FB Mathematik — Winterterm 2009/10, Set 6
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Algebraic Geometry I

Due date: Monday, 07/12/2009, 10:00 Uhr

Exercise 21: Let X be a topological space and B a basis of the topology. Suppose that for every $U \in B$ we have a subring $\mathcal{A}(U)$ of the ring $K^U = \{f : U \longrightarrow K \mid f \text{ function}\}$ of all functions from U to K, and suppose that whenever $U, V \in B$ with $U \subseteq V$ then the restriction

$$\mathrm{res}_{V,U} : \mathcal{A}(V) \longrightarrow \mathcal{A}(U) : f \mapsto f_{|U}$$

is defined, i.e. $f_{|U} \in \mathcal{A}(U)$.

a. Define $\mathcal{A}(U)$ for an arbitrary open subset U of X such that $\mathcal{A}$ is a presheaf.

b. Give a condition on the $\mathcal{A}(V)$ for basic open subsets $V \in B$ such that the presheaf $\mathcal{A}$ is actually a sheaf.

Exercise 22: Let X be a topological space and let $\mathcal{F}$ be a presheaf of rings on X. We denote by $\coprod_{p\in X} \mathcal{F}_p$ the disjoint union of the stalks of $\mathcal{F}$. For an open subset $U \subseteq X$ we call a function

$$s : U \longrightarrow \coprod_{p\in X} \mathcal{F}_p$$

a *section* of $\coprod_{p\in X} \mathcal{F}_p$ over U if $s(p) \in \mathcal{F}_p$ for all $p \in U$ and if, moreover, for every $p \in U$ there is an open neighbourhood V in U and there is a $g \in \mathcal{F}(V)$ such that

$$s(q) = g_q \quad \text{for all } \ q \in V.$$

Show, that if we denote by

$$\mathcal{G}(U) = \left\{ s : U \longrightarrow \coprod_{p\in X} \mathcal{F}_p \;\middle|\; s \text{ is a section} \right\}$$

the set of sections over U, then the collection of the $\mathcal{G}(U)$ together with the obvious restriction maps is a sheaf of rings on X.
Moreover, if $\mathcal{F}$ is a sheaf then the map

$$\mathcal{F}(U) \longrightarrow \mathcal{G}(U) : f \mapsto \left(U \to \coprod_{p\in X} \mathcal{F}_p : p \mapsto f_p \right)$$

is an isomorphism of rings for any $U \subseteq X$ open.

Exercise 23: Let K be algebraically closed, $X = Z(x) \subset \mathbb{A}^2_K$ and $Y = Z(x^2 - y^3) \subset \mathbb{A}^2_K$.

a. Is the K-algebra $K[X]$ isomorphic to $K[Y]$?

b. Is the field $K(X)$ isomorphic to the field $K(Y)$?

Exercise 24: Let $\varphi : X \longrightarrow Y$ be a morphism of affine algebraic sets. Is it true that the image $\varphi(A)$ of a closed subset of $A \subseteq X$ is always closed in Y?