FB Mathematik | Winterterm 2005/06, Set 14
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Algebraic Geometry I

Due date: Friday, 17/02/2006, 18:00 Uhr

Exercise 1: Let X, Y be Noetherian prevarieties. Then $X \times Y$ is a Noetherian prevariety and $\dim(X \times Y) = \dim X + \dim Y$.

Exercise 2: (1) Let $(x_0 : x_1 : \ldots : x_m)$ resp. $(y_0 : y_1 : \ldots : y_n)$ be homogeneous coordinates in $\mathbb{P}^m$ resp. $\mathbb{P}^n$. Let $F_1, \ldots, F_k \in K[x_0, \ldots, x_m, y_0, \ldots, y_n]$ be homogeneous (separately) in the variables x_i, $i = 0, \ldots, m$ and in the variables y_j, $j = 0, \ldots, n$. Then the set

$$V_{\mathbb{P}^m \times \mathbb{P}^n}(F_1, \ldots, F_k) \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{P}^m \times \mathbb{P}^n \mid F_s(x, y) = 0, \quad s = 1, \ldots, k\}$$

is closed in $\mathbb{P}^m \times \mathbb{P}^n$, and every closed subset of $\mathbb{P}^m \times \mathbb{P}^n$ is of this form.
(2) Let $(x_0 : x_1 : \ldots : x_m)$ be homogeneous coordinates in $\mathbb{P}^m$, $(y_1, \ldots, y_n)$ coordinates in $\mathbb{A}^n$. Let $F_1, \ldots, F_k \in K[x_0, \ldots, x_m, y_1, \ldots, y_n]$ be homogeneous in the variables x_i, $i = 0, \ldots, m$. Then the set

$$V_{\mathbb{P}^m \times \mathbb{A}^n}(F_1, \ldots, F_k) \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{P}^m \times \mathbb{A}^n \mid F_s(x, y) = 0, \quad s = 1, \ldots, k\}$$

is closed in $\mathbb{P}^m \times \mathbb{A}^n$, and every closed subset of $\mathbb{P}^m \times \mathbb{A}^n$ is of this form.

Exercise 3: Let X be a separated prevariety, $U, V \subset X$ open affine subsets. Show that $U \cap V$ is an open affine subset of X.

Exercise 4: Let X be a closed subset of $\mathbb{P}^m \times \mathbb{A}^n$. By Exercise 2,(2) there exist polynomials $F_1, \ldots, F_k \in K[x_0, \ldots, x_m, y_1, \ldots, y_n]$ homogeneous in x_i, $i = 0, \ldots, m$ such that $X = V_{\mathbb{P}^m \times \mathbb{A}^n}(F_1, \ldots, F_k)$. Let I denote the ideal generated by $F_1, \ldots, F_k$. Let π denote the projection $\mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$. Prove that

$$\pi(X) = V\Big(\bigcap_{i=0}^{n} (I_{x_i=1}) \cap K[y_1, \ldots, y_n]\Big).$$

Hint. Use Exercises