FB Mathematik — Winterterm 2009/10, Set 1
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Algebraic Geometry I

The exercises on the first set of exercises should not be handed in for marking, they will be discussed in class.

Exercise 1: Let K be any field, $I \trianglelefteq K[x_1,\ldots,x_n] = K[\underline{x}]$ an ideal.
Prove the following statements, or find counterexamples.

a. If $Z(I) = \mathbb{A}^n(K)$, then $I = (0)$.

b. If $Z(I) = \emptyset$, then $I = K[\underline{x}]$.

c. If I is a prime ideal, then $Z(I)$ is irreducible.*

Is it possible to weaken respectively to strengthen the hypotheses such that the results remain respectively become true?

Exercise 2: Let K be an infinite field and let $\emptyset \neq X, Y \subseteq \mathbb{A}^n_K$ be open in the Zariski-topology. Show that the intersection $X \cap Y$ is non-empty. Does the statement still hold if K is finite?

Exercise 3: Let X and Y be two affine algebraic sets in $\mathbb{A}^n_K$ for an algebraically closed field K. Show that

$$I(X \cup Y) = I(X) \cap I(Y)$$

and

$$I(X \cap Y) = \sqrt{I(X) + I(Y)}.$$

Give an example which shows that taking the radical in the second claim is necessary, and interprete this geometrically.

Exercise 4: Let K be an infinite field, $X = Z(y - x^2) \subset \mathbb{A}^2_K$ and $Y = Z(xy - 1) \subset \mathbb{A}^2_K$.
Show that $K[x,y]/I(X) \cong K[t]$ and $K[x,y]/I(Y) \cong K\left[t, \frac{1}{t}\right]$.

*A non-empty topological space V is called *irreducible* if it is not the union of two proper subsets which are both closed.