C \subset \mathbb{P}^2$ be a projective curve. Let $\widetilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the blow-up of $\mathbb{P}^2$ at the point $P = (0 : 0 : 1)$ and let $\pi : \widetilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projection map. Let $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ be the strict transform of C. In each of the following cases, show that the map π induces an isomorphism $\widetilde{C} \xrightarrow{\sim} \mathbb{P}^1$:

a. $C = V_{\mathbb{P}^2}(x_0^3 + x_0^2x_1 - x_1^2x_2)$;

b. $C = V_{\mathbb{P}^2}(x_0^3 - x_1^2x_2)$.

Exercise 3: In each of the following cases, blow-up the singularity and draw the real part of the total transform and of the strict transform in affine charts (using the ray tracer `surf`):

a. $X = V_{\mathbb{P}^2}(x_0^4 - x_1^3)$;

b. $X = V_{\mathbb{P}^3}(x_0^4 - x_1^3 + x_2^2)$.

Exercise 4: Let X be a topological space. Show that X is Hausdorff if and only if the set $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$, where the latter space is endowed with the usual product topology.