FB Mathematik — Winterterm 2009/10, Set 3
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Algebraic Geometry I

Due date: Monday, 16/11/2009, 10:00 Uhr

Exercise 9: [Generalised Division with Remainder]

a. Let R be any commutative ring with one, and let $0 \neq f, g \in R[t]$ be two polynomials with coefficients in R. Show, if $f = \sum_{i=0}^{n} f_i \cdot t^i$ with $f_n \neq 0$, then there exists a natural number k and two polynomials $q, r \in R[t]$ such that

$$a_n^k \cdot g = q \cdot f + r \quad \text{with} \quad \deg(r) < \deg(f).$$

b. Let K be an infinite field and $X = Z(x^2 - y^3) \subset \mathbb{A}_K^2$.
Use part a. to show that $I(X) = \langle x^2 - y^3 \rangle$.

Exercise 10: For a polynomial $f \in K[\underline{x}]$ we define the *basic open set*

$$X_f = \mathbb{A}_K^n \setminus Z(f).$$

Show the following statements:

a. $X_f \cap X_g = X_{f \cdot g}$.

b. $\mathbb{A}_K^n$ and $\emptyset$ are basic open sets.

c. Every open subset in $\mathbb{A}_K^n$ is a finite union of basic open sets.
In particular, the basic open sets form a basis of the topology of $\mathbb{A}_K^n$.

d. Every open subset in $\mathbb{A}_K^n$ is quasi compact, i.e. each open covering contains a finite subcovering.

e. If K is algebraically closed, $X_f \subseteq X_g$ if and only if g divides some power of f.

Exercise 11: Consider $X = Z(xy - 1) \subset \mathbb{A}_K^2$, $\varphi : X \to \mathbb{A}_K^1 : (x, y) \mapsto x$, and the set $Y = \mathrm{Im}(\varphi) \subset \mathbb{A}_K^1$. What is $I(Y)$? Is $Y = Z\big(I(Y)\big)$? Draw a schematic image for $K = \mathbb{R}$.

Note, your answer will depend on the field K! Consider the cases K finite or infinite, or, if you prefer, $K = \mathbb{Z}/2\mathbb{Z}$ and $K = \mathbb{R}$.

Exercise 12:

a. Consider the surface $V(f) \subset \mathbb{C}^3$ defined by the polynomials $f = x^2 + y^2 + xyz$ and consider the planes $H_c = V(x - c) \subset \mathbb{C}^3$ for $c \in \mathbb{C}$ arbitrary. Determine $I(X_c)$ for $X_c = H_c \cap V(f)$.

b. Draw image of the surface $V(x^2 + y^2 + xyz) \subset \mathbb{R}^3$ and its intersecions with the planes $H_c = V(x - c)$ for $c \in \{-1, 0, 1\}$. You may use `surfex`.

Note that if you want to show that a certain ideal is a prime ideal it makes sense to consider the quotient ring by the ideal. – You may use Eisenstein's Criterion for irreducibility without proof.