FB Mathematik — Winterterm 2009/10, Set 5
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Algebraic Geometry I

Due date: Monday, 30/11/2009, 10:00 Uhr

Exercise 17:

a. Let $U = \mathbb{A}^2_{\mathbb{C}} \setminus Z(y^3 - y^2, y^2 + y - 2)$. Are all regular functions on U globally given by rational functions? If so, explain why, if not, give a counter example.

b. Let K be an algebraically closed field. Show that all regular functions on $U = \mathbb{A}^2_K \setminus \{(0,0)\}$ are globally given by rational functions and determine the subring of $K(x,y)$ isomorphic to $\mathcal{O}_{\mathbb{A}^2_K}(U)$.

Exercise 18: Let $X \subseteq \mathbb{A}^n_K$ be an affine algebraic set. If $U \subseteq X$ is open and dense then there exists a basic open subset $X_f \subseteq U$ of X which is dense in X.

Hint, use Prime Avoidance, see Prop. 1.17 in `http://www.mathematik.uni-kl.de/~keilen/download/Lehre/MGSS09/CommutativeAlg.pdf`.

Exercise 19: Let K be an algebraically closed field and $X \subseteq \mathbb{A}^n_K$ be an affine algebraic set with irreducible decomposition $X = X_1 \cup \ldots \cup X_k$. Show that

$$K(X) \cong K(X_1) \oplus \ldots \oplus K(X_k).$$

Hint, use the Chinese Remainder Theorem, see Thm. 1.12 in `http://www.mathematik.uni-kl.de/~keilen/download/Lehre/MGSS09/CommutativeAlg.pdf`.

Exercise 20:

a. Let $K \subset L$ be a field extension, where L is algebraically closed. Let $f = \sum_{|\alpha|=0}^{n} a_\alpha \underline{x}^\alpha \in K[\underline{x}]$ be a polynomial and consider the ideal

$$I = \left\langle \sum_{\beta+\gamma=\alpha} b_\beta \cdot c_\gamma - a_\alpha \;\middle|\; |\alpha| = 0, \ldots, 2n-2 \right\rangle \lhd L\big[b_\beta, c_\gamma \;\big|\; |\beta|, |\gamma| = 0, \ldots, n-1\big],$$

where $a_\alpha = 0$ for $|\alpha| = n+1, \ldots, 2n-2$.

Show that f is irreducible over L if and only if $I = L\big[b_\beta, c_\gamma \;\big|\; |\beta|, |\gamma| = 0, \ldots, n-1\big]$.

Remark: This gives a criterion to check, whether a hypersurfes given by a polynomial $f \in \mathbb{Q}[\underline{x}]$ is irreducible over $\mathbb{C}$.

b. Check, using this criterion and SINGULAR, whether the polynomials $f = x^4 + 4x^3y + 2x^2y^2 - 4xy^3 + y^4 + 2x^2 + 4xy + 1$ and $g = x^4 + 4x^3y + 2x^2y^2 - 4xy^3 + y^4 + 2x^2 + 4xy - 1$ are irreducible over $\mathbb{Q}$ and / or $\mathbb{C}$.

Hint: The command `reduce(1,std(I));` returns zero if and only if the ideal I is the whole ring.