FB Mathematik — Winterterm 2009/10, Set 4
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# Algebraic Geometry I

Due date: Monday, 23/11/2009, 10:00 Uhr

**Exercise 13:** Let $K$ be an algebraically closed field.

a. Let $X \subset \mathbb{A}^3_K$ be the following affine algebraic set

$$X = Z(x^2 - yz, xz - y).$$

Find the irreducible components of $X$ and their vanishing ideals.

b. Let $X = Z\big(yz^2 - yz, xz^2 - xz, xyz + xz - y^2z - yz, y^3z - yz\big) \subseteq \mathbb{C}^3$. Compute

(a) the irreducible componemts of the variety $X$,

(b) $\dim(X)$, and

(c) the dimension of each irreducible component of $X$

without using a computer algebra system. Check your result by using SINGULAR. How many connected components does $X$ have? Draw by hand a picture of the real part of $X$, and compare it to the result you get using `surfex`.

**Exercise 14:**

a. Let $R = K[y]$ and let $f, g \in R[x]$ be such that no polynomial $p \in R[x] \setminus R$ divides both $f$ and $g$. Then $\langle f, g\rangle \cap R \neq \{0\}$.

b. If $f, g \in K[x, y]$ are two coprime polynomials then the ideal $\langle f, g\rangle$ contains a polynomial $0 \neq p \in K[x]$ and a polynomial $0 \neq q \in K[y]$.

In particular, $\#Z(f, g) \leq \deg(p) \cdot \deg(q) < \infty$.

c. If $K$ is algebraically closed then $\dim(K[x, y]) = \dim_c(\mathbb{A}^2_K) = 2$.

Note, the statement in c. holds also if $K$ is not algebraically closed, but we cannot prove this with what we have learned so far in our lecture.

**Exercise 15:** Let $X \subseteq \mathbb{A}^n_K$ be an algebraic variety. Show that $\dim(X) = n - 1$ if and only if there is an irreducible polynomial $f \in K[x_1, \ldots, x_n]$ such that $I(X) = \langle f\rangle$.

**Exercise 16: [Tangent Cone]**

Let $f \in K[x, y] \setminus K$ be a non-constant polynomial and $p = (0, 0) \in Z(f)$. How could one define the tangent lines at $p$? Consider the following examples for $K = \mathbb{R}$: $f = x$, $f = y - x^2$, $f = x^2 - y^3$, $f = x^2 - y^2$, $f = y^2 - x^4$.