FB Mathematik
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Winterterm 2009/10, Set 2

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Algebraic Geometry I

Due date: Monday, 09/11/2009, 10:00 Uhr

Exercise 5: Let K be an infinite field and $X \subseteq \mathbb{A}^3_K$ be the union of the three coordinate axis. Find the ideal $I(X)$ and show that it cannot be generated by less than three elements.

Exercise 6: Let X be a topological space and $\emptyset \neq Y \subseteq X$.

a. If Y is irreducible, then the closure $\overline{Y}$ in X is irreducible.

b. If X is irreducible and Y is open in X, then Y is dense in X, i.e. $\overline{Y} = X$.

c. If Y is irreducible, there is a maximal irreducible subspace Y' in X containing Y, i.e. there is a $Y' \subseteq X$ irreducible such that $Y \subseteq Y'$ and for all irreducible $Y'' \subseteq X$ with $Y' \subseteq Y''$ we have $Y' = Y''$.

   We call these maximal irreducible subsets of X its *irreducible components*.

Exercise 7: Find a polynomial parametrisation of the curve $C = Z(y^2 - x^2 - x^3) \subset \mathbb{A}^2_{\mathbb{R}}$, i.e. find a map

$$\varphi : \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}} : t \mapsto \big(f(t), g(t)\big)$$

whose image is C, where $f, g \in \mathbb{R}[t]$ are polynomials.

Hint, draw the curve $C = Z(y^2 - x^2 - x^3)$ with `surf` (see *my* lecture notes p. 12 for explantions on how to use `surf`) and you will find that it has a double point in $(x, y) = (0, 0)$. Then consider the lines in $\mathbb{R}^2$ through this point. Each such line cuts the curve in precisely one more point. Now consider the line L parallel to the y-axis through the point $(x, y) = (-1, 0)$. Each line through the origin also cuts L in precisely one point. That way you can define a map from L to C which is a parametrisation.

Exercise 8: Consider the three plane curves C_i in $\mathbb{C}^2$ given by the equations $f_i = 0$, $i = 1, 2, 3$, where

$$f_1 = y^2 - 5x^2 - x^3, \quad f_2 = x^4 + y^4 - 2, \quad \text{respectively} \quad f_3 = y^2 + 5x^2 + x^3.$$

How many intersection points do C_1 and C_2 respectively C_1 and C_3 have? How many of these points are real? You may use SINGULAR for the calculations. Verify the real points by drawing the curves using Surf.

Hint, have a look at the SINGULAR example `sing-02` on the web page

`http://www.mathematik.uni-kl.de/~greuel/courses/WS05-06-AG1/index.html`

of Prof. Greuel to see how SINGULAR can be used to compute the intersection points.